Santa Barbara TFT & Rep Theory

Note Title 7/9/2008

Topological Field Theory

Algebraic structure axiomatizing coarse features of quantum field theory.

{M = spacetime}: smooth, oriented n-dimensional manifold [+ possible extra structures: spin etc.]

An n-dim TFT (with one tier) is a multiplicative invariant Z (partition function) on manifolds M as above:

$$M \longmapsto Z(M) \in \mathbb{C} \quad \text{(say)}$$

$$\phi \longmapsto 1, \quad \amalg \longmapsto \cdot$$

functorial: oriented diffeo. invariant.

Caricature: $Z(M) = \int e^{-S(\varphi)} D\varphi$

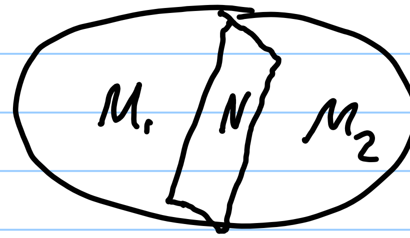
$\{\varphi\}$: fields - local expressions on M (scalars, spinors, sections of a bundle, maps to a target space X, principal bundles, connections, ...)

S (action): integral of local functional of φ.

Express locality of Z in M:

cut & paste formula

$M = M_1 \underset{N}{\amalg} M_2$



$$\Rightarrow \boxed{Z(M) = \langle Z(M_1), Z(M_2) \rangle_{Z(N)}}.$$

A 2-tier TFT assigns in addition

- Z: $(n-1)$-manifolds $\longrightarrow \mathrm{Vect}_{\mathbb{C}}$

  compact, smooth, oriented

  $N \longmapsto Z(N)$

  functorial & multiplicative:

  $\emptyset \mapsto \mathbb{C}$, $(\;)^{op} \mapsto (\;)^*$, $\amalg \mapsto \otimes$

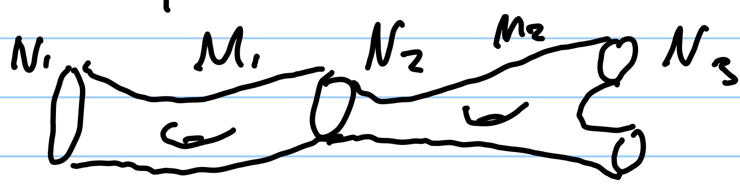
- Z: M^n w/ boundary $\longmapsto Z(M) \in Z(\partial M)$

  so that above gluing rule holds:

  $Z(M_1) \in Z(N)$, $Z(M_2) \in Z(N)^*$

  $\Rightarrow$ pair to give $Z(M)$.

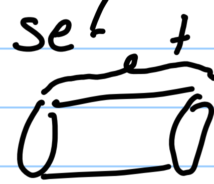
$\Longleftrightarrow$



$Z(M) \in Z(\partial M_{out} \amalg \partial M_{out}^{op})$

$= \mathrm{Hom}(Z(\partial M_{in}), Z(\partial M_{out}))$

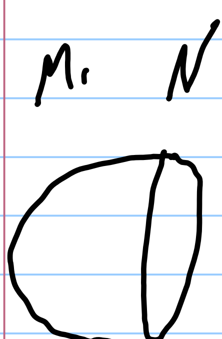
If we think of N as a fixed-time slice of M $\Rightarrow$ $Z(N)$ is the Hilbert space of the field theory & this is just the Hamiltonian formulation of the theory .... except we set the Hamiltonian to be 0:



$$Z(N\times I) = \mathrm{id} \in \mathrm{Hom}(Z(N), Z(N))$$

Construction of TFTs:

$Z(N)$ = linearization/quantization of (ie some kind of functions on) space of boundary values of fields $\varphi|_{\partial M}$ (or jets)



$$Z(M_1)(\varphi_2) = \int_{\varphi|_N=\varphi_2} e^{-S(\varphi)}\, D\varphi$$

Evolution operators:
Push-pull operators

$$\begin{tikzcd} & \mathrm{Fields}(M) \arrow[dl] \arrow[dr] & \\ \mathrm{Fields}(N_1) & & \mathrm{Fields}(N_2) \end{tikzcd}$$

$$v \in Z(N_1) \longrightarrow Z(N_2)$$

$$Z(M)(v)(\varphi_2) = \int_{\varphi|_{N_2} = \varphi_2} v(\varphi|_{N_1})\, e^{-S(\varphi)}\, D\varphi$$

⇝ **Review: Function Theory 101**

X finite set ⇝ $\mathrm{Fun}(X)$ ($\mathbb{C}$-valued)

- Carries commutative unambiguous multiplication
- $\pi : X \to Y$ map $\Rightarrow$

$$\underset{\text{sum along fibers}}{\pi_*} : \mathrm{Fun}(X) \rightleftarrows \mathrm{Fun}(Y) : \underset{\text{pullback}}{\pi^*}$$

- All linear operators $\mathrm{Fun}(X) \to \mathrm{Fun}(Y)$ are integral transforms, with a kernel function $K = K(x,y) \in \mathrm{Fun}(X \times Y)$

$$\begin{tikzcd} & K \\ & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$$K * f(y) = \int f(x) K(x,y)\, dx$$
$$= \pi_{2*}(\pi_1^* f \cdot K)$$

... $\mathrm{Fun}(X) \simeq \mathbb{C}^n$, $\mathrm{Fun}(Y) \simeq \mathbb{C}^m$,

$\mathrm{Fun}(X \times Y) = \mathrm{Mat}_{m \times n}$:

this is just matrix representation of maps..

