Geometric Function Theory

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The Geometric Langlands Conjecture gives a geometric Fourier transform for a reductive group G over an algebraic curve C.

We'll begin with the (classical) Fourier transform for a (locally compact topological) abelian group G (over a point).

To describe representations of G look in functions on G ... e.g. $L^2(G) \circlearrowleft G$.
Let $G^\vee = \mathrm{Hom}(G, U(1))$ group of characters (unitary irreps) of G. eg $\mathbb{R}^\vee = \mathbb{R}$, $(S^1)^\vee = \mathbb{Z}$

The Fourier-Pontrjagin duality describes the decomposition of functions on G under the action of G.

$$\mathbb{F} : L^2(G) \simeq L^2(G^\vee)$$

$\mathbb{R}$: $\cdot$ $f(x) = \int_{\mathbb{R}^\vee} \hat{f}(t)\, e^{ixt}\, dt$

write every function as continuous combination of characters

$\mathbb{F}$ diagonalizes action of G on L^2:

convolution/translation/differentiation $\longrightarrow$ multiplication

by $*g(x)$ $\quad T_\lambda \quad \frac{d}{dx} \longmapsto \cdot\hat{g}(t),\ \cdot e^{i\lambda t},\ \cdot t$

"Spectral decomposition"

- Can extend to different function spaces

eg $e^{\lambda x} \longmapsto \delta_\lambda(t)$

characters $\longleftrightarrow$ points

- Have indecomposable, not irreducible representations of $\mathbb{R} = \mathrm{Lie}(\mathbb{R}) = \mathbb{R} \cdot \frac{d}{dx}$: $xe^{\lambda x}$ generates 2 dim rep $\{e^{\lambda x}, xe^{\lambda x}\}$, $\frac{d}{dx}$ acts as $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

Jordan block $\overset{\mathbb{F}}{\Longleftrightarrow}$ spectral description

t acts as $\mathbb{R}[t]/(t-\lambda)^2$ torsion coherent sheaf

$\mathcal{O}_{2\cdot[\lambda]}$: structure of reps $\rightsquigarrow$ spectral geometry

Geometric Fourier Transform

Want an algebraic version of $\mathbb{F}$ ... e.g. replace $\mathbb{R}$ by $\mathbb{A}^1$. What should replace function spaces? $e^{\lambda x}$ not algebraic fns!

Rep theory suggests look at action of $\frac{d}{dx}$: Let $\mathcal{D} = \mathbb{C}\langle x, \frac{d}{dx}\rangle / (\partial x - x\partial = 1)$

ring of polynomial differential operators

$$e^{\lambda x} \rightsquigarrow \mathcal{D}\cdot e^{\lambda x} \subset C^\infty(\mathbb{R})$$

$$\text{is}$$

$$\mathcal{D}/\mathcal{D}(\partial-\lambda) : \text{ algebraic diff eq}$$

$$\delta_\lambda \rightsquigarrow \mathcal{D}\cdot\lambda \simeq \mathcal{D}/\mathcal{D}(t-\lambda)$$

These $\mathcal{D}$-modules are good algebraic substitutes!

Fourier transform: $\mathcal{P} = \langle e^{ixt}\rangle$

$$\begin{array}{ccc} & \mathbb{A}^1\times\mathbb{A}^1 & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{array}$$

$$M \rightsquigarrow \mathbb{F}(M) = \int M(x)\, e^{ixt}\, dt$$

$$= \pi_{2*}(\pi_1^* M \otimes \mathcal{P})$$

$\mathbb{F} : \mathcal{D}_{\mathbb{A}^1}\text{-mod} \xrightarrow{\sim} \mathcal{D}_{\mathbb{A}^1}\text{-mod}$

Moreover $e^{\lambda x}$, δ_λ satisfy nice algebraic equations $\rightsquigarrow$ get small = holonomic D-modules : "determine the functions up to finite dimensional ambiguities"

(as opposed to $\mathcal{D}$ itself $\simeq \mathcal{D}\cdot$(generic fn))

- D-modules can be decomposed as integrals of characters : $M_t = \mathcal{D} e^{ixt}$

$$M_t|_{x_1 \cdot x_2} \simeq M_t|_{x_1} \otimes M_t|_{x_2}$$

or $\quad \mu^* M_t \simeq M_t \otimes M_t$ character sheaf

Two main features:

- categories of sheaves replacing function spaces (vector-valued functions)
- geometry of D-modules

Geometric Function Theory, aka
Noncommutative Algebraic Geometry

Replace function spaces by categories of modules.

R commutative ring $\Leftrightarrow$ $R\text{-mod} = QCoh(\operatorname{Spec} R)$
recover each other.

Can think of an associative ring as a linear category with one object, or of a linear category as partially defined ring - but lose commutativity!

X scheme $\Rightarrow$ $\Gamma(\mathcal{O}_X) = \operatorname{End}(\mathcal{O}_X)$ too small to be useful, but $(Q)Coh\, X$ interesting — recovers X (Gabriel).

Same linear category can be described in many ways as modules over an algebra .... corresponds to presentations, or
<u>charts</u> on underlying geometry

best analogs for nonaffine schemes of $\mathrm{Fun}(X)$ are <u>derived categories</u> of sheaves

algebra : linear category :: dg algebra : dg category

look at categories where Homs are <u>complexes</u> & compositions dg morphisms.

