Real Groups & Geometric Langlands

Note Title 12/31/2006

Explain work in progress with David Nadler relating geometric Langlands & the representations of real Lie groups.

Localization of Representations

Lie group $G \circlearrowright X$

$\Rightarrow$ $\mathfrak{g}$ acts by vector fields

$U\mathfrak{g}$ acts by differential operators

$$U\mathfrak{g} \longrightarrow \Gamma(\mathcal{D}_X)$$

So get Beilinson functors $\mathcal{D}_X\text{-mod} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} U\mathfrak{g}\text{-mod}$

$\Delta M = M \otimes_{U\mathfrak{g}} \mathcal{D}_X$, just as we have

for $R \longrightarrow \Gamma(\mathcal{O}_X)$ (ie X/R)

$$\mathcal{O}_X\text{-mod} \rightleftarrows R\text{-mod}.$$

Natural place to localize reps of $\mathbb{C}$ reductive Lie algebra $\mathfrak{g}$: flag variety G/B:

Borel-Weil says reps of G found uniquely in fns on G/B: fns on G/N transforming under H as a character.

Beilinson-Bernstein: roughly says

$$\mathcal{D}(G/B) \simeq U\mathfrak{g}\text{-mod} \otimes \mathbb{C}[\mathfrak{h}]$$

at least an equivalence for most $\mathbb{C}[\mathfrak{h}^*/W]$-monodromic/twisted
(actions of $Z\mathfrak{g} \simeq \mathbb{C}[\mathfrak{h}^*]^W$)

eg $\mathcal{D}(G/B) \simeq U\mathfrak{g}_0\text{-mod}$:
center acts as on trivial rep.

We'll ignore the distinction between G/B & G/B
reg vs monodromic / twisted $\mathcal{D}$-modules,
& work with regular dominant inf. character.

Examples: • $\mathcal{D}(B\backslash G/B)$: Harish-Chandra bimodules
(close to BGG Category $\mathcal{O}$ of highest
weight modules) : reps of G as
real Lie group

• $G_{\mathbb{R}} \subset G$ real form, the derived category
of Harish-Chandra modules (admissible reps)
$\simeq D(\mathcal{D}(G_{\mathbb{R}}\backslash G/B))$ Kashiwara-Schmid

(really look at real constructible sheaves : has a holomorphic realization by BB as $\mathcal{D}(K_{\mathbb{C}} \backslash G_{\mathbb{C}} / B)$, related to above by Matsuki : Mirkovic-Uzawa-Vilonen)

.... all such carry Hecke action of "finite Hecke algebra" $\mathcal{D}(B \backslash G / B)$ or variants

This a geometric analogue of the description of reps of a group inside functions on that group; is there a Fourier dual?

<u>Langlands</u> : classified irred admissible reps of reductive real groups .... local Langlands conjecture for $\mathbb{R}$.

Works most naturally for several real forms at once : fix θ quasisplit real form

$1 \to G \to G^{\theta} \to \mathbb{Z}/2 \to 1$

$S = \{\sigma \in G^{\theta} \setminus G : \sigma^2 = id\}/G$,

G_{σ} = fixed points

$\coprod_{\sigma \in S}$ irreps for G_σ $\longleftrightarrow$ Langlands parameters!

Complex analytic data for $G^\vee$ . .

$$1 \to G^\vee \to G^L \to \mathbb{Z}/2 \to 1$$ <u>complex</u> group

$$\begin{array}{ccccc} & & G^L & \to & \mathbb{Z}/2 \\ & & \uparrow & & \| \\ \mathbb{C}^\times & \to & W_{\mathbb{R}} & \to & \mathbb{Z}/2 \end{array}$$

$\Longleftrightarrow$ pairs (δ, λ) $\delta \in G^L \setminus G$, $\lambda \in \mathfrak{g}^{ss}$

$\delta^2 = \exp \lambda$.

Adams-Barbasch-Vogan: promote Langlands' parametrization to a spectral statement on K-groups

Let $\mathcal{HC} = \bigoplus_{\sigma \in S} D(\mathcal{HC}_{G_\sigma})$

$K(\mathcal{HC}) \underset{\text{perfect pairing}}{\longleftrightarrow} K_{G^\vee}(X^\vee)$ for $X^\vee$: "spectral parameter space"

with orbits $\longleftrightarrow$ Langlands parameters

Soergel conjecture: this perfect pairing underlies a (Koszul) equivalence of derived categories: ie a Langlands description for all of $\mathcal{HC}$.

Case of $G_{\mathbb{C}}$ as a red group:
Soergel's conjecture reduces to results of Beilinson - Ginzburg - Soergel giving a (noncanonical) equivalence between $D(B\backslash G/B)$ & $D(B^\vee\backslash G^\vee/B^\vee)$

Our aim: express Soergel's conjecture in a geometric Langlands setting, reproving the $\mathbb{C}$ case & Vogan character duality & deducing general case from a <u>geometric base change</u> conjecture.

General scheme: study geometric Langlands on $\mathbb{P}^1$ in detail.

On $\mathbb{P}^1$ the set of iso classes of bundles is countable & the trivial bundle is <u>open</u>: admits no inf. deformations, has G of automorphisms:

open $BG \subset \mathrm{Bun}_G \mathbb{P}^1$. whole stack looks like $G(\mathcal{O})$ orbits on Gr with ordering reversed.

Tamely ramified case: look at $\mathrm{Bun} = \mathrm{Bun}_G(\mathbb{P}^1, 0, \infty)$

G-bundles + flags at $0, \infty$.

Open $\mathrm{Bun}^\circ \subset \mathrm{Bun}$ trivial bundle locus is isomorphic to $G \backslash G/B \times G/B = B \backslash G/B$

What about real groups? look at <u>antipodal map</u> as real form of $\mathbb{P}^1$

Bun_θ = real G_θ bundles on $(\mathbb{P}^1, \text{antipodal})$ + flag at ∞

<u>Proposition</u> Trivial bundle locus in Bun_θ is precisely $\coprod_{\sigma \in S} G_\sigma \backslash G/B$

Tamely ramified geometric Langlands

$\mathrm{Bun}_G(X, p)$ flag at p: carries Hecke action on $X \setminus p$

at p: carries action of $D(I \backslash G(K) / I)$
affine Hecke category

$$K\left(D(I \backslash G(K)/I)\right) = \text{constr. fn.}\,(I \backslash G(K)/I) = \mathbb{C}[W^{aff}]$$

$$\parallel$$

Kazhdan-Lusztig: $K_{G^\vee}(T^*\mathcal{B}^\vee \times_{\mathfrak{g}^\vee} T^*\mathcal{B}^\vee)$

Steinberg variety $\{(B_1, B_2, n) : n \in \mathfrak{b}_1 \cap \mathfrak{b}_2\}$

Bezrukavnikov:

$$D(I \backslash G(K) / I) = D(St_{G^\vee})$$

$$St_{G^\vee} = \{(B_1, B_2, g) \in g \in B_1 \cap B_2\}/G^\vee$$

derived stack

nilpotence stacky

Langlands meaning of this:

$$\mathrm{Coh}(\mathrm{St}_{G^\vee}) \text{ monoidal} \circlearrowright \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(X, p))$$

local systems with simple pole at "p" + flag at p containing monodromy.

Witten picture:

type of <u>surface singularity</u> in 4d gauge theory: now no longer strictly commute, but <u>braid</u>.

BZ-Nadler: Bondarko $\Rightarrow$

$$D(\mathrm{Bun}_G \mathbb{P}^1, 0, \infty) \simeq D_{\mathrm{Coh}}(\mathrm{St}_{G^\vee})^*$$

$\circlearrowright$ Hecke $\circlearrowright$ — tensor monoidal geometric Langlands for $\mathbb{P}^1, 0, \infty$

$*$ = dualize/saturate/complete.

How to pass from affine flags to finite flags?

S^1

rotate $\mathbb{P}^1, 0, \infty$: action of S^1 on category $Shv(Bun_G)$.

Turns out $\{\text{trivial bundle}\} \subset Bun_G$ = precisely S^1 fixed points!

$\Rightarrow$ can recover $Shv(\text{triv bdle}) = Shv(B\backslash G/B)$ via equivariant localization:

<u>Theorem</u> · $D(B\backslash G/B) \otimes \mathbb{C}[u,u^{-1}]$ $\quad \deg u = 2$

$$\simeq D(Bun_G)^{S^1} \otimes_{\mathbb{C}[u]} \mathbb{C}[u,u^{-1}]$$

· Same for Bun_θ : S^1 action real!

$$D(Bun_\theta)^{S^1} \simeq D\left(\coprod_s G_\sigma \backslash G/B\right)$$

$$\simeq \coprod_{\sigma \in S} HC_{G_\sigma} \qquad (\text{after } \otimes \mathbb{C}[u,u^{-1}])$$

<u>Miracle</u> $St_{G^\vee}$ is actually a loop space, in sense of derived algebraic geometry!

$$St_{G^\vee} = \mathcal{L}\, B^\vee \backslash G^\vee / B^\vee$$

So carries an abstract S^1 action

<u>Theorem</u> $\mathcal{O}(\mathcal{L}X)^{S^1,\circ} \simeq \mathcal{D}(X)$ [illegible]

any X

..... derived interpretation of cyclic homology Hey

<u>Corollary</u> Canonical equivalence of derived categories $\mathcal{D}(B\backslash G/B) \simeq \mathcal{D}(B^\vee \backslash G^\vee / B^\vee)$

nonabelian Fourier transform

Real groups: Pass from $(\mathbb{P}^1, 0, \infty)$ to $(\mathbb{RP}^2, \infty)$, i.e. introduce real structure.

Geometric base change conjecture allows us to pass between covering spaces, predicts

$$\mathcal{D}(\mathrm{Bun}^{\infty}_{G,\theta}\mathbb{RP}^2) \simeq \mathcal{O}(\mathcal{S}t^{\vee}_{\theta^{\vee}})$$

<u>Theorem</u> Geometric base change $\Longrightarrow$ Soergel conj:

i.e. $\mathcal{D}(\mathrm{Bun}_{G,\theta})^{S^1,\circ} \simeq \mathrm{Rep}(G_\theta)$

$$\mathcal{O}(\mathcal{S}t^{\vee}_{\theta^{\vee}})^{S^1,\circ} \simeq \mathcal{D}(G\backslash X^{\vee})$$