Convolution algebras:

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X \times X & X \times X & X \times X \end{tikzcd}$$

Matrix multiplication

$$A * B = \pi_{13*}(\pi_{12}^* A \cdot \pi_{23}^* B)$$

$\mathrm{Fun}(X \times_Y X)$ likewise: block-diagonal matrices

$$\begin{pmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{pmatrix}$$

G group $\rightsquigarrow$ group algebra $\mathrm{Fun}(G) = \mathbb{C}G$

$\mu : G \times G \longrightarrow G$, $f * g = \mu_*(\pi_1^* f \cdot \pi_2^* g)$

Example 2d topological gauge theory:

G a finite gauge group

Fields: principal G-bundles on M

$\Longleftrightarrow$ G-covering spaces of M

$\Longleftrightarrow$ (flat) G connections on M

$\Longleftrightarrow$ $\{\pi_1(M) \longrightarrow G\}/\text{conj.}$

$\underline{\mathcal{M}_G(X)}$

$$Z(\Sigma) = \#\,\text{fields} = \#\{\pi_1(\Sigma) \to G\}/\sim$$

counted w/ auto.

$$= \#\{A_1,\dots A_g\; B_1 \dots B_g \in G : \prod [A_i, B_i] = 1\}/\sim$$

How do we calculate this? (Frobenius, Schur)

- cut & paste:

$$Z(S^1) = \text{linearization of } \mathrm{Fields}(S^1)$$

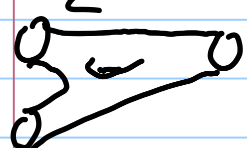
$$= \frac{G}{G} \quad \text{adjoint quotient: conjugacy classes}$$

- clear meaning for functions on a finite set

$$\Rightarrow \quad Z(S^1) = \mathrm{Fun}\left(\frac{G}{G}\right) = (\mathbb{C}G)^G$$

class functions.

Σ



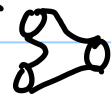
$$Z(\Sigma) : Z(S^1)^{\otimes 2} \longrightarrow Z(S^1)$$

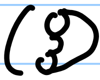
push-pull along fields on Σ.

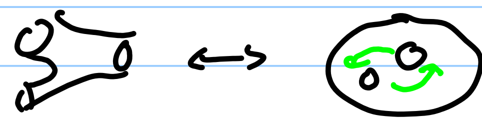
Theorem 2d TFT (2 tiers) $\Longleftrightarrow$ commutative Frobenius algebras (i.e. unital commutative associative algebra with nondegenerate trace)

How? $Z \longmapsto H = Z(S^1)$

$Z($$) = 1 \in H$ $\quad$ $Z($$) = tr : H \to \mathbb{C}$

$Z($$) : H \otimes H \xrightarrow{\mu} H.$

Check $Z($$) = tr(\mu) : H \otimes H \to \mathbb{C}$ nondegenerate. Moreover any Σ^2 can be decomposed into such building blocks, so recover $Z(\Sigma)$ from Frobenius structure.

Commutativity 

Gauge theory example:

$\mathbb{C}G$ is a Frobenius algebra: $tr(f) = f(1)$ nondegenerate trace (dim of a character..)

$\mathbb{C}G^G$ is its center, a commutative Frobenius alg. $\longleftrightarrow$ local operators.

Geometric basis: conjugacy classes.

To <u>solve</u> the system: want to simultaneously diagonalize the local operators $\longleftrightarrow$ spectral decomposition

$\Rightarrow$ calculate Spec $\mathbb{C}G^G$

$= \{$ homomorphisms $\to \mathbb{C}\}$

$= \{$ joint eigenvalues $\} = \hat{G}$

$= \{$ irred characters of $G\}$

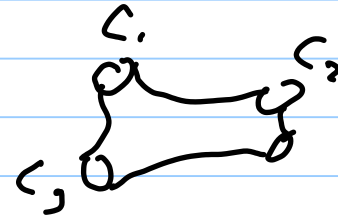
"Quantum mechanics of the vacuum $[H=0]$ is completely integrable" - simultaneously diagonalize all local operators with nondegenerate spectrum!

[GL_n: get quantum many-body systems...]

— in this basis easier to calculate:

$$\rightsquigarrow Z(\Sigma) = \sum_{\substack{\chi \in \text{irred} \\ \text{chars of } G}} \left(\frac{|G|}{\dim \chi}\right)^{2g-2}$$

Similar formula for



fix conjugacy classes at punctures $\Longleftrightarrow$ insert corresponding operators

$$\#\left\{ \prod [A_i, B_i] \prod C_i = 1 \right\}_{C_i \in \{c_i\}}$$

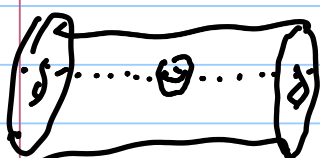
Local operators

In any TFT, $Z(S^{n-1})$ is an algebra, given by "pair of pants"

$n=1$ $n=2$ $n=3$

topological OPE : associative $(n=1)$
commutative $(n>1)$

& $Z(S^{n-1}) \curvearrowright Z(N)$ for any N as local operators :



$N \times I$

$\mathcal{O} \in Z(S^{n-1})$

$$\Rightarrow \int \mathcal{O}_{x,t}(\varphi)\, e^{-S(\varphi)} D\varphi$$

Other labels : e.g. given loop in n-manifold

take boundary of tubular neighborhood $\simeq S^1 \times S^{n-2}$

$\Rightarrow$ can label by any $v \in Z(S^1 \times S^{n-2})$ to get invariant of the pair, or loop operators.

