TFT & Langlands

Note Title 12/23/2006

Kapustin - Witten:

Geometric Langlands is a manifestation of an underlying isomorphism of four-dimensional topological gauge theories $Z_{G,\Psi} \simeq Z_{G^\vee, -\frac{1}{\Psi}}$

- much richer mathematical structure!

<u>TFTs</u> Axiomatize the "coarsest" QFTs, build up tier by tier.

0 tiers

An n-dim TFT $Z/\mathbb{C}$ is an assignment

$M^n \longmapsto Z(M^n)$ M n-dim smooth oriented compact

- oriented diffeo invariant
- multiplicative: $\emptyset \mapsto 1$ $\amalg \mapsto \cdot$

$$Z(M^n) \longleftrightarrow \int_{\varphi \text{ fields on } M} e^{-S(\varphi)} \mathcal{D}\varphi$$

Next express locality (cut & paste) of Z:

1 tier

An n-dim TFT Z assigns (cpt. sm. otd.)

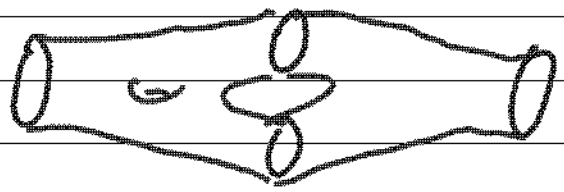
$N^{n-1} \longmapsto Z(N^{n-1}) \in \mathrm{Vect}_{\mathbb{C}}$

- functorial
- multiplicative: $\emptyset \longmapsto \mathbb{C}$, $(\ )^{op} \mapsto *$, $\amalg \mapsto \otimes$

Also M^n manifold with boundary

$$Z : M^n \longmapsto Z(M^n) \in Z(\partial M)$$

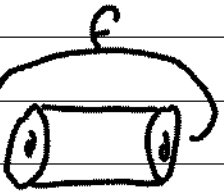
- $Z(N \times I) = Id \in Hom(N,N)$
- functorial under cobordisms



$\in Hom(Z(0), Z(0))$

$\uparrow \langle , \rangle$

$Hom(Z(0), Z(00)) \otimes Hom(Z(00), Z(0))$

$\Rightarrow Z($  $) = Tr(f)$

$$Z(N \times S^1) = \dim Z(N)$$

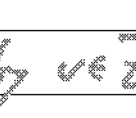
Get map $Z \longmapsto Z_{S^1}$ from nD TFT $\to$ $(n-1)$D TFT :

$$Z_{S^1}(N) = Z(N \times S^1)$$

More general <u>dimensional reduction</u> along Σ^k

$$Z \text{ } n\text{-dim} \longmapsto Z_\Sigma \text{ } n-k \text{ dim}$$

$$Z_\Sigma(M) = Z(M \times \Sigma)$$

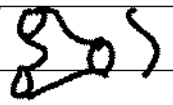
· Labelling convention : v  label by $v \in Z(\partial M)$

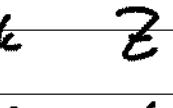
$\Longleftrightarrow$ number $Z(M)(v) \in \mathbb{C}$: i.e. labelling closes off boundary components

<u>Theorem</u> 2D TFT $\Longleftrightarrow$ commutative Frobenius algebras

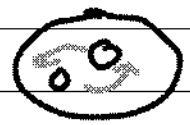
How? $Z \longmapsto H = Z(S^1)$

$Z(\bigcirc) = 1 \in H$ $\qquad Z(\bigcirc) = tr : H \longrightarrow \mathbb{C}$

$Z($  $) : H \otimes H \xrightarrow{\mu} H$.

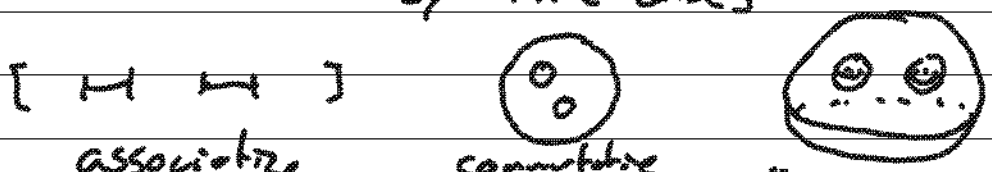
Check $Z($  $) = tr(\mu) : H \otimes H \longrightarrow \mathbb{C}$

nondegenerate. Moreover any Σ^2 can be decomposed into such building blocks, so recover $Z(\Sigma)$ from Frobenius structure.