$\mathcal{A}$ linear category $\rightsquigarrow$

$\mathrm{Com}(\mathcal{A})$ dg category of complexes

$\rightsquigarrow$ $D_{dg}(\mathcal{A})$: add inverses to quasi isomorphisms / kill acyclic complexes : world where different resolutions are identified.

Variants: bounded / unbounded / perfect complexes of (quasi) coherent sheaves / ... : geometric functional analysis

Analogy extends to construction of <u>integral transforms</u>:

operators in analysis $Kf(t) = \int K(x,t) f(x)\, dx$

…. ie "matrices" with matrix entries

$K(x,y)$ (generalized) function on product

Functors between $D(X)$ & $D(Y)$ (in dg sense)

given by sheaves $K \in D(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd} \qquad F \longmapsto \pi_{2*}(\pi_1^* F \otimes K)$$

(all functors are derived!

... badly behaved on abelian level)

$\mathcal{D}$-modules

X smooth variety, $\mathcal{D}_X = \mathcal{O}_X\langle T_X\rangle / \partial f - f\partial = f'$

qcoh sheaf of associative algebras $\qquad \partial_1\partial_2 - \partial_2\partial_1 = [\partial_1, \partial_2]$

A $\mathcal{D}$-module M is a qcoh $\mathcal{O}$-module with a flat connection: ie action of T_X satisfying Leibniz & compatibility with bracket

ie fn $\mathcal{O}$-mod is an "analytic vector-valued fn."
a $\mathcal{D}$-mod is "infinitesimally constant vector-valued fn"

- so $\mathcal{D}$-modules are good analogs for smooth fns like e^x & locally constant functions on p-adic groups

On the other hand $\operatorname{gr}\mathcal{D} = \operatorname{Sym}_{\mathcal{O}} T_X$
$= \mathcal{O}_{T^*X}$ ... in fact $\mathcal{D}_\hbar = \mathcal{O}\langle T\rangle / \hbar\text{-rescaled relations}$ is a flat deformation of $\mathcal{O}_{T^*X}$.

So can think of $\mathcal{D}$-modules as coherent sheaves on a "quantized cotangent bundle"

A $\mathcal{D}$-module M has a "shadow" on T^*X: $\operatorname{gr} M$ for a "good" filtration on M —

well defined cycle $CC(M) = \mathrm{Supp}(\mathrm{gr}\, M)$
in T^*M which is conical (shadow!)
& coisotropic : smallest are Lagrangians

.... "Heisenberg uncertainty principle"
[x, ∂_x can't annihilate the same vector!]
– can't pin down the shadow in both position
& momentum.

Microlocal geometry: try to give a description
of the category of D-modules that's local in T^*X
& a symplectic invariant

$$\begin{tikzcd}
\text{Coisotropics} \arrow[d, phantom, "\cup"] & D\text{-mod} \arrow[l] \arrow[d, phantom, "\cup"] & \\
\text{Lagrangian} \arrow[d, phantom, "\cup"] & \text{holonomic } D\text{-mod} \arrow[l] \arrow[d, phantom, "\cup"] \arrow[dr] & \\
\text{Lagrangian nice at } \infty & \text{regular holonomic } D\text{-mod} \arrow[l, "CC"'] \arrow[r, "\sim", "RH"'] & \text{constructible complexes (perverse)}
\end{tikzcd}$$

– eg $d + \omega$ flat connection on trivial bundle $\Longleftrightarrow$
graph $\Gamma_\omega \subset T^*X$ Lagrangian : depends on trivialization

so need to make many different Lagrangians isomorphic....

Fourier transforms :

A abelian variety

$A^\vee$ = dual abelian variety = geometric characters

$\doteq \mathrm{Hom}(A, B\mathbb{G}_m) \doteq \mathrm{Ext}^1(A, \mathbb{G}_m)$

assignments $x \in A \longmapsto$ line $\mathcal{L}_x$

& $\mathcal{L}_{x+y} \simeq \mathcal{L}_x \otimes \mathcal{L}_y$

$\longleftrightarrow \quad \mu^* \mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$

$\mathcal{P}$
$\downarrow$
$A \times A^\vee$ universal line on the product
$\mathcal{P}|_{x,\mathcal{L}} = \mathcal{L}_x$

$\Rightarrow$ integral transform $\mathbb{F} : D(A) \xrightarrow{\sim} D(A^\vee)$
derived equivalence (Mukai)
$* \longleftrightarrow \otimes$: honest Fourier!

--- any sheaf on A is an integral of character sheaves (deg 0 line bundles)

$\mathcal{D}$-module version (Laumon - Rothstein):
$\mathcal{D}$-mod analog of character is
$\mathcal{L} \in A^\vee$ equipped with flat connection ∇
(infin. constant character)

$$A^\natural = \{ (\mathcal{L}, \nabla) \} \xrightarrow[\text{affine for } T_1^* A]{} A^\vee$$

Same construction gives

$$D(\mathcal{D}_A\text{-mod}) \xrightarrow{\sim} D(\mathcal{O}_{A^\natural}\text{-mod})$$

any $\mathcal{D}$-module is integral of flat characters

$[(A')^\natural = A' / \hat{A'} \longrightarrow (A')^\vee = \bullet / \hat{A'}]$

$A^\natural$ is a flat deformation of $A^\vee \times T_1^* A$ – affine bundle degenerates to associated vector bundle