Order & Disorder operators:

Note that the operators we insert are labelled by functionals on fields on the <u>link</u> of our the point x of insertion.

$\Rightarrow$ in addition to local functionals of fields can have functionals supported on fields with prescribed <u>singularities</u> at x

$\Longleftrightarrow$ changing the path integral from regular fields to singular fields!

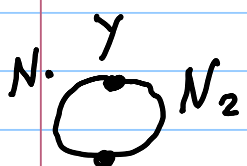
eg in gauge theory: point operators in 2d are all disorder operators, introduce monodromy around this point: ['t Hooft!] but loop operators can calculate trace of monodromy <u>around</u> loop in some representation, Wilson operator.

Extended TFT

Express locality of the vector space $Z(N)$ with respect to cut & paste in N

ie $\quad N = N_1 \coprod_Y N_2 \implies$

$$Z(N) = \langle Z(N_1), Z(N_2) \rangle_{Z(Y)}$$

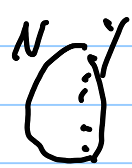


To get a vector space as a pairing $\rightsquigarrow$ let $Z(Y)$ be a linear category & $Z(N) \in \mathrm{Ob}\, Z(\partial N)$.

Put another way: before for

we got a function ($\mathbb{C}$-valued) on space of fields on N from path integral on M: fixing boundary value makes M into a closed n-manifold.

Now



fixing values on ∂N should get a vector space

$\rightsquigarrow$ a vect-valued function, or sheaf, on fields on Y ... $Z(Y)$ = category of sheaves on Fields(Y), a categorified linearization / quantization.

Slightly more formally : 2-category

Objects : $n-2$ manifolds Y 

1-morphisms: $n-1$ manifolds N w/ boundary 

2-morphisms: n manifolds M w/corners 

An n-dim TFT with 3 tiers is a functor to some target 2-category, e.g.

Ob: linear categories

1-Mor: Functors

2-Mor: natural transformations

Explored by Freed, Baez-Dolan, ...

2d: Moore-Segal, Costello : complete picture

Hopkins-Lurie in generality: go down all the way to a point : i.e totally local. All topology disappears, get only algebra.

Theorem (Hopkins-Lurie) An n-dim TFT extended down to a point is completely & freely determined by the self-dual object assigned to a point.

Example (G finite) 2d topological gauge theory

Linear category: {vector bundles on gauge fields on a point}

$= \mathrm{Vect}(BG) = \{\text{linear}_{\mathbb{C}} \text{ reps of } G\}$

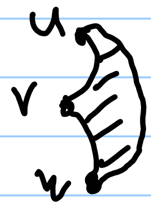
$\simeq \mathbb{C}G\text{-mod}$, modules over the group algebra of G.

Note that vector bundles (or sheaves) on finite sets also have $\otimes$, π^*, π_* like functions $\Rightarrow$ define operations for cobordisms by push-pull.

$$\begin{tikzcd} & \mathrm{Fields}(N) \arrow[dl] \arrow[dr] & \\ \mathrm{Fields}(Y_1) & & \mathrm{Fields}(Y_2) \end{tikzcd}$$

Structures:

$V, W \in \mathrm{Rep}\, G \Rightarrow \mathrm{Hom}_G(V,W) \in \mathrm{Vect}$

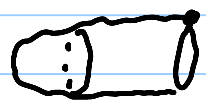


composition

$\mathrm{End}\, V$ is an associative algebra.

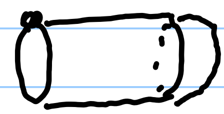
What's special about this category?

Modules over a Frobenius algebra ($\mathbb{C}G$) form a Calabi-Yau category: have nondegenerate traces on $\mathrm{End}\, V$ making them into NC Frobenius algebras:

 : $\mathbb{C} \longrightarrow Z(S^1) = \mathbb{C}G^G \longrightarrow \mathrm{End}\, V$

$= Z(\mathbb{C}G)$

action of center

 =  adjoint:

$V \simeq \bigoplus V_i \otimes \mathbb{C}^{n_i}$ V_i: irreducible $\Rightarrow$

$$\begin{array}{ccccc} \mathrm{End}\, V & \longrightarrow & Z(S^1) = \mathbb{C}G^G & \longrightarrow & \mathbb{C} \\ \text{is} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bigoplus \mathrm{Mat}_{n_i \times n_i} \ni 1_i & \longmapsto & \mathrm{char}\, V & \longmapsto & \dim V \end{array}$$

character map into class functions

Moore-Segal, Costello, Hopkins-Lurie:

$Z(\bullet)$ in a 2d TFT is a CY category

$Z(S^1)$ is its center (HH^*): acts centrally on every object $= \mathrm{End}(\mathrm{Id})$.

<u>Example</u> σ-model on finite orbifold

$$X = \coprod \bullet / H_i$$

$$Z(\Sigma) = \# \operatorname{Map}(\Sigma \to X)$$
$$= \sum_i (\# H_i\text{-bundles})$$

$$Z(S^1) = \operatorname{Fun}(\operatorname{Map}(S^1 \to X))$$
$$= \bigoplus_i \mathbb{C}\left[\frac{H}{H}\right] = \mathbb{C}H^H$$

$$Z(\bullet) = \operatorname{Vect}(\operatorname{Map}(\bullet \to X))$$
$$= \bigoplus \operatorname{Rep} H_i$$

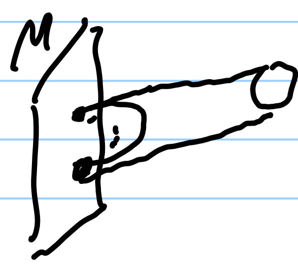
The character map (D-brane $\Longrightarrow$ D-brane stack)

$M \in Z(\bullet) \longmapsto Z($  $)(1_M) \in Z(S^1)$

is the <u>Chern character</u> of a vector bundle.

... $\operatorname{Fun}(LX)^{S^1}$ = cyclic homology (roughly de Rham cohomology)

Stringy picture



open-closed string transition

Local operators in extended TFT

In any TFT, $Z(S^{n-2})$ is an algebra (monoidal category)

$n=1$ $\quad$ $n=2$ $\quad$ $n=3$ $\quad$ $n=4$

associative (monoidal) $\quad$ braided $\quad$ symmetric - - - -

$V \otimes W \xrightarrow{\sim} W \otimes V$

but square is not identity!

$\Rightarrow$ braid group action on $V^{\otimes n}$

note now in codimension 2 have $\underline{3}$ different possibilities for the type of algebra we get!

Structure of 4d TFT:

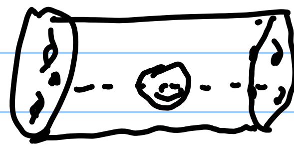
- Σ closed surface $\longmapsto$ category $Z(\Sigma)$.

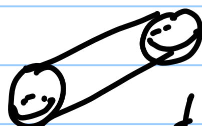
In fact $\Sigma \longmapsto$ 2d TFT Z_Σ with

$Z_\Sigma(C) = Z(\Sigma \times C) \Rightarrow$

$Z(\Sigma)$ is a Calabi-Yau category

- Fields on $\Sigma \times C$ are some kind of maps $C \to$ Fields(Σ):
  - ie Z_Σ is some kind of σ-model on Fields(Σ).

- $S^2 \longmapsto$ tensor (symmetric monoidal) category
- $\forall x \in X$ get action $Z(S^2) \curvearrowright Z(X)$
  from  , all commute
- $x, y \in X$ pick path $x \rightsquigarrow y$

  $\Rightarrow$  get isomorphism of the x & y actions of $Z(S^2)$ depending only on the topology of the corresponding cobordism

  $\Rightarrow$ <u>flat connection</u> on the actions

- categorified quantum mechanical system

"Hilbert space" $Z(X)$, operator algebra $\supset$

$\mathcal{A}(X) := \{Z(S^2)\}_{x \in X}$ large commuting family.

Try to <u>solve</u> - simultaneously diagonalize

- & hope for <u>complete integrability</u>:

nondegeneracy of eigenstates!

Example 4d gauge theory, G finite.

$\text{Fields}(S^2) = BG$, $Z(S^2) = \text{Rep}\, G$.

X surface $\text{Fields}(X) = \mathcal{M}_G(X)$

$= \{G\text{-bundles}\}$

$= \coprod_i \bullet/\text{Aut}\, P_i = \coprod B(\text{Aut}\, P_i)$

$\text{Aut}\, P \subset G$ centralizer of monodromy

$Z(X) = \text{Vect}(\text{Fields}(X))$

$= \bigoplus_i \text{Rep}(\underbrace{\text{Aut}\, P_i}_{\Gamma_i})$

$x \in X$ $V \in \text{Rep}\, G = Z(S^2)$

$\Rightarrow$ Wilson operator $W_{x,V} \curvearrowright Z(X)$

$W_{x,V}(\mathcal{F}) = \mathcal{F} \otimes \underline{W_{x,V}}$

$\underline{W_{x,V}}|_P$ = fiber at x of bundle associated to P in rep. V

$\simeq V|_{\text{Aut}\, P}$: Fibers at $x \underset{\gamma}{\rightsquigarrow} y$

isomorphic but for loop $x \underset{\gamma}{\rightsquigarrow} y$ get monodromy of associated bundle!

Function-sheaf correspondence

Compare X & $X \times S^1$

$Z(X \times S^1) = K(Z(X))$, where

traces live.

Concretely, $\mathrm{Fields}(X \times S^1) = G$-bundles on $X \times S^1 = \coprod \frac{H_i}{H_i}$

$$Z(X \times S^1) = \mathrm{Fun}\left(\coprod \frac{H_i}{H_i}\right) = \bigoplus \mathbb{C}\, H_i^{H_i}$$

$$= \bigoplus \text{Rep ring of } H_i$$

Wilson loops: $x \in X$ get loop ℓ_x in $X \times S^1$. (analog of Frobenius!)

$V \in \mathrm{Rep}\, G \Rightarrow W_{x,V} \circlearrowright Z(X \times S^1)$

multiplication by a function

$W_{x,V}(P) =$ trace of holonomy along ℓ_x in representation V.

Dimensional reduction $\longleftrightarrow$ decategorification:

categories $\xrightarrow{K}$ Vect, Sheaves $\xrightarrow{tr}$ functions

NC Hamiltonian System:

To "solve" the TFT, we try to simultaneously diagonalize all the local operators $\mathcal{H}(X) := \{Z(S^2)_x\}_{x\in X}$ ie perform spectral decomposition.

So need to find joint spectrum of the commutative algebra [category] acting:

- R ring $\Rightarrow$
  $\operatorname{Spec} R\,(k) = \operatorname{Hom}(R, k)$
  & R = functions on $\operatorname{Spec} R$
  R-modules localize as sheaves on $\operatorname{Spec} R$
- $\mathcal{C}$ tensor category $\Rightarrow$
  "Spec" $\mathcal{C}\,(k)$ = Tensor functors$(\mathcal{C}, \mathrm{Vect}_k)$
  & $\mathcal{C}$ = sheaves on "Spec" $\mathcal{C}$ (usually...)
  [$\mathcal{C}$-modules localize on "Spec" R]
  ~ the tensor functors become evaluation

e.g $\mathcal{C} = \operatorname{Rep} G^\vee \Rightarrow$

"Spec" $\mathcal{C} = BG^\vee$ & $\mathrm{Vect}(BG^\vee) = \mathcal{C}$:

Tannakian reconstruction!

Examples: • $\mathcal{C} = \{\text{graded vect}\}$

$\Rightarrow G^\vee$ (Tannaka/Galois group) is $\mathbb{C}^*$

• $\mathcal{C} = \{\text{flat vector bundles on } X\}$

$\Rightarrow G^\vee = \pi_1(X)$.

TFT: Suppose $Z(S^2)$ is Tannakian, ie $Z(S^2) \simeq \operatorname{Rep} G^\vee$ for some group $G^\vee$ $\Leftrightarrow$ "Spec" $Z(S^2) = BG^\vee$

$\Rightarrow$ "Spec" $\mathcal{H}(X) = \mathrm{Loc}_{G^\vee} X$ flat $G^\vee$-bundles on X !

So $Z(X)$ is a module over "functions" (Vect) on $\mathrm{Loc}_{G^\vee} X$

— ie our theory is trying to be $G^\vee$ gauge theory, local operators $\rightsquigarrow$ Wilson!

[G finite $\rightsquigarrow G^\vee = G$, boring]

it is when our quantum system is integrable!

Geometric Langlands conjecture

asserts that a particular 4d TFT Z_G [" $\mathcal{N}=4$ SYM in GL twist with $\psi=0$ "]

is completely integrable in this sense with "Galois" group $G^\vee$ = Langlands dual group:

in our theory Z_Σ = A-model on $T^* \mathrm{Bun}_G \Sigma$

$Z(\Sigma)$ = A-branes on $T^* \mathrm{Bun}_G \Sigma$

- eg Fukaya category (Lagrangians)

$\longleftrightarrow$ $\mathcal{D}$-modules on $\mathrm{Bun}_G \Sigma$

Kapustin-Witten
Nadler-Zaslow

- e.g. holonomic

... singular bundles with flat connection on Bun_G

$Z(S^2) = \mathrm{Shv}(\mathrm{Bun}_G S^2) \simeq \mathrm{Rep}\, G^\vee$

- Geometric Satake Theorem

Mirkovic-Vilonen, (Lusztig, Ginzburg, Drinfeld)

$\Rightarrow$ $\mathrm{Shv}(\mathrm{Bun}_G \Sigma)$ has action of vector bundles on $\mathrm{Loc}_{G^\vee} \Sigma$.

Conjecture: multiplicity one:

$$\mathrm{Shv}(\mathrm{Bun}_G \Sigma) \simeq \mathcal{D}(\mathrm{Loc}_{G^\vee} \Sigma)$$

[RHS get "span" of "Wilson" vector bundles = coherent sheaves]

$\rightsquigarrow$ $Z_\Sigma \simeq$ B-model on $\mathrm{Loc}_{G^\vee} \Sigma$

Motivations for the homotopical / dg world – leaving the toy models!

- <u>Physics</u> TFTs in nature usually come as topological twists of supersymmetric QFTs ("Witten type", or <u>cohomological field theories</u> – as opposed to "Schwarz type" like Chern-Simons, with manifestly topological Lagrangian)

  - have bosons & fermions ⇒ super vector spaces

  - often have charge operator (ghost #) ⇒ $\mathbb{Z}$-graded space of states

  - TFT arises as Q-cohomology of space of states of underlying QFT $Q^2=0$ (BRST operator – conserved supercharge)

  ⇒ underlying dg vector space whose cohomology is ordinary ($\mathbb{Z}$-graded) TFT.

Likewise the space of open string states with given boundary conditions is naturally a complex with BRST differential

So entire TFT picture is naturally refined to a dg version – which is better behaved! ("off shell")

• <u>Math</u> Nice function theories beyond finite sets: want $\circ, \pi^*, \pi_*$

eg for compact oriented manifolds take $H^*(X)$ with $\cup$, π^* & $\int_\pi$.
(or K-theory, or, ...)

Poincaré + Künneth:

$$\mathrm{Hom}(H^*(X), H^*(Y)) = H^*(X) \otimes H^*(Y)$$
$$= H^*(X \times Y) \ni K \quad \text{integral kernels.}$$
$$f \longmapsto \int_{\pi_2} K \cup \pi_1^* f$$

For G compact Lie group can use this instead of Fun

[Freed-Hopkins-Teleman: similar approach to Chern-Simons using twisted K-theory]

Cohomology is the shadow of a complex / quism (de Rham / singular / simplicial / Čech) which is smarter & better behaved:
can't expect to get all the functoriality & gluing we want - especially once we go to codim 2 - without consistently working on cx level...