Commutativity $\longleftrightarrow$ 

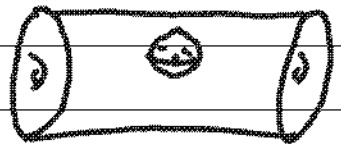
- More generally: Z n-dim

$Z(S^{n-1})$ carries multiplications parametrized by little discs



associative $Z(S^0)$ $\qquad$ commutative $Z(S^1)$ $\qquad$ "more commutative" $Z(S^2)$

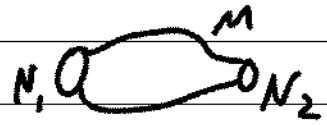
- Moreover $Z(S^{n-1})$ acts on $Z(N^{n-1})$



Generalizations: • Families version: given family of manifolds / base B assign $Z(\underset{B}{\overset{\downarrow}{N}})$ local system on B (parallel transport = isomorphism)

• graded TFT $Z(N) \in \mathrm{grVect}$

• dg TFT / TCFT: assign dg vector spaces to manifolds .... up to quasi isomorphism: weaker topological invariance to quism

Given 

$\Rightarrow$ chain map of degree zero $Z(N_1) \to Z(N_2)$

Given k-chain in $\mathrm{Hom}_{\mathrm{cob}}(N_1, N_2)$
$\Rightarrow$ k-chain in $\mathrm{Hom}_{\mathrm{dg}}(Z(N_1), Z(N_2))$

Now $Z(S^{n-1})$ is an E_n-algebra: multiplication parametrized by (chains on) little discs!

eg $E_1 = A_\infty$-algebra: homotopically weakened notion of dg associative algebra – can strictify, we'll gloss over the dg/A_∞ distinction.

Physics: CY varieties X provide 2 different 2D TFT, the A-model & B-model: both come from the same underlying <u>$N=2$ SUSY σ-model</u>: a conformal field theory ... ie depends on complex structures on our surfaces Σ
– fields are maps $\Sigma \longrightarrow X$ + fermions } "strings moving on X"

<u>topological twist</u>: change natural coordinate dependence of fields to make change of coordinates <u>exact</u> wrt differentials from SUSY $\Rightarrow$ cohomology is a TFT.

A-model: $Z(S^1) = QH^*(X)$ (symplectic invariant)

B-model: $Z(S^1) = \bigoplus \Gamma(\Lambda^i T_X)$ (holomorphic invariant)

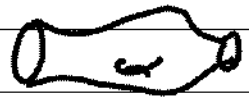
<u>2-tiered theories</u> Express locality of $Z(N^{n-1})$:

$$Z : Y^{n-2} \longmapsto Z(Y^{n-2}) \in \mathrm{Cat}_{\mathbb{C}}$$

$\mathbb{C}$-linear (or dg) category

- 2-functorial, monoidal

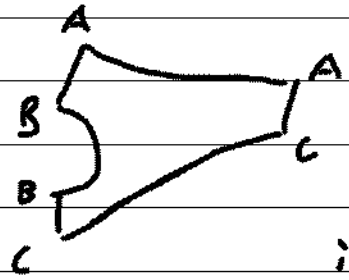
$Z(N^{n-1}) \in \mathrm{Ob}\, Z(\partial N)$ for manifolds w/ ∂.

 $\in \mathrm{Fun}(Z(Y_1), Z(Y_2))$ + composition

natural transformations for n-manifolds with corners:

e.g.

<u>2D</u>

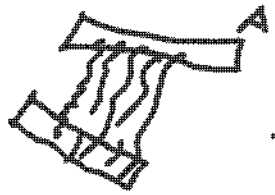


: $A \longleftarrow B$ gives <u>vector space</u> $\mathrm{Hom}(A,B)$

$\Rightarrow$ composition of Hom encoded in such diagram.

