Oxford Topological Field Theory

Note Title 3/14/2007

Formal structure so far:

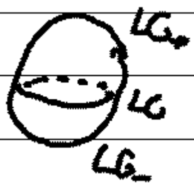
To X Riemann surface $\rightsquigarrow$ assign a dg category

$$Z_{G,0}(X) := D(Bun_G X, \mathcal{D})$$

with an action for every $x \in X$ of the category

$$D(LG_+ \backslash LG / LG_+, \mathcal{D}) \simeq Rep_{dg} G^\vee.$$

Slight reformulation: look at $X = \mathbb{P}^1$



$Bun_G \mathbb{P}^1 = \cancel{LG_-} \backslash LG / LG_+$, combinatorially looks like $LG_+ \backslash LG / LG_+$, & their categories of $\mathcal{D}$-modules are roughly the same, so we'll say

$$Z_{G,0}(\mathbb{P}^1) = \text{Hecke operators on } Z_{G,0}(-)$$

On the other hand we also have another dg category

$$Z_{G^\vee,\infty}(X) := D(Conn_{G^\vee} X, \mathcal{O})$$

which is acted on for each $x \in X$ by $Rep\, G^\vee$

Observe $\mathrm{Conn}_{G^\vee}(\mathbb{P}^1) = \{\text{trivial conn}\} \circlearrowleft \underset{\text{automorphisms}}{G^\vee}$

So $\mathcal{O}$-modules on $\mathrm{Conn}_{G^\vee}(\mathbb{P}^1) = \mathrm{Rep}\, G^\vee$ again!

$Z_{G^\vee,\infty}(\mathbb{P}^1)$ = Hecke operators on $Z_{G^\vee,\infty}(-)$

In this formulation the geometric Langlands conjecture becomes

$$\begin{array}{lccc} & Z_{G,0}(X) & \simeq & Z_{G^\vee,\infty}(X) \\ & \circlearrowleft & & \circlearrowleft \\ (x \in X) & Z_{G,0}(\mathbb{P}^1) & \simeq & Z_{G^\vee,\infty}(\mathbb{P}^1) \end{array}$$

Kapustin and Witten derived this conjecture from a fundamental conjecture in physics:

electric-magnetic duality for supersymmetric four-dimensional gauge theories

[Montonen-Olive S-duality],

which they interpret as giving an isomorphism of two 4-dimensional topological field theories

$$Z_{G,\Psi} \simeq Z_{G^\vee,-1/\Psi}$$

We'll first examine what a 4d TFT is and then how the key features of geometric Langlands emerge.

Topological Field Theory

0 tiers An n-dim TFT $Z/\mathbb{C}$ is an assignment

$$M^n \longmapsto Z(M^n)$$ M n-dim smooth oriented compact

- oriented diffeo invariant
- multiplicative: $\emptyset \mapsto 1$ $\quad \amalg \mapsto \cdot$

$$Z(M^n) \longleftrightarrow \int_{\varphi \text{ fields on } M} e^{-S(\varphi)} \mathcal{D}\varphi$$

Next express locality (cut & paste) of Z:

1 tier An n-dim TFT Z assigns (cpt. sm. otd.)

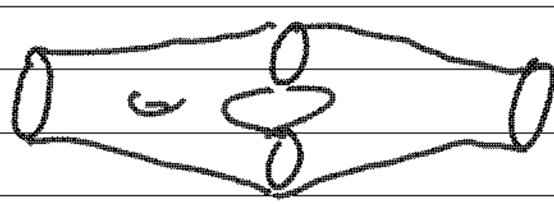
$$N^{n-1} \longmapsto Z(N^{n-1}) \in Vect_{\mathbb{C}}$$

- functorial
- multiplicative: $\emptyset \mapsto \mathbb{C}$, $(\,)^{op} \mapsto {}^*$, $\amalg \mapsto \otimes$

Also M^n manifold with boundary

$$Z: M^n \longmapsto Z(M^n) \in Z(\partial M)$$

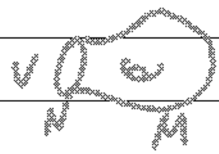
- $Z(N \times I) = Id \in Hom(N, N)$
- functorial under cobordisms

 $\in Hom(Z(0), Z(0))$

$$\uparrow \langle , \rangle$$

$$Hom(Z(0), Z(00)) \otimes Hom(Z(00), Z(0))$$

Notation: $v \in Z(N)$ can use as label to "close off" a boundary component $\cong N$: e.g.

