Oxford Applications

Note Title 3/15/2007

In this lecture we'll explore some relations between geometric Langlands & representation theory. [This is a <u>one-dimensional</u> aspect of the 4D TFT we discussed last time.]

- Crash course in geometric representation theory
- Representations of loop groups & local geometric Langlands
  [Beilinson-Drinfeld, Frenkel-Gaitsgory]
- Tamely ramified Satake correspondence
  [Bezrukavnikov]
- Application to representations of real Lie groups

$\longrightarrow$ S^1 localization

base change $\downarrow$

$$\begin{tikzcd} {}^LG \arrow[r, squiggly] \arrow[d, squiggly] & G \arrow[d, squiggly] \\ {}^LG_{\mathbb{R}} \arrow[r, squiggly] & G_{\mathbb{R}} \end{tikzcd}$$

[Ben-Zvi - Nadler]

Representation Theory of G

Beilinson - Bernstein localization:

$G \curvearrowright X \implies U\mathfrak{g} \to \Gamma(\mathcal{D}_X)$

Get functors $\mathfrak{g}\text{-mod} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathcal{D}_X\text{-mod}$

$$\Delta M = M \otimes_{U\mathfrak{g}} \mathcal{D}$$

Analog of localization for modules over R commutative rings $R\text{-mod} \Longleftrightarrow \mathcal{O}_{\operatorname{Spec} R}\text{-mod}$

This is actually just an aspect of the action of the group algebra $\mathcal{H}_G = D(G, \mathcal{D})$ on $D(X, \mathcal{D})$ . . .

Where should we localize to get an equivalence?

Natural place: flag variety

$\mathcal{B} = G/B = \{ \text{Borel subgroups } B' \subset G \}$

$G = GL_n$: $\mathcal{B} = \{ \text{full flags } 0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n \}$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

SL_2: $\mathcal{B} = \mathbb{P}^1$

Borel-Weil theorem: irreducible reps of G all appear as global sections of unique line bundles on G/B.

These line bundles are not flat: not $\mathcal{D}$-modules, but twisted versions of $\mathcal{D}$-modules: connections with central curvature $\leftrightarrow$ action of center of $U\mathfrak{g}$. [infinitesimal character]
We'll ignore this subtlety, work with regular infinitesimal character.

<u>Theorem</u> (BB) Localization is an equivalence

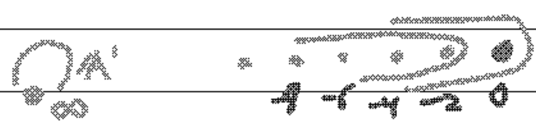
$$\mathfrak{g}\text{-mod}_0 \xrightarrow{\sim} \mathcal{D}_{G/B}\text{-mod}$$

Interesting categories of representations come from looking at $\mathfrak{g}$-modules which integrate to some subgroup $K \subset G$ - Harish-Chandra $(\mathfrak{g}, K)$-modules.

- $\mathcal{H}_{G,B} = D(B\backslash G/B, \mathcal{D})$: [roughly] category $\mathcal{O}$ of highest weight representations of $\mathfrak{g}$ (eg Verma mods)
$\simeq$ category of (∞-dim) representations of G as a real Lie group $\mathcal{HC}_{G_\mathbb{C}}$
[this is the monodromic version of $\mathcal{H}_{G,B}$]

Really these arise as (bi-)monodromic D-modules on $N \backslash G/N$..

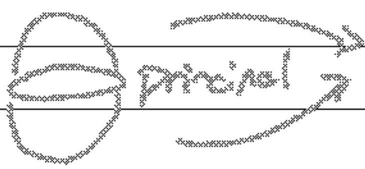
$SL_2\mathbb{C}$:



• = trivial
⊃ = $\mathbb{C}[\mathbb{A}^1]$
⊃ = δ-fns

- $D(G_\mathbb{R} \backslash G/B, D)$ [Kashiwara–Schmid]
  representations of real form $G_\mathbb{R} \subset G$

$SL_2\mathbb{R}$:



The real local Langlands program seeks to provide a description of the category of admissible representations

$$\mathcal{HC}_{G_\mathbb{R}} = D(G_\mathbb{R} \backslash G/B, D)$$

of $G_\mathbb{R}$ in terms of space of "Galois data" in $G^\vee$ Langlands dual [$\mathbb{Z}/2 = \mathrm{Gal}\ \mathbb{C}/\mathbb{R}$ ..] – geometric parameter space $X^\vee$

[Irreducibles: Langlands–Shelstad

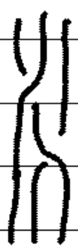
K-groups: Vogan, ABV

Derived categories: Soergel's conjecture]

- Hecke symmetry: $\mathcal{H}_{G,B}$ is an algebra (monoidal category) & acts on $D(X/K, \mathcal{D})$ whenever G acts on X $\Longrightarrow$
  e.g. on $D(G/B, \mathcal{D})$ or $D(K\backslash G/B, \mathcal{D})$.

  - classical intertwining operators between representations

$\mathcal{H}_{G,B}$ is a categorical version of the Artin braid group B_G of G:



$B_{SL_n} = B_n$ braid group on n strands
$\twoheadrightarrow S_n$ symmetric group = Weyl group W_{SL_n}.

Bruhat decomposition $B\backslash G/B \longleftrightarrow W$

W_G has generators s_i $(i = 1, \dots, \mathrm{rk}\, G)$
& relations $s_i s_j s_i = s_j s_i s_j$ when $\overset{i}{\bullet}\!-\!\overset{j}{\bullet}$
+ $s_i^2 = 1$. [variants for $\bullet\!=\!\bullet$, $\bullet\!\equiv\!\bullet$]

B_G: drop $s_i^2 = 1$ relation.

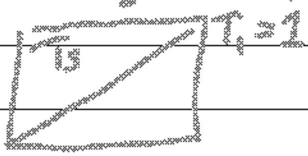
To model $\mathcal{H}_{G,B}$ we replace $\mathbb{C} \rightsquigarrow \mathbb{F}_q$
& look at convolution algebra

$$H_{G,B,q} = \mathbb{C}[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)]$$

- quotient of $\mathbb{C}B_G$, depending polynomially on q : generators T_s satisfy $(T_s+1)(T_s-q)=0$

$q=1 \Rightarrow \mathbb{C}W$.

SL_2 : calculate as convolution on $\mathbb{C}[{}_{SL_2}\backslash \mathbb{P}^1\times\mathbb{P}^1]$



$$T_s * T_s(x,y) = \sum_z T_s(x,z)T_s(z,y)$$

$$= \#\{z \neq x \ \& \ z\neq y\}$$

$$= \begin{cases} q & x=y \\ q-1 & x\neq y \end{cases} \Rightarrow T_s^2 = (q-1)T_s + 1$$

$\mathcal{H}_G$ likewise has "generators" T_s (standard sheaves on corresponding orbits) satisfying braid relations
$\Rightarrow$ $\mathcal{H}_G$ action gives action of braid group on category.

So we wish to describe $\mathcal{H}\mathcal{C}_{G,R}$ with action of braid group B_G ....

Loop Group Representations

$LG = G(\mathbb{C}((z)))$. "Group algebra" [informal]

$\mathcal{H}_{LG} = D_{sm}(LG, \mathcal{D})$

analog of smooth compactly supported functions on $G(\mathbb{F}_q((z)))$ or $G(\mathbb{Q}_p)$.

$\mathcal{H}_{LG}$-modules $\leftrightsquigarrow$ dg categories with LG action

Examples: • $\hat{\mathfrak{g}}$-mod, [smooth] representations of affine Kac-Moody algebra

(LG acts through adjoint action on $\hat{\mathfrak{g}}$, fixed objects = integrable representations

• $D(Z, \mathcal{D})$ whenever LG acts on Z.

Study by taking K-invariants $K \subset G(\mathcal{O})$ for smaller & smaller ["compact open"] K, as modules for $\mathcal{H}_{LG,K} = D({}_K\backslash LG/K, \mathcal{D})$

• $(\hat{\mathfrak{g}}\text{-mod})^K = (\hat{\mathfrak{g}}, K)\text{-mod}$: integrable to K

• $D(Z/K, \mathcal{D})$ $\quad Z \circlearrowleft LG$.

Automorphic theory:

look at functions on $\mathrm{Bun}_{G,\{K_i\}} X$

G-bundles with level structures

(e.g. flags or jets of trivializations)

at finitely many points $G(\mathbb{C}(X)) \backslash G(\mathbb{A}_X) / \prod K_i \cdot \prod_{x \neq x_i} LG_+$

... approximations to full level structure $G(\mathbb{C}(X)) \backslash G(\mathbb{A}_X)$.

$D(\mathrm{Bun}_{G,\{K_i\}} X, \mathcal{D})$ carries action of

$\mathrm{Rep}_{ds} G^\vee = \mathcal{H}_{LG, LG_+}$ at all $x \notin \{x_i\}$

$\Rightarrow$ "sheafifies" over Galois spectrum

$\mathcal{C}onn_{G^\vee}(X \setminus \{x_i\})$

& action of $\mathcal{H}_{LG, K_i}$ at x_i.

The ramified geometric Langlands program seeks to describe this module in terms of $\mathcal{O}$-modules over a space of $G^\vee$ connections on X with level structure at the x_i.

Problem: describe noncommutative Hecke algebras $\mathcal{H}_{LG,K}$ and their modules

Tamely ramified story

Iwahori subgroup $I \subset LG_+$ $\quad$ $\mathcal{F}\ell = LG/I$ affine flag manifold

$$\begin{array}{ccc} I & \subset & LG_+ \\ \downarrow & & \downarrow \text{ eval at } 0 \\ B & \subset & G \end{array}$$

— loop group analog of G/B

(G/B bundle over Gr)

Bruhat decomposition: I orbits all cells,
& $I \backslash LG / I \longleftrightarrow W_{aff} = W \ltimes \Lambda$
affine Weyl group. $[G_{\mathcal{O}} \backslash Gr \longleftrightarrow \Lambda / W]$

Affine Hecke algebra $H_{LG, I, q}$
convolution algebra of smooth cptly supp. functions on $I \backslash G(\mathbb{F}_q((z))) / I$
or p-adic analog — controls representations of $G(\mathbb{Q}_p)$ with I-fixed vector.

- generated by $H_{G,B,q}$ with one extra T_{s_0} satisfying braid relations & $(T_s+1)(T_s-q)=0$

$\rightarrow$ quotient of group algebra of <u>affine</u> braid group $\hat{B}_G$.

Likewise $\mathcal{H}_{LG,I}$ has "generators" T_s obeying $\hat{B}_G$

Langlands dual description:

$\mathcal{C}onn_{G^\vee}(X, \{x_i\})$ parabolic $G^\vee$-connections on X, x_i

connections with simple pole at x

and a flag at x preserved by residue.

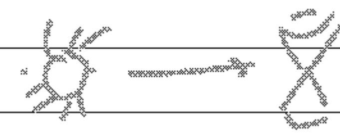
Local data at x is up to $G^\vee$ action

$$T^*\mathcal{B}^\vee = T^* G^\vee/B^\vee \simeq \left\{ (n, B') \;\; \begin{array}{l} B' \in \mathcal{B}^\vee \\ n \in [\mathfrak{b},\mathfrak{b}] \end{array} \right\}$$

$=$ flags + nilpotent preserving it

Springer resolution: $T^*\mathcal{B}^\vee \longrightarrow \mathfrak{g}^\vee$ moment map for $G^\vee$ action

$\mathcal{N}^\vee$ = cone of nilpotent elements $\searrow \mathcal{N}^\vee \subset$

$\mathfrak{sl}_2$: $T^*\mathbb{P}^1 =$  blowup of $\mathcal{N}^\vee = \{a^2 - bc = 0\}$

What acts on this local data, preserving the connection data (residue & $G^\vee$ action)?

Steinberg variety: $\mathcal{S}t^\vee = T^*\mathcal{B}^\vee \times_{\mathcal{N}^\vee} T^*\mathcal{B}^\vee$

$$= \{ n, B_1', B_2' : n \in B_1' \cap B_2' \text{ nilpotent} \}$$

Carries action of $G^\vee \times \mathbb{C}^*$ ($\mathbb{C}^*$ rescales n)

& "functions" on $\mathcal{S}t^\vee$ are a matrix algebra relative to $\mathcal{N}^\vee$

Theorem (Kazhdan-Lusztig)

$$H_{LG,I,q} \simeq K_{G^\vee \times \mathbb{C}^*}(St), \quad (q=1) \ \mathbb{C}W_{aff} = K_{G^\vee}(St)$$

- compare with Satake $H_{LG,LG_+,q} \simeq \underline{Rep}\, G^\vee = K_{G^\vee}(\bullet)$

Theorem (Bezrukavnikov) [roughly]

Monodromic (twisted) D-modules

$$\mathcal{H}^{mon}_{LG,I} \simeq D(St^\vee/G^\vee, \mathcal{O}) \text{ as algebras}$$

much subtler than commutative case of LG_+!

$G^\vee$ side are families of matrix algebras $/\mathcal{N}^\vee \Rightarrow$ easy to describe modules: "functions" (K or D) of $T^*\mathcal{B}^\vee$ or Springer fibers $\subset T^*\mathcal{B}^\vee$

K-L Thm $\Rightarrow$ tamely ramified p-adic local Langlands conjecture (Deligne-Langlands conj.)

Bezrukavnikov $\Rightarrow$ e.g. affine braid group actions on $D(T^*\mathcal{B}^\vee, \mathcal{O})$ or $D(\mu^{-1}(n), \mathcal{O})$.

Springer geometry at heart of representation theory (eg of quantum groups, modular representations, ...) $\longrightarrow$ host of consequences: proofs of conjectures of Lusztig....

Bezrukavnikov's theorem makes precise the formulation of tamely ramified geometric Langlands:

roughly $D(\mathrm{Bun}_G(X,\{x_i\}),\mathcal{D}) \xrightarrow{\sim} D(\mathrm{Conn}_{G^\vee}(X,\{x_i\}),\mathcal{O})$

compatible with $\mathcal{H}_{sph}$ at $x \notin \{x_i\}$

& $\mathcal{H}_{LG,I}$ at x_i (on both sides!)

It also implies the conjecture for $(\mathbb{P}^1, 0, \infty)$:

in fact $\mathrm{Bun}_G(\mathbb{P}^1,0,\infty) = I_- \backslash LG / I$

combinatorially opposite to $I \backslash LG / I$

& $\mathrm{Conn}_{G^\vee}(\mathbb{P}^1,0,\infty) = \left\{ (g, B_1, B_2) : \begin{array}{l} B_1, B_2 \in \mathcal{B}^\vee \\ g \in B_1 \cap B_2 \end{array} \right\}$

group version of Steinberg

Prop. (BZ-Nadler)

Tamely ramified geometric Langlands holds on $(\mathbb{P}^1,0,\infty)$: canonical Hecke-compatible equivalence

$$D(\mathrm{Bun}_G(\mathbb{P}^1,0,\infty),\mathcal{D}) \simeq \widehat{D}(\mathrm{Conn}_{G^\vee}(\mathbb{P}^1,0,\infty),\mathcal{O})$$

I_+, I_--monodromic (completed version)

From LG to G to $G_{\mathbb{R}}$

Work with D. Nadler: Link affine Hecke theory ($\mathcal{H}_{LG,I}$ & its module categories) to finite Hecke theory ($\mathcal{H}_{G,B}$ & its module categories) & apply to study representations of real groups $\mathcal{HC}_{G_{\mathbb{R}}}$.

Two basic ingredients:

- S^1 localization $\quad (LG \rightsquigarrow G)$
- Automorphic base change $\quad (G \rightsquigarrow G_{\mathbb{R}})$

S^1 localization

M manifold with S^1 action $\Rightarrow$ S^1 equivariant cohomology of M $H^*_{S^1}(M)$ is a module over $H^*_{S^1}(\cdot) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[u]$, $|u|=2$.

Localized cohomology $H^*_{S^1}(M) \otimes_{\mathbb{C}[u]} \mathbb{C}[u,u^{-1}]$ agrees with cohomology of the fixed points $H^*(M^{S^1}, \mathbb{C}[u,u^{-1}])$.

[Note we only recover $\mathbb{Z}/2$ graded version of $H^*(M^{S^1},\mathbb{C})$]

Now loop spaces. LZ carry S^1 actions & fixed points $(LZ)^{S^1} = Z$ are constant loops. So morally S^1 localization bridges topology of LZ & Z [Witten].

We'll apply this to relate $\mathcal{H}_{LG}$ & $\mathcal{H}_G$.

To be more precise, I'll state a result for $I\backslash LG/I = Bun_G(\mathbb{P}^1, 0, \infty)$, much more general...

Theorem (BZ-N)

$$D_{mon}(I\backslash LG/I, \mathcal{D})^{S^1, loc} \simeq D_{mon}(B\backslash G/B, \mathcal{D})$$

On the spectral side, we also have an S^1 action on connections by rotation... but $St^\vee/G$ doesn't look like a loop space! eg finite dimensional

In fact we find $St^\vee/G$ is a loop space [inertia stack] in the sense of derived algebraic geometry - a world where homotopy theory & varieties happily coexist ...

Loop spaces: $\mathcal{L}X = \mathrm{Map}(S^1, X) = X \underset{X \times X}{\overset{*}{\times}} X$

"derived homotopy fiber product"

... these are "fast loops":

$S^1 =$  two points & two zip-lines between them, don't have time to move along the variety!

- e.g. $X = BG = \bullet/G$

  $\mathcal{L}X = \{\text{object} + \text{automorphism}\} = G/G$

- eg X smooth scheme,

  $\mathcal{L}X = \Delta \overset{L}{\cap} \Delta$ derived self intersection of the diagonal

  $= T_X[-1]$ odd tangent bundle

- $X = B^\vee \backslash G^\vee / B^\vee$

  $\mathcal{L}X = St^\vee / G^\vee$.

- S^1 acts on $\mathcal{L}X$. Case of X smooth scheme $\Rightarrow$ de Rham differential, as <u>odd vector field</u> on $T_X[-1]$

Theorem $D(\widehat{\mathcal{L}X}, \mathcal{O})^{S^1} \underset{H^*(BS^1)}{\otimes} \mathbb{C}[u,u^{-1}]$

$$\simeq D(X, \mathcal{D}) \underset{\mathbb{C}}{\otimes} \mathbb{C}[u,u^{-1}]$$

($\widehat{\mathcal{L}X}$ = small loops = formal completion :
e.g. $\widehat{\mathcal{L}}(BG) = \widehat{G}/G$)

So we have a diagram

$$\begin{tikzcd}
D_{mon}(Bun_G(\mathbb{P}^1, 0, \infty)) \arrow[r, leftrightarrow, "\text{Bezrukavnikov}"] \arrow[d, "S^1 \text{ localization}"'] & D(St^\vee/G^\vee, \mathcal{O}) \arrow[d] \\
D_{mon}(B\backslash G/B, \mathcal{D}) \arrow[r, leftrightarrow] \arrow[d, equal] & D(B^\vee\backslash G^\vee/B^\vee, \mathcal{D}) \arrow[d, equal] \\
\mathcal{HC}_{G_{\mathbb{C}}} & \mathcal{H}_{G^\vee, B^\vee}
\end{tikzcd}$$

$\mathcal{H}_{G,B} \underset{[BGS]}{\simeq} \mathcal{HC}_{G_{\mathbb{C}}}$

Theorem There is a canonical equivalence
($\mathbb{Z}/2$-graded) $\mathcal{HC}_{G_{\mathbb{C}}} \simeq \mathcal{H}_{G^\vee, B^\vee}$

Compatible with finite Hecke symmetry
- strong form of complex local Langlands / Soergel conjecture

Real groups

[BZ-Nadler] show how to understand the real local Langlands program (in particular Vogan duality & Soergel's conjecture) inside the geometric Langlands program.

Basic idea: <u>base change</u>.

$$\begin{array}{l} X \quad \text{covering space of curves} \\ \downarrow \Gamma \qquad \Rightarrow \quad \mathrm{Conn}_{G^\vee}(Y) = \left[\mathrm{Conn}_{G^\vee}(X)\right]^{\Gamma} \\ Y \end{array}$$

<u>Proposition</u> $D(\mathrm{Conn}_{G^\vee}(Y), \mathcal{O}) = \left[D(\mathrm{Conn}_{G^\vee}(X), \mathcal{O})\right]^{\Gamma\text{-b.c.}}$

- impose the condition that Hecke actions at x & $\gamma \cdot x$ are identified for $\gamma \in \Gamma$.

<u>Conjecture</u> $D(\mathrm{Bun}_G(Y), \mathcal{D}) = \left[D(\mathrm{Bun}_G(X), \mathcal{D})\right]^{\Gamma\text{-b.c.}}$

impose same condition on automorphic side.

- follows from geometric Langlands conjecture!

We apply a twisted form of base change to $X = \mathbb{P}^1$, $Y = \mathbb{RP}^2$, $\Gamma = \{\eta\}$ antipodal map, acts on G by a real involution θ

$G_{\mathbb{R}} = G^{\theta}$. — base change to $Bun_G(\mathbb{P}^1, 0, \infty)$ should then produce D-modules on the stack of parabolic real (G,θ)-bundles on $(\mathbb{P}^1, \text{antipodal})$.

- Base change on the spectral side produces $\mathcal{O}$-mods on the "Langlands parameter spaces"
- S^1 localization of $Bun_{G,\theta}$ produces precisely the categories $HC_{G_{\mathbb{R}}}$ of real Harish-Chandra modules
- S^1 localization of $\mathcal{O}$-mods on Langlands parameter spaces produces precisely the categories of D-modules appearing in Soergel's conjecture (whose K-groups appear in Vogan duality), ie Langlands parameters
- We can prove the automorphic base change $\mathbb{P}^1 \to \mathbb{RP}^2$ on K-groups.

Theorem (Affine Vogan duality / real Kazhdan-Lusztig theorem)

$\exists$ canonical $K(\mathrm{Bun}_{G,\theta}, \mathcal{D}) \simeq K(\mathrm{St}^\vee_{\sigma^\vee}/G^\vee, \mathcal{O})^*$
as affine Hecke modules
producing Vogan's character duality
$$K(\mathcal{HC}_{G,\theta}) \simeq K(\mathcal{D}(_{K^\vee}\backslash \mathcal{B}^\vee))^*$$
on S^1-localization.

$$\begin{array}{ccc}
\mathcal{D}(\mathrm{Bun}, \mathcal{D}) & \xleftrightarrow{\text{Bezrukavnikov}} & \mathcal{D}(\mathrm{St}^\vee/G^\vee, \mathcal{O}) \\
?\downarrow \text{base change} & & \downarrow \text{base change} \\
\mathcal{D}(\mathrm{Bun}_\theta, \mathcal{D}) & & \mathcal{D}(\mathrm{St}^\vee_{\sigma^\vee}/G^\vee, \mathcal{O}) \\
\downarrow S^1 \text{ localization} & & \downarrow S^1 \text{ localization} \\
\mathcal{D}(\coprod G_{\mathbb{R}}\backslash G/B, \mathcal{D}) & & \mathcal{D}(\coprod K^\vee\backslash G^\vee/B^\vee, \mathcal{D}) \\
\| \text{ localization} & & \| \\
\mathcal{HC}_\theta & \xleftrightarrow[\text{Soergel conjecture}]{?} & \text{Langlands parameters}_{\theta^\vee}
\end{array}$$