Moore-Segal: axioms for 2D TFT with labellings Λ of corners $\Longleftrightarrow$ Frobenius category: $Z(S^1)$ linear category with traces & compatibilities.

- Note $Z^2(S^0)$ monoidal $Z^3(S^1)$ braided $Z^4(S^k)$ $k \geq 2$ symmetric
- $Z(Y \times S^1)$ = categorical trace: $K_{\mathbb{C}}(Z(Y)) = HH^*(Z(Y))$

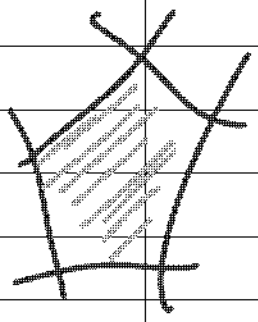
Interpretation: 

strings w/ contained 2 ... $\Lambda \leftrightarrow$ <u>D-branes</u>

<u>Physics</u>: To a CY can associate two categories in a natural way, which are the allowable (topological) boundary conditions for the A- & B- models

- A-model: $Fuk(X)$ Fukaya category of X

  Objects: Lagrangians $L \subset X$
  + decorations (grading, unitary local system)



  Morphisms: intersection points & Floer homology on count discs w/ marked points "Fuk"(X): potential big Fukaya, "coisotropic branes"

- B-model: $D^b(X)$ bounded derived category of coherent sheaves on X

Both are same kind of object: a dg category — or weaker, A_∞ category — with a <u>trace</u> $Hom(A,A) \to \mathbb{C}[\dim]$ on objects coming from holomorphic top form (B-model) — NC Calabi-Yaus

Thm (Costello): 2D dgTFT $\Longleftrightarrow$ NC Calabi-Yaus
(rough paraphrase)

[More precisely given set Λ of labels
Open TFT with Λ-labels $\Longleftrightarrow$ A_∞ CYs with Objects $= \Lambda$
NC CYs $\Longrightarrow$ 2D TCFT which have universality properties ..]

So the structure of 2dTFT captures a variety in the sense of noncommutative geometry
$\approx$ birational type in 3d (Bridgeland)
up to discrete symmetries more generally (Anel-Toen)

Thus a 4d (dg)TFT Z produces
a functor C surface $\longrightarrow Z_C$ NC CY
where Z_C = category of branes $Z(C)$,
with action (for $p \in C$) of symmetric
monoidal category $Z(S^2)$

Electric-Magnetic Duality

G_c compact Lie group
$\rightsquigarrow$ 4d TFT: $N=4$ Supersymmetric Yang-Mills with gauge group G_c, in GL twist.

- $N=4$ $D=4$ SYM is a gauge theory: fields include connection on principal G_c-bundle + 6 adjoint scalars + fermions: comes from $N=1$ $D=10$ SYM: connection A + adjoint spinors. require translational invariant in 6 dimensions
$\Rightarrow$ A_μ + ϕ^i $i=1,\dots,6$ (+ fermions)

Theory has a complex parameter τ^{eff}! combination of coeffs of YM action $\int F\wedge *F$ (coupling constant) $\left(\frac{4\pi i}{e^2}\right)$
& topological (Chern) term $\frac{\theta}{2\pi}\int F\wedge F$ (theta angle)

S-duality

Key feature: Theories for group G & group $G^\vee$ are isomorphic .... generalization of E-M symmetry of Maxwell. Exchanges $\tau \leftrightarrow -\frac{1}{\tau}$
– part of $SL_2\mathbb{Z}$ symmetry with $\tau \mapsto \tau+1$

[Goddard-Nuyts-Olive / Montonen-Olive]

GL Topological twist: change action of change of coordinates on $\mathbb{R}^4$ on ϕ^i : $\phi^{1,\dots 4}$ 1-forms, $\phi^{5,6}$ scalars ...

i.e $\mathbb{R}^{10} \simeq T^*M \times \mathbb{R}^2$.

So $A, \varphi \iff$ complexified connection.