Geometric Function Theory

A new kind of functional & harmonic analysis in algebraic geometry, where we replace function spaces (L^2, C^∞, distributions, ...) by various dg categories - forms of the derived category - of sheaves on algebraic varieties or stacks [orbifolds]

i.e. Fun $\rightsquigarrow$ Shv

codim 1 $\rightsquigarrow$ codim 2

spaces of states $\rightsquigarrow$ categories of branes

[eg Fourier-Mukai]

[Note: triangulated categories are <u>bad</u>. need to keep open string BRST complexes, not just their cohomology!]

<u>Examples</u>:

$\left.\begin{matrix}\text{bounded} \\ \text{unbounded}\end{matrix}\right\}$ derived category of $\left\{\begin{matrix}\text{perfect complexes} \\ \text{coherent sheaves} \\ \text{quasicoherent sheaves} \\ \text{D-modules}\end{matrix}\right.$

Largest, best behaved: unbounded quasicoherent [∞-rank version of coherent] - analog of distributions or L^2

- easiest to prove theorems there, then prove "regularity" results. D-modules: harder!

Properties

Kontsevich's NC geometry

- Refined invariant: $D(X)$ detects deformation theory of X .... detects X up to discrete ambiguity (Toën-Vaquié)
  Can recover $H^*_{dR}(X)$ ($\mathbb{Z}/2$ graded) & H^*_{Dol} & Hodge structure (notion of smooth & proper) & K-theory
  CY case - equivalent to 2d dg TFT
- Interesting invariant: nontrivial dualities like T-duality $\longleftrightarrow$ Fourier-Mukai
- Has a "commutative ring" structure given by $\otimes$.
  Moreover $X =$ "Spec" $D_{coh}(X)$
  (BZ-Francis-Nadler Tannakian theorem)
  $\Rightarrow$ theory of spectral decomposition for $D(X)$-modules
- $f: X \to Y$ have
  $$f_* : D(X) \rightleftarrows D(Y) : f^*$$

– don't make sense e.g. on vector bundles, need derived versions to get well behaved operations

• "Schwarz kernel theorem":

All "continuous" operators $D(X) \longrightarrow D(Y)$ are given by integral transforms $K \in D(X \times Y)$

$$K * F = \pi_{2*}(\pi_1^* F \otimes K)$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

[Toën for varieties,
BZ-Francis-Nadler for stacks, eg Bun_G Σ]

• Künneth & Poincaré: [BZ-Francis-Nadler]

$$D(X \times Y) \simeq D(X) \otimes D(Y)$$

$$D(X) \simeq Hom(D(X), D(\cdot))$$

All these tools mean we can build TFTs by push-pull operations for varieties & stacks just like we did for finite sets & orbifolds

[Costello construction of B-model,
Hopkins-Lurie construction of string topology: M manifold,
$Z_M(pt) = D_{loc\,const}(M)$, $Z_M(S^1) = C_*(LX)$
– need one outgoing boundary compact]

Cohomological (or dg) TFT

2 tiers n-Cob : category [or homotopical version thereof...]

objects = closed $n-1$ manifolds

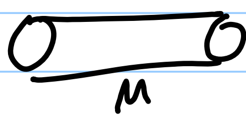
$\mathrm{Hom}(N_1, N_2)$ = <u>Space</u> of n-manifolds M giving cobordisms N_1  N_2

Very versatile : spaces give Vect (H_0) graded Vect (H_*), or chain complexes / quasi-isom (C_*) $\Rightarrow$ can easily generalize TFT :

dg TFT : functor

$$\mathrm{Chains}(n\text{-Cob}) \longrightarrow dg\ \mathrm{Vect}$$

ie • $N \longmapsto Z(N)$ complex

• N_1  N_2 is a <u>0-cycle</u> of $n\mathrm{Cob}(N_1, N_2)$

$\Rightarrow$ get chain map of degree 0

$$Z(M) : Z(N_1) \longrightarrow Z(N_2)$$

• Given k-dim family $\{M(x)\} : N_1 \to N_2$

– ie k-chain on $n\mathrm{Cob}(N_1, N_2)$ –

get chain map of degree k $Z(N_1) \to Z(N_2)$

- eg homotopic M's induce chain homotopic maps..

$\rightsquigarrow$ $C_*(\mathcal{M}\mathrm{Cob}(N_1, N_2)) \longrightarrow \mathrm{Hom}(Z(N_1), Z(N_2))$
map of complexes.

Can pass to H_* or H_0 $\Rightarrow$ graded or usual TFT

eg $n=2$ spaces of 2-manifolds with given boundary (e.g. empty) are homotopic to $\mathcal{M}_{g,n}$ moduli of Riemann surfaces $\simeq B\,\mathrm{Diff}\,\Sigma_{g,n}$ (since Teichmüller space is contractible)

$\Rightarrow$ we are constructing cochains / cohomology classes on moduli of Riemann surface:

Topological conformal field theory: this is precisely the structure of twisted $N=2$ SCFT (coupled to 2d gravity), $\rightsquigarrow$ topological strings.

What about branes?

Branes in topological string theories naturally form dg categories [of which triangulated categories are a pale shadow]: spaces of open string states come as graded (ghost #) vector spaces with BRST operator (differential): ie get category where Homs are complexes (& everything understood up to quasi-isom)

3 tiers: Same story: 2-category

$Ob = n\text{-}2$ manifolds

$Hom(Y_1, Y_2)$ = topological category of cobordisms $Y_1 \xrightarrow{N} Y_2$

$2Hom(N_1, N_2)$ = space of cobordisms w/corners $N_1 \xrightarrow{M} N_2$

[cf. Lurie: Morse Lectures, IAS '08 notes on my webpage]

dgTFT: $Z(Y)$ dg category
$Z(N)$ dg functors
$Z(M)$ natural transformations

Classification in two dimensions

Costello, Kontsevich-Soibelman classify
2d dg TFTs going down to a point (3 tiers)
Generalized by Hopkins-Lurie (ongoing)
to ndim TFTs going down to a point
valued in <u>anything</u> for <u>any n</u>.

<u>Theorem</u> 2d dg TFT $\Longleftrightarrow$
NC Calabi-Yau

[strictly get dim=0 CYs unless we twist
TFT definition by determinant line
$\Longleftrightarrow$ central charge]

Noncommutative Calabi-Yau of dim. n :
dg category with a nondegenerate trace
$\mathrm{Hom}(A,A) \longrightarrow \mathbb{C}[n]$

$[\ \mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B,A[n])^*\ ]$
Serre duality is given by shift by n, $\omega = 0$

Geometric construction: X CY $\Rightarrow D_{dg}(X)$
NC CY, "immersion" of space of
CYs into space of NC CYs.

So 2d TFTs <u>are</u> essentially Calabi Yaus.

4d TFT gives in particular

- Σ surface $\Longrightarrow$ Z_Σ NC CY

  ...roughly a CY variety, which is Fields(Σ). The 2d TFT Z_Σ is then a σ-model into this NC CY.

- S^2 $\Longrightarrow$ $Z(S^2)$ has an algebra structure, & acts on Z_Σ for every $x \in X$. "'t Hooft / Wilson loops"

Geometric Langlands: our 4d TFT is a particular 4d gauge theory:

$\text{Fields}(\Sigma) = T^* \text{Bun}_G(\Sigma)$ roughly Calabi-Yau

Z_Σ = some dg category of sheaves on this (D-branes in σ-model).

$Z(S^2)$ is [roughly] Rep $G^\vee$ dual group

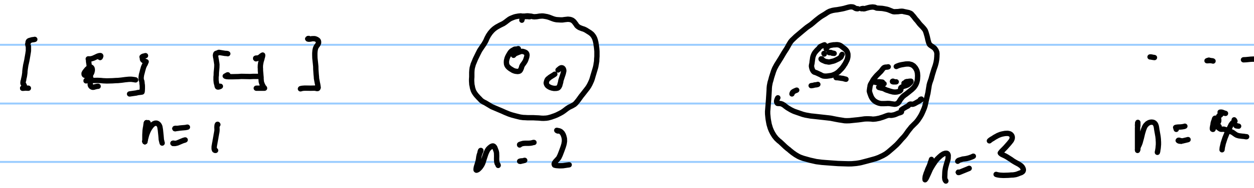
"Spec" $\{Z(S^2)\}_{x \in \Sigma} = \text{Loc}_{G^\vee} \Sigma$

Geometric Langlands Conjecture (strong form)

These operators have simple spectrum: $Z(\Sigma) \simeq D(\text{Loc}_{G^\vee} \Sigma)$

Local operators

What structure does $Z(S^{n-1})$ have?



E_n-algebra: multiplication parametrized by space of discs ($\Longleftrightarrow$ points) in n-disc : structure of n-dim OPE.

Even on level of graded Vect, all n are different : binary operations given by H_*(2-pt configs in $\mathbb{R}^n$)
$$= H_*(S^{n-1}) = \begin{cases} \mathbb{C} & \deg n-1 \\ \mathbb{C} & \deg 0 \end{cases}$$

$\Rightarrow$ An E_n graded vector space $V^\bullet$ ($n\geq 2$) has a commutative product (deg 0) & a Poisson bracket of deg $n-1$!

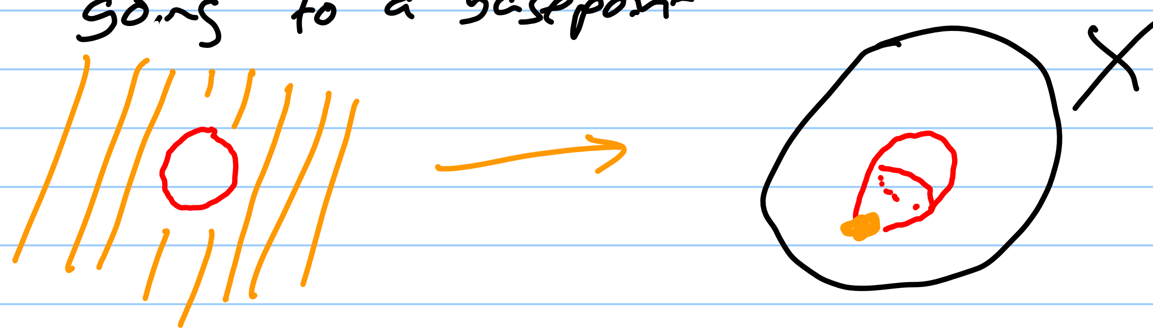
$n=2$ this is why vertex algebras look both like comm & Lie algebras!

- vertex algebras are a conformal refinement of E_2 algebras, where OPE can depend holomorphically, not just homotopically, on positions.

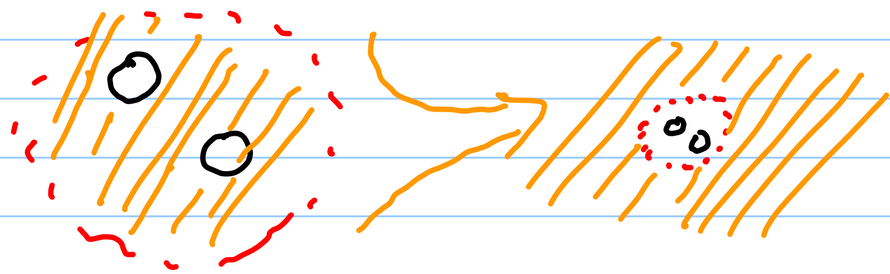
	Vect	Cat	gr Vect	dg Vect	dg Cat	Space
1	Assoc	Monoidal	Assoc	A_∞	E_1	loop space
2	Comm	Braided	Gerstenhaber (odd Poisson)	E_2	E_2	double loop space
3		Symmetric	even Poisson	E_3	E_3	triple loop space
			odd Poisson $\vdots$	$\vdots$	$\vdots$	
∞	↓	↓	Comm	E_∞	E_∞	infinite loop space. [connective] spectrum

Commutativity of n-dimensional OPE $\Longleftrightarrow$ Commutativity of π_n of spaces :

e.g $n=2$. $\pi_2(X)$ = components of $\Omega^2 X$
= iterated based loops in X
= maps from $D^2 \longrightarrow X$ with boundary going to a basepoint



Group structure on π_2 comes from composition on iterated loops $\Omega^2 X$, which is an E_2 multiplication: labelled by little 2-discs:



rotate

up to homotopy: commutative since, we can one around the other!

Vertex algebras: E_2 with holomorphic rather than homotopical dependence on positions $\rightsquigarrow$ hence vertex operators appear both as Lie & as commutative algebras

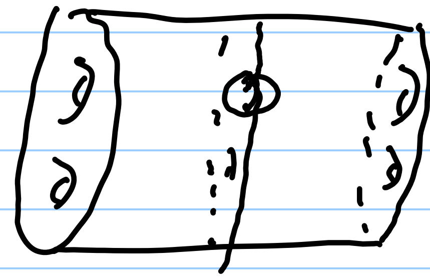
[Beilinson-Drinfeld, Gaitsgory-Lurie]

Theorem (May, Segal, ...)

A space Y is an n-fold loop space $Y \sim \Omega^n X$ for some X $\Longleftrightarrow$ Y has an "n-dimensional OPE" structure (May version: E_n space, Segal version: literally n-dim OPE)

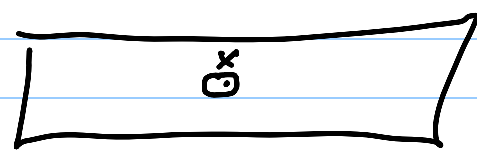
Example: Geometric Satake theorem, or 't Hooft operators

Study local operators on G-bundles (G complex or compact) on a Riemann surface



Solutions of Bogomolnyi equations with singularity

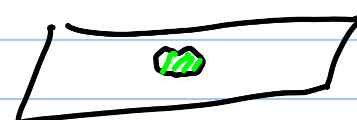
zoom & slice
(lose some symmetry)



local operators on bundles on a surface (singularities of gauge fields)

Gr = G-bundles with trivialization outside of x

$\cong LG/LG_+$

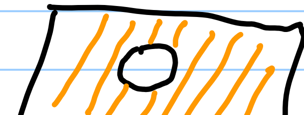


- clutching functions up to reparametrization

$\cong \Omega K$ based loops in the compact form of G

$\cong \Omega^2(BK)$ double based loops in the classifying space

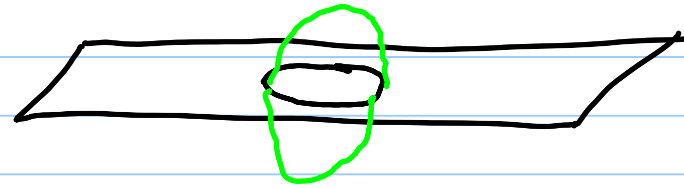
= maps $D^2 \longrightarrow BK$ with boundary going to basepoint



= K-bundles trivialized outside disc

[For 3d picture:

our modifications are labelled by



$LG_+ \backslash LG / LG_-$ double cosets = relative positions of pairs of bundles $\Longleftrightarrow$ instantons on the two-sphere.

These have an E_3, not just E_2, product.]

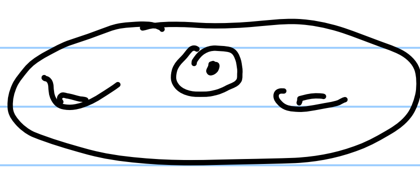
Hint at the CFT-Langlands connection?

the Kac-Moody vertex algebra arises by linearizing the affine Grassmannian:

$Gr \rightsquigarrow$ δ-functions at the basepoint on Gr

$\simeq$ Vacuum K-M vertex algebra

Construction of spaces of conformal blocks $\longleftrightarrow$ insertion of "objects" of the trivial surface operator



$$Z(\Sigma) = \langle Z(\Sigma \setminus p), Z(D) \rangle_{Z(S^1)}$$

Chern-Simons (3d analogy):

$Z(\Sigma)$ = vector space of conformal blocks
= coinvariants of some representation insertions (eg all vacuum)

$Z(S^1)$ = integrable level k reps of LG

Can construct vectors in $Z(\Sigma)$ from vectors in basic representation.

Likewise in our 4d TFT construct objects of $Z(\Sigma)$ from objects in $Z(D)$ - which are reps of Kac-Moody!