 $= Z(M)(v) \in Z(\emptyset) = \mathbb{C}$

<u>Theorem</u> 2D TFT $\Longleftrightarrow$ commutative Frobenius algebras

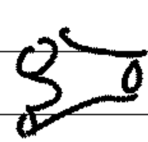
(i.e. unital commutative associative algebra with nondegenerate trace)

How? $Z \longmapsto H = Z(S^1)$

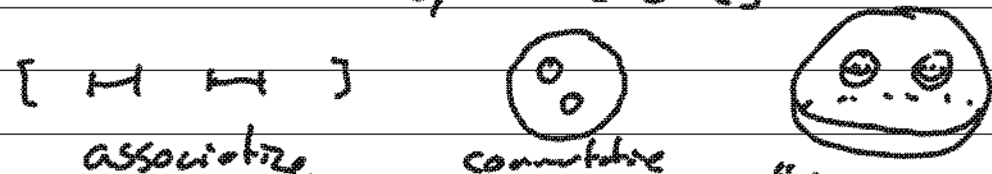
$Z(\ ) = 1 \in H$ $\qquad Z(\ ) = tr : H \longrightarrow \mathbb{C}$

$Z(\ ) : H \otimes H \xrightarrow{\mu} H$.

Check $Z(\ ) = tr(\mu) : H \otimes H \longrightarrow \mathbb{C}$ nondegenerate. Moreover any Σ^2 can be decomposed into such building blocks, so recover $Z(\Sigma)$ from Frobenius structure.

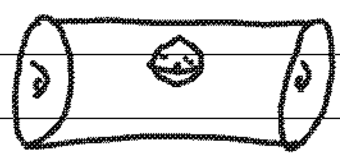
Commutativity  $\longleftrightarrow$ 

- More generally: Z n-dim
  $Z(S^{n-1})$ carries multiplications parametrized by little discs



associative $Z(S^0)$ — commutative $Z(S^1)$ — "more commutative" $Z(S^2)$

- Moreover $Z(S^{n-1})$ acts on $Z(N^{n-1})$



choose point $x \in N$ and a time & remove ball around x to get action.

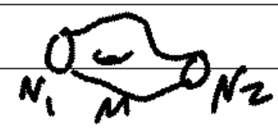
Variants

- Behavior in families: require local constancy under deformations, so given family $N^{n-1} \hookrightarrow \mathcal{N} \downarrow B$ get local system $Z(\mathcal{N}) \downarrow B$

- dg version: make $Z(N)$ a dg vector space, isoms ⇝ quisms — more homotopical variant of TFTs.

Cobordisms(N_1, N_2) is a <u>space</u>,
so can incorporate topology in morphisms!

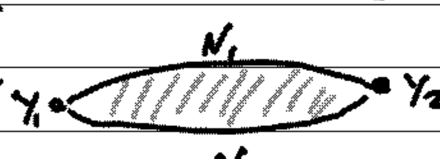
$$C^*(\text{Cobord}(N_1, N_2)) \longrightarrow \text{Hom}^*(N_1, N_2)$$

generalizes  $\longmapsto Z(M) \in \text{Hom}^0(N_1, N_2)$

$\Rightarrow Z(S^{n-1})$ is an E_n algebra,
really gets more & more commutative with n.
(e.g. $E_1 = A_\infty$ homotopy associative, $\underline{\sim}$ dga)

<u>2 tiers</u> Express locality of $Z(N^{n-1})$
by cutting along Y^{n-2} :
Z assigns Y^{n-2} a (linear, or dg) category
so Homs are $\mathbb{C}$ (dg) vector spaces
· (Z functorial, monoidal ...)

$Z\left(Y_1 \xrightarrow{N} Y_2 \right)$ is a functor $Z(Y_1) \to Z(Y_2)$

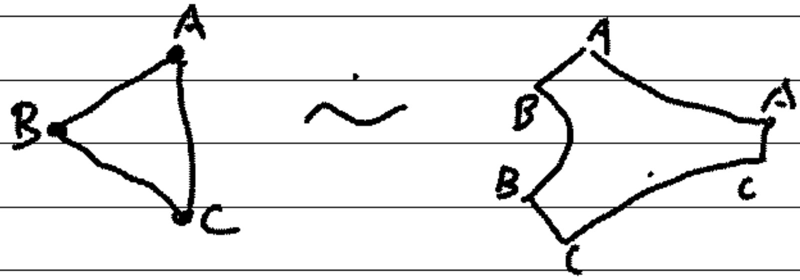
Z () is a natural transformation
$Z(N_1) \longrightarrow Z(N_2)$

$Z(S^0)$ monoidal, $Z(S^1)$ braided, $Z(S^{\geq 2})$ symmetric

Careful studies of 2-tiered 2-dim theories
by Moore-Segal & Costello

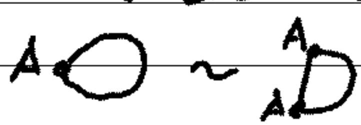
example with labellings : $A, B, C \in Ob(\mathcal{C} = Z(\cdot))$

$\overset{A}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{B}{\bullet} \in Vect$ is $Hom_{\mathcal{C}}(A,B)$



$\in Hom_{Vect}\left(Hom(A,B) \otimes Hom(B,C), Hom(A,C) \right)$

is (associative / A_∞) composition law



: $Hom_{\mathcal{C}}(A,A) \longrightarrow \mathbb{C}$ trace

$\Rightarrow$ $Hom_{\mathcal{C}}(A,A)$ is a (NC) Frobenius algebra.

Theorem (Costello) [Maxim!] 2tier 2d TFTs
are Non-Commutative Calabi-Yaus :
dg categories that look like $D(X, \mathcal{O})$
for X with holomorphic volume form

[ie have trace maps $Hom^{\bullet}_{\mathcal{C}}(A,A) \longrightarrow \mathbb{C}$ [dim X]]

Dimensional reduction: Σ k-manifold

Z n-dim TFT $\Longrightarrow$ Z_Σ $(n-k)$dim TFT:

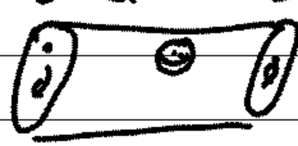
$$Z_\Sigma(M) := Z(M \times \Sigma)$$

So a 2-tier 4dim theory Z gives rise to

X surface $\longmapsto Z_X(\cdot) = Z(X)$ NC CY

Moreover $Z_{S^2}(\cdot) = Z(S^2)$ is a symmetric (E_3) monoidal category and acts on $Z(X)$

given any $x \in X$



Note: $T^*\mathrm{Bun}_G$, $\mathrm{Conn}_{G^\vee}$ are CY

(in fact holomorphic symplectic)

& $\mathcal{D}_{\mathrm{Bun}_G}$ is a NC CY ...

So geometric Langlands indeed looks like a statement about 4d TFTs!

How does this arise in physics?

2d TFTs

X Calabi-Yau manifold (Kähler)

$\longrightarrow$ two 2d TFTs, the A-model & the B-model. Both come from the same conformal field theory: a theory that depends on the complex structure of Σ^2 (& a spin_c structure), known as the $N=2$ SUSY σ-model

Fields in the theory ~ $Map(\Sigma \to X)$ + fermions

"strings moving on X"

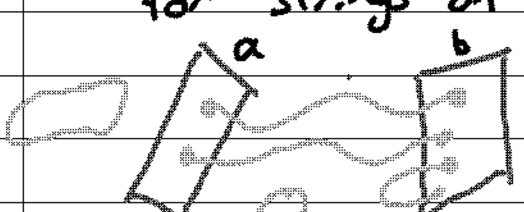
Two ways to take topological twist: change transformation properties of the fields so that changes of coordinates are exact wrt differentials coming from SUSY

$\Rightarrow$ homotopically topological field theory.

$Z_{X,A}$ A-model: depends only on symplectic structure of X

$Z_{X,B}$ B-model: depends only on complex structure of X

Category $Z(\bullet)$: boundary conditions for strings on X [D-branes]



$Hom_{\mathcal{C}}(a,b)$ = states of strings from a to b

(quantize maps with ∂ conditions)

B-branes are coherent sheaves
(roughly vector bundles on submanifolds):

$$Z_{X,B}(\cdot) = D(X,\mathcal{O})$$

A-branes more complicated: objects ~ coisotropic submanifolds with local systems

$$Z_{X,A}(\cdot) \supset \mathrm{Fuk}(X) \quad \text{derived Fukaya category}$$

.... objects are Lagrangians $\subset X$ (+ "grading" & local system)

Morphisms: intersections

+ higher corrections to associativity from pseudo holomorphic discs

- Homological mirror symmetry (example):

$$\begin{tikzcd} A \arrow[dr] & & A^\vee \arrow[dl] \\ & B & \end{tikzcd}$$

dual ($\mathbb{R}$) torus fibrations

$$Z_{A,A} \simeq Z_{A^\vee,B}$$

- KW suggest that for M complex, A-branes (T^*M) are $D(M,\mathcal{D})$ ... in particular $\mathrm{Fuk}(T^*M)$ are holonomic $\mathcal{D}$-modules (Lagrangian support).

Nadler-Zaslow: prove real analytic version of the latter.

Electric-Magnetic Duality

G_c compact Lie group, (G complexification)
$\Rightarrow$ 4d (Hg) TFT $Z_{G,\Psi}$ ($\Psi \in \mathbb{P}^1$ parameter)
called $N=4$ SUSY Yang-Mills in GL twist

This is a gauge theory: fields are connections A on a principal G_c bundle on M, Higgs field (adjoint 1-form) φ & two scalars + fermions
(bosons come from gauge fields on 10-manifold $T^*M \times \mathbb{R}^2$ invariant in 6 dimensions)
[Particular coordinate dependence $\Longleftrightarrow$ topological twist of $N=4$ YM on flat space]

Parameter Ψ: controls choice of BRST operator Q (differential made out of SUSYs) & action S
(combination of YM $\int F \wedge * F$ & Chern $\int F \wedge F$ terms, F = curvature of A)

S-Duality Conjecture $Z_{G,\Psi} \simeq Z_{G^\vee, -\frac{1}{\Psi}}$
- part of $SL_2\mathbb{Z}$ symmetry

$G = GL_1$: electric-magnetic duality of Maxwell theory.

What does a gauge theory assign to $M^4, N^3, C^2, \ldots$? [caricature]

- $Z(M^4) \sim$ # solutions to the classical field equations (Yang-Mills-Higgs)
- $Z(N^3) \sim$ cohomology (or cochains) of moduli space of dimensionally reduced ("monopole") equations
- $Z(C^2) \sim$ category of branes on moduli space of dimensionally reduced equations

C surface, $Z_C(\Sigma) = Z(C \times \Sigma)$:
Think of C as very small $\Longrightarrow$ field configurations on $C \times \Sigma$ $\leadsto$ maps from Σ to moduli space $\mathcal{M}_C$ of solutions on C ... ie σ-model on $\mathcal{M}_C$.

Our case: $\mathcal{M}_C$ is Hitchin moduli space

Hitchin's equations on C:

$$F - \phi \wedge \phi = 0 \qquad (F = dA)$$
$$d_A \phi = 0 = d_A * \phi$$

Hyperkähler moduli space $\Longrightarrow$ I, J, K Kähler structures

So have a wealth of σ-models: A or B in any of I, J, K

KW: Ψ parameter $\longleftrightarrow$ $\mathbb{P}^1$ family of σ-models

$\Psi = 0$: A-model in K symplectic structure

$\simeq T^* \mathrm{Bun}_G^{ss} X$ symplectically

so $Z_{G,0}(C) = \text{A-branes}(T^* \mathrm{Bun}_G X)$

$\longleftrightarrow D(\mathrm{Bun}_G X, \mathcal{D})$

$\Psi = \infty$: B-model in J complex structure

$\simeq \mathrm{Conn}_G^{ss} X$

so $Z_{G,\infty}(C) = \text{B-branes}(\mathrm{Conn}_G X)$

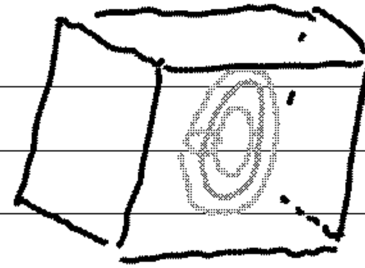
$\longleftrightarrow D(\mathrm{Conn}_G X, \mathcal{O})$

Other Ψ : branes for generalized complex structures, interpolating complex J $\longleftrightarrow$ symplectic K

"Quantum Langlands Conjecture"

Kapustin: B-model on I $\sim T^* \mathrm{Bun}_G X \Rightarrow$ "Classical Langlands" (not TFT)

Line operators



$L \subset M$ loop
$\Rightarrow$ have two kinds of observables:

- Wilson loop: calculate trace of holonomy of connection in representation $V \in \underline{\mathrm{Rep}\,G} = K(\mathrm{Rep}\,G) \rightsquigarrow W_{L,V}$

- 't Hooft loop: create singularity in gauge field along L (magnetic monopole)
  – labelled by geometry of connection in transversal S^2, which is a YM solution on $S^2 \longleftrightarrow V^\vee \in \underline{\mathrm{Rep}\,G^\vee}$
  $\rightsquigarrow H_{L,V^\vee}$

Kapustin

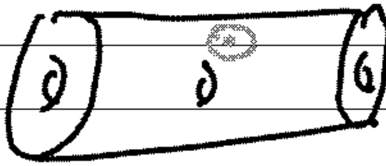
Moreover S-duality interchanges $\underset{G}{W_{L,V}} \longleftrightarrow \underset{G^\vee}{H_{L,V}}$

Note: in TFT line operators labelled by $Z(\partial(\text{tubular nbhd of } L))$
$= Z(S^2 \times S^1)$ crossing with S^1
" $= K(Z(S^2))$ " taking "trace of Id"
$\longrightarrow$ HH or K-theory

[In topological theory with parameter Ψ only subset of line operators survives: $\Psi = 0$ all 't Hooft, $\Psi = \infty$ all Wilson]

In two dimensions: 

line operators become point operators


P_1 P_2 $\Rightarrow$ get new picture of Hecke correspondences as giving field configurations on $C \times I$ (solutions of Bogomolny monopole equations) with prescribed boundary conditions P_1, P_2 and singularities at point x at fixed time.