The resulting theory has two odd symmetries Q_1, Q_2 with $Q_i^2 = 0 = \{Q_1, Q_2\}$,

cohomology of either Q is a TFT

$\Rightarrow$ $\mathbb{P}^1$ – family of twisted TFTs

labelled by $Q = aQ_1 + bQ_2$

... in fact dg TFTs if take Q-complex.

Resulting TFT actually depends on a particular combo of t, τ called $\Psi \in \mathbb{P}^1$

$$S = \frac{i\psi}{4\pi} \int F \wedge F + \{Q, -\}$$

4d Gauge theories as TFTs: • $Z(M^4) = \#$ solns of field equations (YM equations + modification — Higgs)

• $Z(N^3) = H^*$(moduli of solns of "monopole" eqns on N)

$\boxed{\text{KW}}$ • $Z(C^2)$ = Category of branes on moduli of solns on C^2:

$$C \Rightarrow \text{2d TFT} : \quad Z_C(\Sigma) = Z(C \times \Sigma)$$

Approximation

Think of C as very small: gauge fields on $C \times \Sigma$ $\longleftrightarrow$ maps from Σ to moduli of solns of field eqns on C, ie σ-model on $\mathcal{M}(C)$ moduli space.

Thus $Z(C)$ = category of branes on $\mathcal{M}(C)$!
— on NC CY.

What is $\mathcal{M}(C)$? Hitchin moduli space

Hitchin's equations on C:

$$F - \phi \wedge \phi = 0 \qquad (F = dA)$$
$$d_A \phi = 0 = d_A * \phi$$

Moduli of Higgs bundles, a hyperkähler manifold $\rightsquigarrow$ in particular CY.

$\mathcal{M}_{Hit}(C)$ has a $\mathbb{P}^1$ of Kähler structures
$\operatorname{span}\{I, J, K\} \Rightarrow$ a wedge of
σ-models: here A, B models in each,
& in fact linear combs of both corresponding
to interpolating complex $\xrightarrow[\text{gen. cx str.}]{}$ symplectic

[$\mathbb{C}^*$ symmetry identifying all but $\pm I$]

KW: $\Psi = \infty \Leftrightarrow J_B$ $\quad \Psi = 0 \Leftrightarrow K_A$

$\mathcal{M}_{Hit}(C)_J$ = moduli of stable flat
G connections on C $\subset \mathcal{L}oc_G(C)$

$$Z_G(C)_{\psi=\infty} = D^b(\mathcal{L}oc_G\, C)$$

$\mathcal{M}_{Hit}(C)_K \underset{\text{symplectically}}{=}$ moduli of stable
Higgs bundles $\subset T^*\mathrm{Bun}_G(C)$

$$Z_G(C)_{\psi=0} = \text{"Fuk"}(T^*\mathrm{Bun}_G\, C)$$

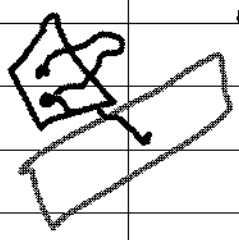
S-duality: $Z_{G,\psi=0} \simeq Z_{G^\vee,\psi=\infty}$

$\Rightarrow$ "Fuk"$(T^*\mathrm{Bun}_G C) \simeq D(\mathrm{Coh}\, \mathcal{L}oc_{G^\vee} C)$

From Fukaya categories to $\mathcal{D}$-modules:

(Kapustin, Witten; Nadler, Zaslow)

KW: T^*Bun_G carries canonical coisotropic (in fact space-filling) brane A, get functor from branes to $\mathrm{Hom}(A,A)$-modules



- calculate that this gives $\mathcal{D}$-modules on Bun_G.

$$A\text{-branes}(T^*Bun_G\, C) \simeq \mathcal{D}_{Bun_G}\text{-mod}$$

Nadler-Zaslow: prove a version of this statement in spirit of microlocal study of $\mathcal{D}$-modules,

X real analytic manifold $\quad \mathrm{Fuk}(T^*X) \simeq \mathcal{D}_{rh}(X)\text{-mod}$

't Hooft & Wilson loops

Operators in 4d gauge theory come in two major types

Wilson: given loop L in M^4 can calculate holonomy of connections along L & take trace in any representation of G: $V \in \mathrm{Rep}\, G \implies \mathrm{tr}_V \, \mathrm{hol}_L(A) = W_V(A)$

In quantum theory can insert this into path integral: $\langle W_V \rangle = \int_{\mathrm{conn}\, A} W_V(A) e^{-S(A)} DA$

't Hooft: can introduce magnetic monopole along L & make all calculations in the presence of this monopole:

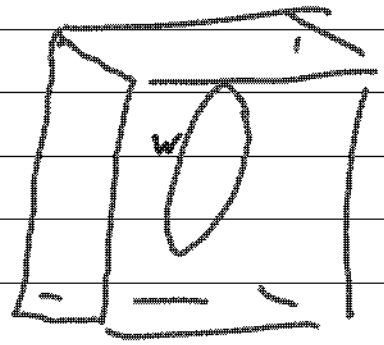
$$\int_{\substack{\mathrm{conn.}\, A \\ \mathrm{with} \\ \mathrm{sing.}}} e^{-S(A)} DA \quad \left(\text{"disorder operator"}\right)$$

What are these labelled by? conical singularity of soln of YM in $\mathbb{R}^3 \setminus 0 \iff$ solution on S^2

$\Longleftrightarrow$ $G_{\mathbb{C}}$-bundles on $\mathbb{P}^1$ $\Longleftrightarrow$ $\mathbb{C}^* \longrightarrow G_{\mathbb{C}}/\sim$

$\Longleftrightarrow$ Rep $G^\vee$.

From formal POV this is simply calculate some allowable labellings of a loop:



- i.e. states in $Z(S^1 \times S^2)$

give operators on $Z(Y^3)$ for any loop in Y^3.

$$\boxed{\text{S-duality exchanges Wilson \& 't Hooft loops}}$$

Topological twist: not all operators defined for all values of the parameters.

	't Hooft	Wilson
$\Psi = 0$	✓	×
$\Psi = \infty$	×	✓

So <u>Rep $G^\vee$</u> acting on $Z_{\Psi=0}(Y^3)$ by 't Hooft loops

$\longleftrightarrow$ <u>Rep $G^\vee$</u> acting on $Z_{\Psi=\infty}(Y^3)$ by Wilson loops

Down to 2 dimensions: $Z(S^1 \times S^2) = K_{\mathbb{C}}\, Z(S^2)$

G $\quad \Psi = 0$: $\quad Z_G(S^2) = \mathcal{D}\text{-mod}(\mathrm{Bun}_G\, \mathbb{P}^1)$

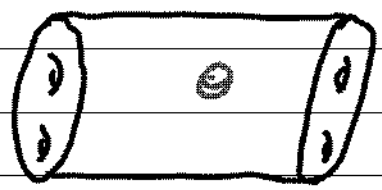
$$\simeq \mathrm{Rep}\, G^\vee \quad (\text{Satake})$$

$G^\vee$ $\quad \Psi = \infty$: $\quad Z_{G^\vee}(S^2) = \mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee}\, \mathbb{P}^1)$

$$= \mathrm{Rep}\, G^\vee$$

So 't Hooft loops $\longleftrightarrow$ Hecke operators ...

Dynamic POV:



describe Hecke correspondences in terms of solutions to Bogomolny equations on $C \times I$ with prescribed singularities:

Bogomolny $\longleftrightarrow$ holomorphic type on leaves stays constant away from singularity.

[Wilson loops analogous to Frobenius characteristic polynomial]

<u>Eigenbranes</u> In $Z_{G,\psi=0}(C) = \mathrm{Fuk}(T^*\mathrm{Bun}_G C)$

have "obvious" objects, given by $\mathbb{C}^\times$ line bundles on smooth Hitchin fibers ($\leftrightarrow$ flat unitary conns).

These correspond to <u>points</u> on $\mathcal{L}\mathfrak{g}_G C$ under S-duality $\Rightarrow$ are <u>eigenbranes</u> for the 't Hooft operators: