Date: 01/17

We'll be talking about SL_2 over different fields: $k = \mathbf{C}, \mathbf{R}, \mathbf{F}_p, \mathbf{Q}_p$. The most fundamental group in Lie theory: simplest nonabelian group, building block for other simple groups.

Appears:

- number theory: modular forms ($SL_2(\mathbf{R})$, $SL_2(\mathbf{Q}_p)$)
- hyperbolic geometry: $PSL_2(\mathbf{R}) = Iso(\mathbf{H}^2)$, $PSL_2(\mathbf{C}) = Iso(\mathbf{H}^3)$
- algebraic geometry: Lefschetz theorem: $H^*(X, \mathbf{C})$ for X compact Kähler smooth projective variety over $\mathbf{C}$) carries an action of $SL_2(\mathbf{C})$.
- analysis: special functions (Bessel, hypergeometric, etc) have interpretation in terms of SL_2. Huygens principle holds in n dimensions if some rep of SL_2 is finite-dimensional. Reference: Howe-Tan, Nonabelian Harmonic Analysis
- physics: Lorentz group is $SL_2(\mathbf{C})$. Particles are reps of $SL_2(\mathbf{C})$.

References: Fulton-Harris, Representation Theory; Etingof, Representation Theory; Carter-Macdonald-Segal, Lie groups; Lang, $SL_2(\mathbf{R})$.

G a group, V a vector space over $\mathbf{C}$. (V, ρ), where $\rho : G \to \mathrm{Aut}(V)$ a homomorphism.

Let V, W be two reps of G, a map $\phi : V \to W$ is a G-homomorphism if $g \cdot \phi(v) = \phi(g \cdot v)$.

$\mathrm{Hom}_G(V, W)$ = collection of such ϕ, a vector space.

Definition. *G a topological group (Lie group) if G is a topological space (manifold) and $G \times G \to G$ and $G \xrightarrow{i} G$ are continuous (smooth).*

For now V is finite-dimensional, so no topological issues there.

A representation of a topological group is a map $G \to \mathrm{Aut}\, V$, which is continuous.

In other words, let $v \in V$, $v^* \in V^*$. Can construct a function f_{v,v^*} on G: $f_{v,v^*}(g) = v^*(g \cdot v) \in \mathbf{C}$ called a matrix element.

$V \cong \mathbf{C}^n$, $\{e_i\}$ is a basis, $\{e_i^*\}$ the dual basis. $f_{e_j, e_i^*}(g) = ij$ entry of matrix $\rho(g) \in \mathrm{Aut}(\mathbf{C}^n) \subset \mathrm{Mat}_{n \times n}(\mathbf{C})$.

This gives a map $V \otimes V^* \to \mathrm{Fun}(G)$. Can also identify $\mathrm{End}(V) \cong V \otimes V^*$.

To specify a class of representations, we specify the class of functions on G that we allow.

V, W two reps of G, can construct $V \oplus W$ a new representation.

This leads to a basic question: is a representation a direct sum?

Definition. *V is irreducible if it has no nontrivial subrepresentations (i.e. no G-invariant subspaces).*

Definition. *V is indecomposable if we can't write $V \cong W_1 \oplus W_2$.*

Let $G = \mathbf{C}$, take $V = \mathbf{C}^2$. $G \subset GL_2(\mathbf{C})$ as upper-triangular matrices. Then W spanned by the first basis vector is a subrep, but it has no complement. So, V is indecomposable, but not irreducible.

The standard representation of $SL_2(\mathbf{C})$ on $\mathbf{C}^2$ is irreducible.

compact abelian groups	compact
abelian	Lie groups$(SL_2, ...)$

For compact groups irreducible = indecomposable. For abelian groups irreps are 1-dimensional.

Theorem (Schur's Lemma)**.** (1) *V_1, V_2 are two finite-dimensional irreps. Let $\phi : V_1 \to V_2$ be a G-map, then ϕ is either 0 or ϕ is an isomorphism.*
(2) *Let $\phi : V_1 \to V_1$, then $\phi = \lambda \cdot \mathrm{id}$, $\lambda \in \mathbf{C}$.*

Proof. (1) Observe, that $\mathrm{Ker}\,\phi \subset V_1$ and $\Im\phi \subset V_2$ are G-invariant subspaces. Since V_1 and V_2 are irreps, then these are either 0 or everything.
(2) Let $\lambda \in \mathbf{C}$. Then $\mathrm{Ker}(\phi - \lambda \cdot \mathrm{id})$ is a G-invariant subspace. There is at least one eigenvalue, so this is nonzero, i.e. everything for some λ.

□

For $\mathbf{C}^2$ and $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ the representation, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutes with everything, but is not scalar.

Corollary. *G abelian, V a finite-dimensional irrep, then* $\dim V = 1$.

Proof. For any $g \in G$, $\rho(g) \in \mathrm{Aut}_G(V)$ commutes with everything, so by Schur's lemma is scalar. Any subspace is G-invariant, so V is 1-dimensional. □

Corollary. *More generally, V is an irrep and $z \in Z(G)$ (center of G), then z acts by scalar multiplication.*

I.e. V has a central character: $\chi : Z(G) \to \mathbf{C}^*$, s.t. $z \cdot v = \chi(z)v$.

Definition. *V is completely decomposable (semisimple) if $V \cong V_1 \oplus V_2 \oplus ... \oplus V_k$ a direct sum of irreducibles.*

Proposition. *G a finite group, then any rep V is completely decomposable.*

Definition. *V is unitarizable if $\exists\langle,\rangle$ a Hermitian positive-definite inner product, which is G-invariant.*

Here G-invariance means $\langle g\cdot v, g\cdot w\rangle = \langle v, w\rangle$.

Proposition. *A unitarizable representation is semisimple.*

Proof. $W \subset V$, then $W^\perp \subset W$ is also G-invariant, then $V = W \oplus W^\perp$. By induction, it is semisimple. □

Proposition. *G finite, V finite-dimensional, then V is unitarizable.*

Proof. Start with any $\langle,\rangle_0$ on V. Define $\langle,\rangle = av(\langle,\rangle_0) = \frac{1}{|G|}\sum_{g\in G} g\cdot\langle,\rangle_0$, which is G-invariant.

$\langle v,v\rangle = \frac{1}{|G|}\sum\langle gv, gv\rangle_0$ is nonzero, so it is nondegenerate. □

For G compact, the same holds. Replace $\frac{1}{|G|}\sum_g$ by integration with respect to the Haar measure: $\int_G d\mu$. E.g. $G = S^1$ has $d\mu = \frac{1}{2\pi}d\theta$.

Think of the integral as a map $C(G,\mathbf{R}) \to \mathbf{R}$, which satisfies some properties, but in particular is left-invariant.

Let G be a Lie group. Take any volume form at the identity $dx_1 \wedge \dots \wedge dx_n \in \wedge^{top} T_1^* G$ and then translate it using $L_{g^{-1}}$. Get a top form on G and then normalize it to have $\int_G d\mu = 1$.

For G compact $d\mu$ is actually both left and right-invariant:

$$\int_G f(\cdot g) d\mu : G \to \mathbf{R}_+^*,$$

which has to be trivial.

How to decompose representations. First look for the trivial rep. For V assign its G-invariants $V^G = \{v \in V : g\cdot v = v \quad \forall g \in G\}$. $V^G = \mathrm{Hom}_G(\mathbf{C}, V)$.

For G compact, we have $av_V : V \to V^G$, where $av_V(v) = \int_G g\cdot v d\mu$. It is an idempotent: $av_V \circ av_V = av_V$. Then $V = V^G \oplus \mathrm{Ker}(av_V)$.

Given V a rep and W an irrep. Consider $\mathrm{Hom}_G(W,V)$, these are all injective.

$$\mathrm{Hom}_G(W,V)\otimes W \overset{ev}{\to} V \tag{1}$$

$$\phi, w \mapsto \phi(w) \tag{2}$$

Proposition. *ev is injective*

So, $\mathrm{Hom}_G(W,V)\otimes W \hookrightarrow V$. If V is completely decomposable, $V = \dots \oplus W \oplus W \oplus \dots \oplus W \oplus \dots$, we picked out all W summands.

$$V \cong \bigoplus_{W irreducibles} \mathrm{Hom}_G(W,V)\otimes W.$$

Here $\mathrm{Hom}_G(W, V)$ is the multiplicity space and $\mathrm{Hom}_G(W, V) \otimes W$ is a W-isotipic component.

Date: 01/19. Fourier series.

Reference: Terry Tao, The Fourier Transform (on his blog).

$G = \mathbf{T} = S^1 = U(1) = SO(2)$.

On S^1 we have a Haar measure $\frac{1}{2\pi} d\theta$.

Look at $H = L^2(\mathbf{T})$, a representation of $\mathbf{T}$ by left translation. Explicitly, for $f \in H, \alpha \in \mathbf{T}$ we have

$$(\alpha \cdot f)(\theta) = f(\theta - \alpha),$$

pull back function f under the translations $\mathbf{T} \to \mathbf{T}$. These are lots of commuting operators, can try to simultaneously diagonalize them. Equivalently, decompose H into irreducibles. $H = W_1 \oplus W_1 \oplus ...W_2 \oplus ...W_3 \oplus ...$

What are the irreducible representations of $\mathbf{T}$? It is a homomorphism $\mathbf{T} \to \mathbf{C}^\times$. Since $\mathbf{T}$ is compact, it factors through $\chi : \mathbf{T} \to U(1)$. Such a homomorphism is called a (unitary) character.

$\chi(\theta_1 + \theta_2) = \chi(\theta_1)\chi(\theta_2)$. The solutions are given by $\chi_n(\theta) = e^{2\pi i n\theta}$ for $n \in \mathbf{Z}$. Here we think $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

$H_{\chi_n} = \chi_n$-eigenspace (isotypic component) in H, here each α acts by $\chi_n(\alpha)$.

$(\alpha \cdot \chi_n)(\theta) = e^{2\pi i n(\theta - \alpha)} = e^{-2\pi i n\alpha} e^{2\pi i n\theta} = \chi_{-n}(\alpha)\chi_n(\theta)$. So, $H_{\chi_n} = \mathbf{C}\chi_{-n}$.

$$H \supset \bigoplus_{n \in \mathbf{Z}} \mathbf{C}\chi_{-n} = \oplus H_{\chi_n}.$$

$f \in H = L^2(S^1)$. Can take its component by projecting f onto χ_{-n}: $f \mapsto \langle f, \chi_{-n} \rangle \chi_{-n}$.

$$\langle f, \chi_{-n} \rangle = \int f(\theta)\overline{\chi_{-n}(\theta)} d\theta = \int f(\theta)\chi_n(\theta) d\theta = av(f\chi_n).$$

Digression: V, W two representations of a group G, can construct a new representation $V \otimes W$ (for concreteness both are finite-dimensional). The action is given by $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$.

Now, let V be a rep of $\mathbf{T}$, construct $V \otimes (\mathbf{C}\chi_n)$. As a vector spaces, these are the same. The action is new: $(g \cdot v)_{new} = (g \cdot v)\chi_{-n}(g)$.

So, $V_{\chi_n} = (V \otimes \mathbf{C}\chi_n)^{\mathbf{T}}$.

$$\begin{array}{ccc}
(V \otimes \mathbf{C}\chi_n)^{\mathbf{T}} & \overset{\sim}{\leftarrow} & V_{\chi_n} \\
{\scriptstyle av}\uparrow & & \twoheaduparrow \\
V \otimes \mathbf{C}\chi_n & \longleftarrow & V
\end{array}$$

If V a finite dimensional,

$$f = \sum_{n \in \mathbf{Z}} \hat{f}(n)\chi_{-n},$$

where $\hat{f}(n) = \langle f, \chi_{-n} \rangle = \int f(\theta)e^{2\pi i n\theta}d\theta$.

Let $H = L^2(\mathbf{T})$ or any other representation of $\mathbf{T}$. Can produce the finite part

$$H^{fin} = \{v \in H : \quad span(\alpha \cdot v) \text{ is finite-dimensional}\} = \oplus V_{\chi_n}.$$

In the case $L^2(\mathbf{T})^{fin} = \mathbf{C}[z, z^{-1}]$, where $z = e^{2\pi i\theta}$. $L^2(\mathbf{T})^{fin} \subset L^2(\mathbf{T})$ is dense.

We constructed

$$L^2(S^1) \to L^2(\mathbf{Z}) = \ell^2$$
$$f \mapsto \hat{f} = \{\hat{f}(n) : n \in \mathbf{Z}\}$$

For any function space (or any V) $V^{fin} \subset V$ is dense.

For example, in $C^\infty(S^1)$ $\hat{f}(n) \to 0$ is faster than any polynomial. In $C^\omega(S^1)$ $\hat{f}(n) \to 0$ exponentially.

Dually, in $C^{-\infty}(S^1)$ (distributions) $\hat{f}^n) \to \infty$ at most polynomially. For hyperfunctions $C^{-\omega}(S^1)$ we have $\hat{f}(n) \to \infty$ at most exponentially.

Generalization: Pontryagin duality. The same for locally-compact abelian (LCA) groups.

We'll look at G being finite, compact, discrete.

- Finite: products of $\mathbf{Z}/n$'s.
- Tori: products of S^1's.
- Lattices: products of $\mathbf{Z}$'s.
- Euclidean space: products of $\mathbf{R}$'s.
- $\mathbf{Q}_p, \mathbf{Z}_p, \mathbf{A}$

For G an LCA, construct $\hat{G} = \mathrm{Hom}(G, \mathbf{T})$, collection of continuous unitary characters. For G compact $\hat{G} = \mathrm{Hom}(G, \mathbf{C}^\times)$.

$\hat{G}$ is a topological group: the topology is given by a compact-open topology from $Map(G, \mathbf{C})$. $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$, also abelian. Can prove that it's again locally-compact.

Rep theory POV: χ is the same as a 1d unitary representation of the group. $L_{\chi_1} \otimes L_{\chi_2} = L_{\chi_1\chi_2}$.

For example, $G = \mathbf{Z}/n$, think of G as a subset of $\mathbf{C}^\times$. The characters are given by raising to the k'th power, these are parametrized by $k \in \mathbf{Z}/n$. So,

$$\hat{G} \cong \mathbf{Z}/n.$$

Therefore, for G finite $G \cong \hat{G}$.

For $G = S^1 = \mathbf{T}$ we have $\hat{G} = \mathbf{Z}$.
For $G = \mathbf{Z}$, $\hat{G} = \{\mathbf{Z} \to S^1\} = S^1$ since $\mathbf{Z}$ is cyclic.

Theorem. $G \xrightarrow{\sim} \hat{\hat{G}}$

The map is given as follows. $g \in G$, $\chi \in \hat{G}$. Let $\chi(g) = \langle \chi, g \rangle \in S^1$. This is a function $G \times \hat{G} \to \mathbf{T}$.

Claim: $\chi \mapsto \langle \chi, g \rangle$ as a S^1-valued function on $\hat{G}$ is a character of the group $\hat{G}$. Indeed, $\langle \chi_1 \chi_2, g \rangle = \chi_1(g) \cdot \chi_2(g) = \langle \chi_1, g \rangle \langle \chi_2, g \rangle$.

For G compact, $\hat{G}$ is discrete.

Let's compute $\hat{\mathbf{R}}$. An element $t \in \hat{\mathbf{R}}$ is a map $\mathbf{R} \to \mathbf{T}$. The answer is $\chi_t(x) = e^{2\pi i t x}$.

So, for V a vector space $\hat{V} = V^*$ under $v, v^* \mapsto e^{2\pi i \langle v^*, v \rangle}$.

$$G \xleftarrow{\ \pi_1\ } G \times \hat{G} \xrightarrow{\ \pi_2\ } \hat{G}$$

Can do an integral transform $\pi_1^*(f)(g, \chi) = f(g)$. Explicitly,

$$\mathrm{Fun}(G) \to \mathrm{Fun}(\hat{G})$$
$$f \mapsto \hat{f}(\chi) = \int_G f(g) \langle \chi, g \rangle dg$$

Theorem (Plancherel). *$L^2(G) \xrightarrow{\sim} L^2(\hat{G})$ is an isometry.*

For $G = \mathbf{T}$, the Fourier coefficients were defined as $\hat{f}(n) = \int f(\theta) \chi_n(\theta) d\theta$, where we can think of $n \in \mathbf{Z} = \hat{S^1}$.

If G is compact, $\hat{G}$ is discrete, so a function on $\hat{G}$ is a discrete list of numbers. Then $f \mapsto \hat{f}$ is just writing (finite) functions in the basis of eigenfunctions, i.e. decomposing

$$L^2(G)^{fin} = \bigoplus_{\chi \in \hat{G}} \mathbf{C}\chi.$$

Date: 01/24.

Reference: Ramakrishnan-Valenza, Fourier analysis on number fields (GTM 186, Springer).

Fourier transform/Pontrjagin duality.

The goal: "solve" representation theory of a locally compact abelian group G. I.e. any $g \in G$ acts on $\mathrm{Fun}(G)$. We would like to simultaneously diagonalize all these. The same idea works for any other representation.

Let V be a finite-dimensional vector space. $\mathcal{O}_1, ..., \mathcal{O}_k$ matrices in $\mathrm{End}\, V$, all commute. Want to find a set X (join spectrum) and a bunch of functions $f_1, ..., f_k \in \mathrm{Fun}(X)$, and decompose V over X:

$$V = \bigoplus_{x \in X} V_x,$$

such that $\mathcal{O}_i$ acts on V by multiplication by f_i: write $v \in V$ as $v = \oplus v_x$, where $v_x \in V_x$, then $\mathcal{O}_i \cdot v = f_i v = \oplus f_i(x) v_x$.

For example, let $\mathcal{O}_i$ are diagonal matrices on $\mathbf{C}^n$. Then $X = n$ points, then $\mathbf{C}^n = \oplus_{x \in X} \mathbf{C}$.

$X =$ all common eigenvalues of $\mathcal{O}$, i.e. for any $x \in X$ we have an assignment $\mathcal{O}_i \mapsto f_i(x) \in \mathbf{C}$

$$V_x = \{v \in V : \mathcal{O}_i \cdot v = f_i(x) v\}.$$

For X a set, $\mathrm{Fun}(X)$ is a ring and is diagonalized in the basis of δ-functions δ_x for $x \in X$. That is, $f \delta_x = f(x) \delta_x$.

Let $G = S^1$, V a finite-dimensional representation. Then $V = \oplus_{n \in \mathbf{Z}} V_n$ (here $X = \mathbf{Z}$). For $\theta \in S^1$ and $v \in V_n$, $\theta \cdot v = e^{2\pi i n \theta} v$. For $\theta \in S^1$ we have a function $n \mapsto e^{2\pi i n \theta} \in \mathrm{Fun}(\mathbf{Z})$.

More generally, for $V = \mathrm{Fun}(S^1)$ we know that $V = \hat{\oplus} V_n$, where $V_n \cong \mathbf{C} e^{-2\pi i n \theta}$.

For G an LCA group (finite, S^1, $\mathbf{R}$).

Definition. *The Pontrjagin dual group* $\hat{G} = \mathrm{Hom}_{cont}(G, S^1)$.

For $g \in G$ we have a function on $\hat{G}$: $\langle -, g \rangle$. For any $\chi \in \hat{G}$ we assign $\langle \chi, g \rangle = \chi(g)$.

Lemma. *This assignment defines a group homomorphism* $G \to \mathrm{Fun}(\hat{G})^\times$.

Proof. Check: $g = hk$. $\langle -, hk \rangle = \langle -h \rangle \langle -, k \rangle$.

Similarly, $\langle -, g^{-1} \rangle = \{\chi \mapsto \chi(g^{-1}) = \chi(g)^{-1}\} = \langle , g \rangle^{-1}$. □

$V = \mathrm{Fun}(\hat{G})$ forms a representation of G: $g \cdot f = \langle -, g \rangle f$ in which all g's are simultaneously diagonalized. Then $V = \oplus_{\chi \in \hat{G}} V_\chi$, where $V_\chi = \mathbf{C} \cdot \delta_\chi$.

Recall the Plancherel theorem: $L^2(G) \xrightarrow{\sim} L^2(\hat{G})$ as G-representations. Here isometry means

$$\int_G dhdg = \int_{\hat{G}} \hat{f}\hat{h}d\hat{g}.$$

So, the Fourier transform diagonalizes the regular representation.

For $f \in \mathrm{Fun}(G)$ the Fourier transform $\hat{f}(\chi) = \int_G f(g)\overline{\chi(g)}dg = \langle f, \chi \rangle_{L^2}$.

Defining properties:

(1) Characters$\in \mathrm{Fun}(G)$ are mapped to δ-functions$\in \mathrm{Fun}(\hat{G})$. I.e. $\chi \mapsto \delta_\chi$ $(\chi \in \hat{G})$.
   δ-functions$\in \mathrm{Fun}(G)$ are mapped to characters$\in \mathrm{Fun}(\hat{G})$. I.e. $g \in G$ is mapped to $\langle -, g \rangle \in (\hat{\hat{G}})$. $\langle \chi_1\chi_2, g\rangle = \langle \chi_1, g\rangle\langle \chi_2, g\rangle$.

(2) Translations on G are mapped to multiplication operators (by characters) on $\hat{G}$.

One might object that neither $e^{2\pi ixt}$ nor δ_t are L^2-functions. Instead, consider the space $\mathcal{S}(\mathbf{R})$ of Schwarz functions. Then both $e^{2\pi ixt}$ and δ_t are in $\mathcal{S}(\mathbf{R})^*$ (tempered distributions).

Question: why is $()^\vee : \mathrm{Fun}(G) \to \mathrm{Fun}(\hat{G})$ a map of representations?

$g \in G$, $(\tau_g f)(h) = f(hg^{-1})$. Want to compute

$$\hat{\tau_x f}(t) = \int_G \tau_x f(y)\langle t, y\rangle dy = \int_G f(y-x)\langle t, y\rangle dy.$$

Let $z = y - x$, then the integral is

$$\begin{aligned} &= \int_G f(z)\langle t, z+x\rangle dz \\ &= \int_G f(z)\langle t, z\rangle\langle t, x\rangle dz \\ &= \langle t, x\rangle \hat{f}(t) \end{aligned}$$

Group algebras. G finite. $\mathrm{Fun}(G) = \mathbf{C}G$ has an associative multiplication by convolution:

$$(f \star h)(g) = \sum_{g_1g_2=g} f(g_1)h(g_2) = \sum_{g_1} f(g_1)h(gg_1^{-1}).$$

Convolution is characterized by $\delta_{g_1} \star \delta_{g_2} = \delta_{g_1g_2}$. $f = \sum f(g_i)\delta_{g_i}$.

As a vector space, $\mathbf{C}G$ is a linear span of $g \in G$.

We have the following maps:

$$G \times G \xrightarrow{\mu} G$$

$$\mathrm{Fun}(G \times G) \xleftarrow{\mu^*} \mathrm{Fun}(G)$$

$$\mathrm{Fun}(G \times G)^* \overset{\mu_*}{\mathrm{Fun}} (G)^* \cong \mathrm{Fun}(G)$$

$$f, h \mapsto f \star h$$

Let V be a representation of G. Then for any $g \in G$ $g \cdot v = \delta_g \star v$. Can extend this to the group algebra:

$$\sum_{g \in G} f(g) g \cdot v =: f \star v,$$

this makes V into a module over $\mathbf{C}G$.

Exercise: check that $(h \star f) \star v = h \star (f \star v)$.

Conversely, given a $\mathbf{C}G$-module, we get a representation of G via action by δ-functions.

Let G be a locally compact with μ, a Haar measure. Can define the convolution:

$$(f \star h)(x) = \int_G f(y) h(x-y) dy.$$

This makes $C_c(G)$ (continuous functions with a compact support) or $L^1(G)$ into associative algebras.

Proposition. *Convolution is mapped to multiplication under the Fourier transform.*

$\widehat{f \star g} = \hat{f}\hat{g}$.

For $f = \delta_g$, the convolution $f \star - = \tau_g$, so for finite groups if follows from what we had before.

That is, the Fourier transform diagonalizes the action of the entire group algebra on $\mathrm{Fun}(G)$.

$\mathcal{F} = \mathrm{Fun}(G)$ for G finite, or $\mathrm{Fun}(G)^{fin}$ (finite linear combinations of characters) for G compact. For example, for S^1 $\mathcal{F} \cong \mathbf{C}[z, z^{-1}]$ for $z = e^{2\pi i x}$.

$(\mathcal{F}, \star)$ is a commutative ring. For $\chi \in \hat{G}$ a character of G, can think of it as a 1-dimensional representation of G. Equivalently, it is a homomorphism $\chi : \mathcal{F} \to \mathbf{C}$ (extend by linearity). The other direction works as well (by restriction). Another way to think of it, it's a maximal ideal in $\mathcal{F}$ ($\mathrm{Ker}(\chi) \subset \mathcal{F}$). I.e. a point of $\mathrm{Spec}(\mathcal{F})$.

More generally, for $R = \mathrm{Fun}(X)$ a ring with multiplication. $x \in X$ gives rise to a maximal ideal in R (kernel of the evaluation at x).

E.g. $G = S^1$, $\mathcal{F} = (\mathbf{C}[z, z^{-1}], \star) = (span\{e^{2\pi i n x}\}, \star)$. Moreover, $\mathcal{F} \cong \mathrm{Fun}_{fin}(\mathbf{Z})$.

For V a representation of G we get an $\mathcal{F}$-module, i.e. a module for the ring of functions on $\hat{G} = \operatorname{Spec} \mathcal{F}$.

Date: 01/26. Spectral decomposition

V finite-dimensional vector space. $T \in \mathrm{End}(V)$. Make V into a $\mathbf{C}[x]$-module: x acts as T.

$\mathrm{Spec}\,\mathbf{C}[x] = \mathbf{C}$, affine line: maximal ideals are of the form $(x-\lambda)$ for $\lambda \in \mathbf{C}$. Equivalently, these are homomorphisms $\mathbf{C}[x] \stackrel{eval_\lambda}{\to} \mathbf{C}$ (aka 1-dimensional modules).

$$I \subset \mathbf{C}[x] \twoheadrightarrow \mathbf{C}[T] \subset \mathrm{End}\,V,$$

so $\mathbf{C}[T]$ polynomial functions on spectrum $\mathrm{Spec}\,T \subset \mathbf{C}$.

Jordan form: $V \cong \oplus_i \mathbf{C}[x]/(x-\lambda_i)^{n_i}$.

One of these pieces is a Jordan block of the form

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \dots & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

Remark: it is a principal example of a module which is not semisimple, but indecomposable.

V finite-dimensional vector space with an endomorphism T. It is the same as a finite-dimensional representation of $\mathbf{R}$: $x \in \mathbf{R}$ acts on V as $e^{ixT} = \mathrm{id} + ixT + \frac{(ixT)^2}{2!} + \dots$

$iT = \frac{d}{dx}\rho(x)|_{x=0}$.

Remark: T is self-adjoint $\Leftrightarrow$ ρ unitary.

$\mathbf{C}e^{isx} \subset \mathrm{Fun}(\mathbf{R})$ is an eigenspace for translations: $\tau_y e^{isx} = e^{-isy}e^{isx}$.

$$\begin{aligned} \tau_y(xe^{isx}) &= (x-y)e^{is(x-y)} \\ &= e^{-isy}(xe^{isx}) - ye^{isy}(e^{isx}), \end{aligned}$$

i.e. xe^{isx} is a part of a 2-dimensional representation.

$$\tau_y \begin{pmatrix} e^{isx} \\ xe^{isx} \end{pmatrix} = \begin{pmatrix} e^{-isy} & 0 \\ -ye^{-isy} & e^{-isy} \end{pmatrix} \begin{pmatrix} e^{isx} \\ xe^{isx} \end{pmatrix}$$

So,

$$-i\frac{d}{dy}(\tau_y)|_{y=0} = -i \begin{pmatrix} -is & 0 \\ -1 & -is \end{pmatrix}.$$

$\widehat{e^{ist}} = \delta_s(t)$, then $(t-s)\delta_s(t) = 0$. So, this is the same as $\mathbf{C}[t]/(t-s)$ as a $\mathbf{C}[t]$-module.

Question: what's the analytic realization of $\mathbf{C}[t]/(t-s)^2$?

$\delta'_s(f(t)) = -\delta_s(f'(t)) = -f'(s)$.

$$\begin{aligned} t \cdot \delta_s'(t) &= t\partial\delta_s(t) \\ &= \partial t\delta_s(t) - \delta_s(t) \\ &= s\delta_s' - \delta_s \end{aligned}$$

Alternatively,

$$\begin{aligned} t\delta_s'(f(t)) &= -(tf)'(s) \\ &= -sf'(s) - f(s) \end{aligned}$$

That means:

$$t \begin{pmatrix} \delta_s \\ \delta_s' \end{pmatrix} = \begin{pmatrix} s & 0 \\ -1 & s \end{pmatrix} \begin{pmatrix} \delta_s \\ \delta_s' \end{pmatrix}$$

Under Fourier transform,

$$\begin{pmatrix} e^{isx} \\ xe^{isx} \end{pmatrix} \leftrightarrow \begin{pmatrix} \delta_s \\ \delta_s' \end{pmatrix} \in \mathcal{S}(\mathbf{R})^*.$$

Recall, that

$$\begin{aligned} \text{characters} &\leftrightarrow \text{points (delta-functions)} \\ \text{translations } \tau_y &\leftrightarrow \text{multiplication by a phase} \\ \text{convolution} &\leftrightarrow \text{multiplication} \\ \text{differentiation} &\leftrightarrow \text{multiplication by coordinate} \end{aligned}$$

Case of S^1: $\hat{\partial}_x = 2\pi it\cdot$. Then

$$\partial_x e^{2\pi inx} = 2\pi in e^{2\pi inx}.$$

Under Fourier transform, $S^1 \to \mathbf{Z}$, so we are multiplying by a function on $\mathbf{Z}$ (i.e. via the inclusion $\mathbf{Z} \hookrightarrow \mathbf{C}$).

Case of $\mathbf{R}$: Weyl algebra = polynomial differential operators in one variable. It is generated by x, ∂_x. I.e. it is

$$\mathcal{D} = \mathbf{C}\langle x, \partial\rangle/(\partial x - x\partial = 1).$$

$\mathcal{D}$ acts on any function space closed under x, ∂: $C^\infty(\mathbf{R}), \mathcal{S}(\mathbf{R}), \mathcal{S}(\mathbf{R})^*$.

Fourier transform gives an automorphism of $\mathcal{D}$:

$$L \,\hat{\cdot}\, f = \hat{L} \cdot \hat{f},$$

where $L \in \mathcal{D}$.

So,

$$\begin{aligned} x &\mapsto i\partial \\ i\partial &\mapsto -x \end{aligned}$$

$$e^{isx} \in \mathcal{S}(\mathbf{R})^*$$
$$\mathcal{D} \cdot e^{isx} \subset \mathcal{S}(\mathbf{R})^*$$

We get a module over $\mathcal{D}$. Actually, $\mathcal{D}e^{isx} \cong \mathcal{D}/\mathcal{D}(\partial - is) \cong \mathbf{C}[x]$ (the last isomorphism is as vector spaces).

Similarly, $\mathcal{D}\delta_s \cong \mathcal{D}/\mathcal{D}(x-s) \cong \mathbf{C}[\partial]$.

If M is a module for a ring $\mathcal{D}$, then an automorphism $\mathcal{D} \overset{()^h}{\to} \mathcal{D}$ induces a new module M^h: as a vector space, this is the same. $(L \cdot m)_{new} := L^h \cdot m$ for $L \in \mathcal{D}$.

$$M = \mathcal{D}/\mathcal{D}(\partial - is) \leftrightarrow M^h = \mathcal{D}/\mathcal{D}(x-s)$$
$$\mathcal{D} \cdot e^{ixs} \leftrightarrow \mathcal{D} \cdot \delta_s$$
$$\mathcal{S}(\mathbf{R})^* \overset{h}{\to} \mathcal{S}(\mathbf{R})^*$$

Analytic story. Let H be a Hilbert space and $T \in \operatorname{End} H$. Suppose T is normal: $TT^* = T^*T$.

$A_T \subset \operatorname{End} H$ smallest self-adjoint closed subalgebra. This is an analog of $\mathbf{C}[T]$.

$$\operatorname{Spec} T = \{x \in \mathbf{C} : \lambda \operatorname{id} - T \notin A_T^*\} \subset \mathbf{C}.$$

Theorem (Spectral theorem)**.** $A_T \cong C(\operatorname{Spec} T)$.

Under this identification, T goes to the coordinate function x on $\mathbf{C}$. We smear H over $\operatorname{Spec} T \subset \mathbf{C}$.

λ-eigenspace: $H/\text{ideal} I_\lambda \subset A_T = C(\operatorname{Spec} T)$.

If A is now a commutative C^*-algebra (Banach algebra with a conjugate-linear involution). Can do the Gelfand transform:

$$A \to \hat{A} = \operatorname{Spec} A,$$

a Hausdorff topological space. These are closed maximal ideals or homomorphisms $A \to \mathbf{C}$.

Theorem (Gelfand)**.** $A \overset{\sim}{\to} C(\hat{A})$.

Gelfand-Naimark theorem: the category of commutative C^*-algebras is equivalent to the category of locally-compact Hausdorff topological spaces.

Fourier transform: $\hat{G} = \operatorname{Spec}(C_c(G), \star)$. Then

$$(C_c(G), \star) \leftrightarrow (C_c(\hat{G}), \cdot)$$

So, representation theory of G is identified with the geometry of space $\hat{G}$.

Date: 02/07. Lie algebras vs Lie groups.

To a Lie group G we associate a Lie algebra $\mathrm{Lie}(G) = \mathfrak{g} = T_1G$.

Lie is a functor Lie groups $\to$ Lie algebras, i.e. T_1G has a canonical Lie algebra structure. If $\phi : G \to H$ a map of Lie groups, there is a map $\mathrm{Lie}(\phi) : \mathfrak{g} = T_1G \to \mathfrak{h} = T_1H$ of Lie algebras. Here $\mathrm{Lie}(\phi) = d\phi|_1$.

Theorem (Lie's theorem)**.** Lie *is an equivalence of categories:*

$$\{1-\mathit{connected\ Lie\ groups}\} \xrightarrow{\sim} \{\mathit{Lie\ algebras}\}.$$

Here 1-connected means connected and simply-connected.

Basic ingredients. 1-parameter subgroups of G is a homomorphism $\psi : \mathbf{R} \to G$.

It is functorial:

$$\begin{array}{ccc} \mathbf{R} & \longrightarrow & G \\ & \searrow & \downarrow \phi \\ & & H \end{array}$$

Theorem. *There is a bijection between 1-parameter subgroups and* $\mathfrak{g}$.

For example, $G \subset GL_n\mathbf{R} \subset M_n\mathbf{R}$. Then $\mathfrak{g} = T_1G \subset T_1GL_n\mathbf{R} = M_n\mathbf{R}$, so Lie algebras of matrix groups are matrix algebras.

Then we have

$$\exp : M_n\mathbf{R} \to GL_n\mathbf{R}$$
$$A \mapsto e^A$$

We can also map

$$A \mapsto \{e^{tA}\}_{t\in\mathbf{R}},$$

which is a 1-parameter subgroup.

$M_n\mathbf{R} \xrightarrow{\sim} GL_n\mathbf{R}$. Calculate differential, it is invertible, so exp is a diffeomorphism near 0.

Note, that $\exp(tA)$ is the unique solution of the ODE:

$$f'(t) = A \cdot f(t)$$
$$f(0) = \mathrm{id}$$

for $f : \mathbf{R} \to M_n\mathbf{R}$.

One can obtain this from differentiating $f(t+s) = f(t)f(s)$ at $s = 0$.

Intrinsically,

$$T_1G \xrightarrow{\sim} \{left-invariant\ vector\ fields\ on\ G\}.$$

Left-invariant vector fields are vector fields ξ, such that for every $h \in G$ the derivative of the left translation by h preserves ξ:

$$\xi(hg) = d(h \cdot -) \cdot \xi(g).$$

If G is a Lie group, then the tangent bundle $TG \xrightarrow{\sim} G \times T_1 G$.

So, for any $x \in \mathfrak{g}$ we have a left-invariant vector field l_x, get an ODE for $\psi : \mathbf{R} \to G$

$$d\psi|_{t\in\mathbf{R}}(1) = l_x(\psi(t))$$
$$\psi(0) = 1$$

There is a unique solution to this ODE.

Lie algebra structure on $T_1 G = \mathfrak{g}$: by mapping $x \mapsto l_x$ we embedded $\mathfrak{g} \subset \mathrm{Vect}(G)$ (as vector spaces). This is in fact a Lie subalgebra by naturality: the bracket is coordinate-independent, so the left translation preserves the brackets.

$$\exp(tA)\exp(tB) = \exp\left(tA + tB + \frac{1}{2}t^2[A,B] + ...\right).$$

Baker-Campbell-Hausdorff formula gives the rest of the Taylor series in terms of $[,]$.

Alternatively, consider the group commutator $h^{-1}g^{-1}hg$. Then

$$\frac{1}{t}\frac{d}{dt}h_t^{-1}g_t^{-1}h_t g_t|_{t=0} = [A,B].$$

Examples of Lie algebras.

$$GL_n\mathbf{C} \xrightarrow{\det} GL_1\mathbf{C} = \mathbf{C}^\times$$
$$\mathfrak{gl}_n\mathbf{C} \xrightarrow{\mathrm{tr}} \mathfrak{gl}_1\mathbf{C} = \mathbf{C}.$$

So, the Lie algebra of $SL_n\mathbf{C}$ is $\mathfrak{sl}_n\mathbf{C}$, the Lie algebra of traceless matrices.

If $K = e^{tA} \in O_n$, i.e. $KK^t = \mathrm{id}$, then $A + A^t = 0$, so $\mathrm{Lie}(O_n)$ consists of skew-symmetric matrices.

Let G be a 1-connected group. If we have a finite-dimensional representation of $\mathrm{Lie}(G)$, think of it as $\mathfrak{g} \to \mathfrak{gl}_n\mathbf{C}$, then we can associate a finite-dimensional representation of G: $G \to GL_n\mathbf{C}$ by Lie's theorem.

For example, $G = SL_2\mathbf{C}$ is topologically $S^3 \times \mathbf{R}^3$, $K = SU_2$ is topologically S^3. So, both are 1-connected.

Any $g \in SU_2$ has an axis. g lives in a one-parameter subgroup:

$$\begin{array}{ccc} \mathbf{R} & \longrightarrow & SU_2 \\ \downarrow & \nearrow & \\ S^1 & & \end{array}$$

So, $\exp : \mathfrak{su}_2 \to SU_2$ (this is true for any compact G).

$SL_2\mathbf{R}$ is not simply-connected: it acts on $\mathbf{H}$ by Mobius transformations, and the stabilizer of i is S^1. So,

$$\mathbf{H} \cong SL_2\mathbf{R}/S^1.$$

Weyl's unitary trick: finite-dimensional representations of $SL_2\mathbf{C}$, $SL_2\mathbf{R}$, SU_2 are all the same.

Given any real Lie algebra $\mathfrak{g}$, construct $\mathfrak{g}_\mathbf{C} = \mathfrak{g} \otimes_\mathbf{R} \mathbf{C}$, the bracket extends by $\mathbf{C}$-linearity.

Representation of $\mathfrak{g}$ are maps $\mathfrak{g} \to \mathfrak{gl}_n\mathbf{C}$, but $\mathfrak{g} \subset \mathfrak{g}_\mathbf{C}$, so the representation uniquely extends.

$\mathfrak{su}_2 \otimes \mathbf{C} = \mathfrak{sl}_2\mathbf{C}$.

Last time: V irreducible finite-dimensional representation of $\mathfrak{sl}_2\mathbf{C} = \mathbf{C}\langle e, h, f\rangle$ is V_n for some positive integer n.

Corollary. *Any finite-dimensional representation of $SL_2\mathbf{C}$ (or SU_2, etc) is a direct sum of copies of V_n.*

Date: 02/09.

Last time we classified all irreducible finite-dimensional representations of ${}_2\mathbf{C}$. They weights $-n, -n+2, ..., n-2, n$ with respect to h. e and f move from one weight space to another.

If V, W are two reps of G, then $V \otimes W$ is also a rep of G:

$$g \cdot (v \otimes w) = gv \otimes gw.$$

Let $g = e^{tx}$, then differentiating we get

$$x \cdot (v \otimes w) = xv \otimes w + v \otimes x \cdot w.$$

V^* is also a representation: require the pairing $V^* \times V \to \mathbf{C}$ to be invariant. Then

$$(g \cdot v^*)(v) = v^*(g^{-1}v).$$

For a Lie algebra,

$$x \cdot v^*(v) = v^*(-x \cdot v).$$

So, $V^{\otimes n}$ is also a representation of G. It is never irreducible: $V^{\otimes n} \supset \operatorname{Sym}^n V$.

Note, that S_n acts naturally on $V^{\otimes n}$. One also has an action of $GL(V)$. These two actions commute, i.e. $GL(V) \times S_n$ also acts. For example, $\operatorname{Sym}^n V = (V^{\otimes n})^{S_n}$ (trivial S_n-isotypic piece). Since S_n is finite, we also have a map $V^{\otimes n} \to \operatorname{Sym}^n V$.

$$V^{\otimes n} \to \operatorname{Sym}^n V$$
$$W \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(w)$$

So, can think of $\operatorname{Sym}^n V$ as coinvariants.

For example, if $V = \operatorname{span}\{x, y\}$, then $\operatorname{Sym}^n V = \{\sum a_m x^m y^{n-m}\}$.

Since the $GL(V)$ action commutes with the S_n action, $GL(V)$ also acts on $\operatorname{Sym}^n V$. Similarly, $GL(V)$ acts on $\wedge^n V$. Here $\wedge^n V$ is the sign-isotypic component of S_n.

Any irreducible representation W of S_n defines the Schur functor $V \to (V^{\otimes n})^{S_n}_W$.

Back to $SL_2\mathbf{C}$. We have the standard 2-dimensional representation $\mathbf{C}^2 = \operatorname{span} v_1, v_2$. $hv_1 = v_1$, $hv_2 = -v_2$. This is what we called V_1.

$SL_2\mathbf{C}$ acts on $\operatorname{Sym}^n \mathbf{C}^2$. It is spanned by

$$\begin{matrix} v_2^n & v_2^{n-1}v_1 & \ldots & v_1^{n-1}v_2 & v_2^n \\ -n & -n+2 & \ldots & n-2 & n \end{matrix}$$

So, we get V_n.

Now, consider $(\mathbf{C}^2)^*$, the dual representation. Denote the basis as x and y. $\mathrm{Sym}^n(\mathbf{C}^2)^\times = \mathrm{span}\{x^n, x^{n-1}y, ..., y^n\}$, homogeneous polynomials of degree n on $\mathbf{C}^2$.

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x &= -\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= -\begin{pmatrix} a & b \end{pmatrix} \\ &= -(ax+by)\end{aligned}$$

Note, that $e \cdot y = 0$ and $e \cdot x = -y$, so $e = -y\frac{\partial}{\partial x}$. This gives $e \cdot x^m y^n = -mx^{m-1}y^{n+1}$.

Similarly, $f \cdot x = 0$ and $f \cdot y = -x$, so $f = -x\frac{\partial}{\partial y}$.

Finally, $h \cdot x = -x$, $h \cdot y = y$, so $h = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$.

These formulas can be obtained as follows: $SL_2\mathbf{C}$ acts on $\mathbf{C}^2\backslash 0$, which gives a map $\mathfrak{sl}_2\mathbf{C} \to \mathrm{Vect}(\mathbf{C}^2\backslash 0)$. For example,

$$\exp(te) = \begin{pmatrix} 1 & e^t \\ 0 & 1 \end{pmatrix},$$

so $x \mapsto x + e^{-t}y$. Taking a derivative at $t = 0$, one gets $ey = -y\frac{\partial}{\partial x}(x)$.

Similarly,

$$\exp(th) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Now, a geometric way. $SL_2\mathbf{C}$ acts on the *manifold* $\mathbf{C}^2$. Then $SL_2\mathbf{C}$ acts on $\mathrm{Fun}(\mathbf{C}^2)$ for some class of functions. For example, consider polynomial functions $\mathbf{C}[x, y]$.

Get a decomposition

$$\mathbf{C}[x, y] = \oplus_n \text{homogeneous polynomials of degree } n = \oplus_n V_n,$$

all irreducible representations occur once.

To put it differently, besides $SL_2\mathbf{C}$ one also has a $\mathbf{C}^\times$-action on $\mathbf{C}^2$. The decomposition of $\mathbf{C}[x, y]$ with respect to the $\mathbf{C}^\times$ is the same:

$$\mathbf{C}[x, y] = \oplus_n \mathbf{C}[x, y]_n.$$

Note, that representations of S^1 are the same as *holomorphic* representations of $\mathbf{C}^\times$. For example, $z \mapsto z^p\overline{z}^q$ does not come from an S^1 representation.

Since $\mathbf{C}^2\backslash 0$ is a homogeneous space for $SL_2\mathbf{C}$, we can identify $\mathbf{C}^2\backslash 0 = SL_2\mathbf{C}/N$, where N consists of upper-triangular matrices in $SL_2\mathbf{C}$.

Polynomial functions on $\mathbf{C}^2\backslash 0$ are the same as polynomial functions on $\mathbf{C}^2$ (an analog of Hartogs theorem): $f(z, w)/g(z, w)$ has poles on a codimension 1 variety, while the origin has codimension 2.

$SL_2\mathbf{C}$ has an action of $SL_2\mathbf{C}$ from both sides. If we mod out by N, we still have a left action of $SL_2\mathbf{C}$ on $SL_2\mathbf{C}/N$. From the right we still have an action of the normalizer of N is $SL_2\mathbf{C}$: this is precisely the Borel subgroup $B = \mathbf{C}^\times \ltimes \mathbf{C}$, where $\mathbf{C}^\times = H$ is the Cartan subgroup.

The map $\mathbf{C}^2 \backslash 0 \to \mathbf{CP}^1$ is just $SL_2\mathbf{C}/N \to SL_2\mathbf{C}/B$.

Let $z = x/y$ a global coordinate on $\mathbf{CP}^1$.

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{1}{y}\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} = -\frac{x}{y^2}\frac{\partial}{\partial z}.$$

So,

$$\begin{aligned} e &\mapsto -\frac{\partial}{\partial z} \\ h &\mapsto -2z\frac{\partial}{\partial z} \\ f &\mapsto z^2\frac{\partial}{\partial z}. \end{aligned}$$

Problem: there are no holomorphic functions on $\mathbf{CP}^1$. x, y are homogeneous coordinates on $\mathbf{CP}^1$, so $\mathbf{C}[x,y]$ is the homogeneous coordinate ring of $\mathbf{CP}^1$.

Given a point $l \in \mathbf{CP}^1$, i.e. a line $l \subset \mathbf{C}^2$. $x, y \in (\mathbf{C}^2)^* \to l^*$. So, $x|_l, y|_l \in l^*$. Taking a ratio, get a number (since l^* is a line).

Get a line bundle $\mathcal{O}(1)$ on $\mathbf{CP}^1$. It is an assignment $l \in \mathbf{CP}^1 \mapsto$ line l^*. It is locally trivial:

$$\mathcal{O}(1)|_{\mathbf{CP}^1\backslash\infty} \overset{/y}{\cong} (\mathbf{CP}^1\backslash\infty) \times \mathbf{C}.$$

Similarly, $\mathcal{O}(1)|_{\mathbf{CP}^1\backslash 0} \cong (\mathbf{CP}^1\backslash 0) \times \mathbf{C}$.

On the overlap $\mathbf{C}^\times$, the two differ by $z\cdot = x/y\cdot$.

More line bundles: $l \in \mathbf{CP}^1 \mapsto (l^*)^{\otimes n} = \operatorname{Sym}^n l^*$. $\operatorname{Sym}^n((\mathbf{C}^2)^*)$ surjects onto $(l^*)^{\otimes n}$ by restriction. So, $\operatorname{Sym}^n l^*$ are homogeneous polynomials of degree n on l.

Theorem (Borel-Weil)**.** *Every irreducible representation of* $SL_2\mathbf{C}$ *(i.e.* V_n*) appears as holomorphic sections of a line bundle (*$\mathcal{O}(n)$*) on* $\mathbf{CP}^1$*.*

Date: 02/14.

Recall from last time: we had the line bundles $\mathcal{O}(n)$, whose fiber at $l \in \mathbf{CP}^1$ is $(l^*)^{\otimes n}$. They are locally trivial: over $\mathbf{CP}^1\backslash\infty$ and $\mathbf{CP}^1\backslash 0$ can trivialize $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ using homogeneous coordinates y and x: they are sections of $\mathcal{O}(1)$ away from ∞ and 0 respectively.

So, it makes sense to speak of C^∞, analytic or algebraic sections of $\mathcal{O}(n)$. These are the same as functions f_0 on $\mathbf{CP}^1\backslash 0$ and f_∞ on $\mathbf{CP}^1\backslash\infty$, such that on $\mathbf{C}^\times$ differ by $(z = x/y)^n$.

$H^0(\mathbf{CP}^1, \mathcal{O}(n)) = \Gamma(\mathcal{O}(n))$, the space of holomorphic (the same as algebraic in this case) sections of $\mathcal{O}(n)$.

The functions f_∞ are spanned by $1, z, z^2, z^3, \ldots$. The condition is that $f_0 = f_\infty \cdot z^n$ is holomorphic at ∞. So, the only allowed powers are $1, z, ..., z^n$. In terms of homogeneous coordinates, these are $y^n, y^{n-1}x, ..., x^n$, i.e. exactly the homogeneous polynomials of degree n.

Theorem (Borel-Weil for $\mathfrak{sl}_2$)**.** *Every irreducible representation of $SL_2\mathbf{C}$ appears as $\Gamma(\mathcal{L})$, global holomorphic sections of a line bundle on $\mathbf{CP}^1$ (namely, $V_n \cong \Gamma(\mathcal{O}(n))$).*

In fact, every holomorphic line bundle on $\mathbf{CP}^1$ is $\mathcal{O}(n)$ for some $n \in \mathbf{Z}$. Take any meromorphic section s of $\mathcal{L}$, then n =number of zeros of s - number of poles of s.

For any line bundle on $\mathbf{CP}^1$ has $\Gamma(\mathcal{L}) \cong$ an irrep of $SL_2\mathbf{C}$ and all irreps appear once (except for 0).

Meta theorem: representation theory of $SL_2\mathbf{C}$ is equivalent to the geometry of $\mathbf{CP}^1$.

For every line bundle $\mathcal{L}$ we have the space $H^0(\mathbf{CP}^1, \mathcal{L})$. We can also consider $H^1(\mathbf{CP}^1, \mathcal{L})$. This is the same as holomorphic sections of $\mathcal{L}$ on $\mathbf{C}^\times$ up to sections that extend to 0 and sections that extend to ∞.

$H^1(\mathbf{CP}^1, \mathcal{O}(n))$ contains $..., z^{-n}, ..., z^{-1}, 1, z, z^2, ...$ Nonnegative powers $1, z, z^2...$ are holomorphic on $\mathbf{CP}^1\backslash\infty$. z^k for $k \le n$ also extends to ∞ (after multiplication by z^{-n}).

So, $H^1(\mathbf{CP}^1, \mathcal{O}(n)) = 0$ for $n \ge 0$. For negative degrees we have $H^1(\mathbf{CP}^1, \mathcal{O}(-n)) \cong V_{n-2}$, this is $n-1$ dimensional.

$$\begin{array}{c|ccccccc} H^0 & 0 & 0 & 0 & V_0 & V_1 & V_2 & V_3 \\ H^1 & V_1 & V_0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

On any Riemann surface one has the canonical line bundle ω: sections are 1-forms. The sections are dz. In terms of $w = 1/z$ one has $dw = -1/z^2 dz$. So, dz has a second-order pole at ∞, i.e. $\omega = \mathcal{O}(-2)$.

Serre duality: one has an isomorphism

$$H^1(\mathbf{CP}^1, \mathcal{L}) \cong H^0(\mathbf{CP}^1, \mathcal{L}^* \otimes \omega)^*.$$

This is a version of the residue pairing: given a section $f \in \mathcal{L}(\mathbf{C}^\times)$ and $g \in \mathcal{L}^* \otimes \omega(\mathbf{CP}^1)$ one has

$$\langle f, g\rangle \in \omega(\mathbf{C}^\times) \overset{\mathrm{Res}}{\mapsto} \mathbf{C}.$$

So, $H^1(\mathbf{P}^1, \mathcal{O}(-n)) = H^0(\mathbf{P}^1, \mathcal{O}(n-2))^* = (V_{n-2})^* = V_{n-2}$.

Induced representations. Suppose G is a finite group and H is a subgroup. Then

$$\operatorname{Ind}_H^G : \operatorname{Rep} H \to \operatorname{Rep} Gw \qquad \mapsto v = \operatorname{Ind}_H^G W$$

As a vector space,

$$\operatorname{Ind}_H^G W \cong \bigoplus_{[g]\in G/H} [g]W.$$

The action of G is the natural one on the cosets.

Definition.

$$\begin{aligned}\operatorname{Ind}_H^G W &= \operatorname{Map}_H(G, W)\\ &= \{f : G \to W | f(gh^{-1}) = h \cdot f(g)\}.\end{aligned}$$

$g \in G$ acts by left translation on G: $g \cdot f(g_1) = f(gg_1)$.
Can reinterpret the definition in terms of the group algebra:

$$\operatorname{Map}(G, W) = \operatorname{Hom}_{\mathbf{C}}(\mathbf{C}G, W).$$

Note, that $\mathbf{C}G$ is a representation of H via the right action of H on G.
So, it makes sense ro write

$$\operatorname{Hom}_H(\mathbf{C}G, W),$$

maps of H-representations. Equivalently, this is $\operatorname{Hom}(\mathbf{C}G, W)^H$. This is a representation of G since G acts on $\mathbf{C}G$ on the left.
Let V be a representation of G. Then it is a representation of H, this is the restriction functor $\operatorname{Res}_G^H : \operatorname{Rep} G \to \operatorname{Rep} H$.

Theorem (Frobenius reciprocity)**.** $\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) = \operatorname{Hom}_H(\operatorname{Res}_G^H V, W)$.

Proof.

$$\begin{aligned}\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) &= \operatorname{Hom}_G(V, \operatorname{Hom}_H(\mathbf{C}G, W))\\ &\subset \operatorname{Hom}_{\mathbf{C}}(V, \operatorname{Hom}_{\mathbf{C}}(\mathbf{C}G, W))\\ &= \operatorname{Hom}(V \otimes \mathbf{C}G, W)\\ &= \operatorname{Hom}(\mathbf{C}G, \operatorname{Hom}(V, W))\end{aligned}$$

We took the subspace of $G \times H$-invariant morphisms. Then

$$\begin{aligned}
\operatorname{Hom}_G(V, \operatorname{Hom}_H(\mathbf{C}G, W)) &= \operatorname{Hom}_{\mathbf{C}}(\mathbf{C}G, \operatorname{Hom}_{\mathbf{C}}(V, W))^{G \times h} \\
&= \operatorname{Hom}_G(\mathbf{C}G, \operatorname{Hom}_H(V, W)) \\
&= \operatorname{Hom}_H(V, W).
\end{aligned}$$

□

This is equivalent to saying that Ind is the right adjoint of the restriction functor.

Let R be a ring (not necessarily commutative). Let M be a right R-module and N is a left R-module.

Then one has the tensor product:

$$M \otimes_R N = M \otimes N / \sim,$$

where we identify $mr \otimes n \sim m \otimes rn$.

It satisfies the following universal property:

$$\begin{array}{ccc}
M \times N & \longrightarrow & P \\
\downarrow & \nearrow & \\
M \otimes_R N & &
\end{array}$$

Claim: $\operatorname{Ind}_H^G W = \mathbf{C}G \otimes_{\mathbf{C}H} W = \oplus_{g \in G} g \cdot W / \sim$, where $(gh)w \sim g(hw)$.

Note, that $\mathbf{C}G$ is both functions and measures on G.

$$\begin{aligned}
\operatorname{Hom}_G(\operatorname{Ind}_H^G W, V) &= \operatorname{Hom}_G(\mathbf{C}G \otimes \mathbf{C}HW, V) \\
&= \operatorname{Hom}_G(\mathbf{C}G, \operatorname{Hom}_H(W, V)) \\
&= \operatorname{Hom}_H(W, V).
\end{aligned}$$

So, we also get

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G W, V) = \operatorname{Hom}_H(W, \operatorname{Res}_G^H V).$$

Geometric interpretation. Construct a vector bundle $G \times_H W$ on G/H. One naturally has $G \to G/H$, a principal H-bundle.

Then $G \times_H W = G \times W / \sim$, where $(g, w) \sim (gh^{-1}, hw)$ for $h \in H$.

So, get a copy of W for every coset.

Suppose G acts on X and $\mathcal{W} \to X$ a vector bundle.

Definition. *$\mathcal{W}$ is said to be an G-equivariant vector bundle, if for any $g \in G, x \in X$ we are given $\phi_g : \mathcal{W}|_{g \cdot x} \xrightarrow{\sim} \mathcal{W}|_x$.*

Another way to say that: $\phi_g : g^*\mathcal{W} \xrightarrow{\sim} \mathcal{W}$.

We also require associativity: $\phi_{gh} = \phi_h\phi_g : \mathcal{W}|_{ghx} \to \mathcal{W}|_x$.

Sections of $\mathcal{W}$ are assignments $x \mapsto f(x) \in \mathcal{W}|_x$. Then the space of sections $\Gamma(\mathcal{W})$ is a representation of G:

$$(g \cdot f)(x) = \phi_g(f(gx)).$$

$\mathcal{W} = G \times_h W$ is G-equivariant: for $k \in G$

$$k \cdot (g \times w) = kg \times w \cong kgh^{-1} \times hw.$$

Finally, $\Gamma(G \times_H W) = \mathrm{Map}_H(G, W) = \mathrm{Ind}_H^G W$.

Date: 02/16

Let $G \supset H$ be Lie groups. Then G/H is a manifold and $G \times_H W$ is an equivariant vector bundle.

$\operatorname{Lie} G = \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ as vector spaces, where $\mathfrak{h} = \operatorname{Lie} H$.

An open neighborhood of the identity in G looks like open neighborhoods of 0 in vector spaces $\mathfrak{s}, \mathfrak{h}$ using the exponential map. That is,

$$G \supset U \ni g = s \cdot h$$

with $s \in \exp(\mathfrak{s})$ and $h \in \exp(\mathfrak{h})$.

Open neighborhood of $[1] \in G/H$ is diffeomorphic to an open neighborhood of $0 \in \mathfrak{s}$.

We are getting a splitting of $G \to G/H$ near the identity.

For finite groups we had $\operatorname{Ind}_H^G =$ sections of the G-equivariant vector bundle $G \times_H W$. Need to put some adjective for sections: $C^\infty, C_c^\infty, C, C_c, L^2$, analytic, algebraic. Get different kinds of inductions: smooth induction, compact induction, unitary induction, holomorphic induction etc.

Consider $G = SL_2\mathbf{C}$ and

$$H = B = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \cong \mathbf{C}^\times \ltimes \mathbf{C}.$$

Let's induce 1-dimensional representations of B to G.

Remark: any algebraic irrep of B is 1-dimensional. The reason being that the algebraic representations of $N = \mathbf{C}$ are trivial: $\exp(itx)$ is not algebraic unless $t = 0$. Then we know that the only irreps of $\mathbf{C}^\times$ are 1-dimensional.

$$\begin{array}{ccc} B & \to & GL_1(\mathbf{C}) = \mathbf{C}^\times \\ \downarrow & \nearrow & \\ B/[B,B] = B/N = H & & \end{array}$$

So, only need a representation of H, i.e. consider $\{\mathbf{C}_n\}_{n \in \mathbf{Z}}$.

Then $\operatorname{Ind}_B^G \mathbf{C}_n$ are sections of the line bundle $SL_2\mathbf{C} \times_B \mathbf{C}_n$. This is the same as $SL_2\mathbf{C}/N \times_H \mathbf{C}_n = (\mathbf{C}^2 \backslash 0) \times_{\mathbf{C}^\times} \mathbf{C}_n = \mathcal{O}(-n)$.

Using holomorphic induction, we get $\operatorname{Ind}_B^G \mathbf{C}_n \cong V_{-n}$ for $n \leq 0$ and 0 otherwise.

The same works for $SL_n\mathbf{C}$: any irreducible finite-dimensional representation of $SL_n\mathbf{C}$ is a holomorphic induction of a 1-dimensional representations of

$$B = \begin{pmatrix} \star & ... & \star \\ & ... & \star \\ 0 & & ... \end{pmatrix}.$$

This is the same as holomorphic sections of a line bundle on G/B, the flat variety.

Reverse engineering Borel-Weil.

V is an irrep of $SL_2\mathbf{C}$, $\mathbf{P}(V) \cong \mathbf{CP}^n$. There is a line bundle $\mathcal{O}(1)$ on $\mathbf{P}(V)$ (the dual of the tautological line bundle). Then $\Gamma(\mathcal{O}(1)) = V^*$.

Suppose we have $v \in V$ the higher weight vector, i.e. $x \cdot v = 0$ and $h \cdot v = \lambda v$. Then $N \cdot v = v$ and $H \cdot \mathbf{C}v = \mathbf{C}v$. Then $B[v] = [v]$, where $[v] = \mathbf{C}v \in \mathbf{P}(V)$. Note, that λ has to be integral since we have an action of the group H.

Consider

$$SL_2\mathbf{C} \to \mathbf{P}(V)$$
$$g \mapsto g[v]$$

It factors through B:

$$\mathbf{CP}^1 = SL_2\mathbf{C}/B \hookrightarrow \mathbf{P}(V).$$

Since $\mathbf{CP}^1$ does not have nontrivial covers, $\mathbf{CP}^1 \cong \mathbf{O}_{[v]} \subset \mathbf{P}(V)$.

Now assume V is irreducible. Then $\mathbf{O}_{[v]}$ is not contained in any hyperplane, i.e. $\mathbf{P}(W) \subset \mathbf{P}(V)$.

One finds that $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^n$ has degree n. Then

$$V^* = \Gamma(\mathbf{CP}^1, \mathcal{O}(1)) \to \Gamma(\mathbf{CP}^1, \mathcal{O}_{\mathbf{P}^n}(1)|_{\mathbf{P}^1}).$$

Since it is not contained in any hyperplane, this map is injective. It turns out that it is also surjective. Since $\mathbf{P}^1$ has degree n, $\mathcal{O}_{\mathbf{P}^n}(1)|_{\mathbf{P}^1} = \mathcal{O}(n)$.

One has a natural map $\mathbf{P}1 \hookrightarrow \mathbf{P}^n$, the Veronese map. More generally, one has

$$\mathbf{P}(W) \to \mathbf{P}(\operatorname{Sym}^n W)$$
$$[w] \mapsto [w^n]$$

For the Veronese map you map $[x, y] \to [x^n, x^{n-1}y, ..., y^n]$, precisely the basis we found previously.

Decomposing representations. Suppose we have $V_5 = \operatorname{Sym}^5 \mathbf{C}^2$ and $V_3 = \operatorname{Sym}^3 \mathbf{C}^2$. What is the tensor product $V_5 \otimes V_3$.

From the picture, we see that

$$V_5 \otimes V_3 = V_8 \oplus V_6 \oplus V_4 \oplus V_2.$$

One certainly has $V_5 \otimes V_3 \to V_8$, which reflects the fact that $\mathrm{Sym}^\bullet$ is a ring: $\mathrm{Sym}^n \otimes \mathrm{Sym}^m \to \mathrm{Sym}^{n+m}$.

Universal enveloping algebra. $G : \mathbf{C}G$ is the same as $\mathfrak{g} : U\mathfrak{g}$.

Representations of G are the same as representations for $\mathbf{C}G$.

Want the same: Lie algebra representations of $\mathfrak{g}$ are the same as representations of the associative algebra $U\mathfrak{g}$.

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{gl}(V)) = \mathrm{Hom}_{Assoc}(U\mathfrak{g}, \mathrm{End}(V)).$$

Recall, if A is any associative algebra, then can construct the Lie algebra A^{Lie} with the bracket $[a, b] = ab - ba$. Get a functor

$$\mathrm{Lie} : Assoc \to \mathrm{Lie}\,.$$

We are asking for

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A^{\mathrm{Lie}}) = \mathrm{Hom}_{Assoc}(U\mathfrak{g}, A).$$

That means that U is the left adjoint to the functor Lie.

Given any vector space V, can construct the tensor algebra $TV = \oplus_n V^{\otimes n}$, which is an associative algebra. It satisfies the following universal property:

$$\mathrm{Hom}_{\mathrm{Vect}}(V, Forget(A)) = \mathrm{Hom}_{Assoc}(TV, A).$$

Suppose we have a representation $\mathfrak{g} \xrightarrow{\phi} \mathrm{End}(V)$. This is a map of vector spaces, so get a map $T^\bullet\mathfrak{g} \to \mathrm{End}(V)$. But, $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$. Therefore, the map $T^\bullet\mathfrak{g} \to \mathrm{End}(V)$ factors through $U\mathfrak{g} \to \mathrm{End}(V)$, where

$$U\mathfrak{g} = T^\bullet\mathfrak{g}/\sim,$$

where we identify $\mathfrak{g}^{\otimes 2} \ni x \otimes y - y \otimes x \sim [x, y] \in \mathfrak{g}^{\otimes 1}$.

For example, if $\mathfrak{g}$ is abelian, $U\mathfrak{g} = \mathrm{Sym}\,\mathfrak{g}$. In general, get a deformation of $\mathrm{Sym}\,\mathfrak{g}$.

Question: how big is $U\mathfrak{g}$? Looks like $\mathrm{Sym}\,\mathfrak{g}$.

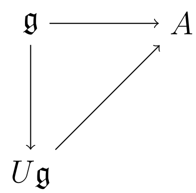
Theorem (Poincare-Birkhoff-Witt)**.** *There is an identification* $U\mathfrak{g} \xrightarrow{\sim} \mathrm{Sym}\,\mathfrak{g}$ *as vector spaces once we have an ordered basis of* $\mathfrak{g}$.

For example, in degree 2 we have the basis $e^2, eh, ef, h^2, hf, f^2$. But, for example, $fe = ef - h$.

We have a map $\mathrm{Sym}^n\,\mathfrak{g} \hookrightarrow U\mathfrak{g}$ given by taking an element of $\mathrm{Sym}^n\,\mathfrak{g}$ and writing it in the lexicographic order.

Date: 02/21

Last time: universal enveloping algebra is defined by the following universal property:



PBW theorem: pick an ordered basis $x, y, z...$ of $\mathfrak{g}$, then we get a basis of $U\mathfrak{g}$ consisting of lexicographically ordered monomials $x^l y^m z^n...$

I.e. as a vector space $U\mathfrak{g} \xrightarrow{\sim} \operatorname{Sym}\mathfrak{g}$. If $\mathfrak{g}$ is abelian, then $U\mathfrak{g} \cong \operatorname{Sym}\mathfrak{g}$ as algebras.

Deformation $U_\hbar\mathfrak{g}$ for $\hbar \in \mathbf{C}$:

$$U_\hbar\mathfrak{g} = T^*\mathfrak{g}/\sim,$$

where $x \times y - y \otimes x \sim \hbar[x, y]$. This is a family of algebras over the complex plane.

So, for $\hbar = 1$ we get $U\mathfrak{g}$; for $\hbar = 0$ we get $\operatorname{Sym}\mathfrak{g}$. The PBW theorem also works with $\hbar$. One can see that

$$U_\hbar\mathfrak{g} \cong U\mathfrak{g}$$

as algebras for any $\hbar \cong 0$. Indeed, for any $x \in \mathfrak{g}$ define $\tilde{x} = x/\hbar$. Then

$$[\tilde{x}, \tilde{y}] = [x/\hbar, y/\hbar] = \frac{1}{\hbar^2}[x, y] = \frac{1}{\hbar}\widetilde{[x, y]}.$$

$U\mathfrak{g}$ is a filtered associative algebra:

$$U\mathfrak{g} = \cup_{n \in \mathbf{Z}} U\mathfrak{g}_{\leq n},$$

where $U\mathfrak{g}_{\leq n}$ consists of sums of monomials of degree $\leq n$. The filtration comes from the natural filtration on $T\mathfrak{g}$, which descends to $U\mathfrak{g}$.

Filtered algebra means that

$$U\mathfrak{g}_{\leq n} \cdot U\mathfrak{g}_{\leq m} \to U\mathfrak{g}_{\leq n+m}.$$

If you have a filtered algebra, one can construct the associated graded algebra:

$$\operatorname{gr} U\mathfrak{g} = \oplus_n U\mathfrak{g}_{\leq n}/U\mathfrak{g}_{\leq n-1} \xrightarrow{\sim} \operatorname{Sym}\mathfrak{g}.$$

Picture of $U\mathfrak{sl}_2$:

$\operatorname{Sym}^2\mathfrak{g}$	e^2	eh	ef, h^2	hf	f^2
$\mathfrak{g}$		e	h	f	
$\mathbf{C}$			1		

What is this as a representation of $\mathfrak{g}$? $\mathfrak{g}$ acts on $U\mathfrak{g}$ by $x \circ u = xu - ux$ for $x \in \mathfrak{g}$ and $u \in U\mathfrak{g}$.

It is easy to see that $\mathrm{Sym}^2\,\mathfrak{g} \cong V_4 \oplus V_0$. So, $(\mathrm{Sym}^2\,\mathfrak{g})^{\mathfrak{g}} \cong \mathbf{C}$.

Claim: in $U\mathfrak{g}_{\leq 2}$ there is a 1-dimensional space of $\mathfrak{g}$-invariants (other than $\mathbf{C} \cdot 1$). This is the quadratic Casimir:

$$C = ef + fe + \frac{1}{2}h^2$$

$$2ef - h + \frac{1}{2}h^2$$

$(U\mathfrak{g})^{\mathfrak{g}}$ consists of expressions $z \in U\mathfrak{g}$, such that $zx = xz \quad \forall x \in \mathfrak{g}$. But then it coincides with the center $Z\mathfrak{g}$ of $U\mathfrak{g}$.

$$\begin{aligned} Ce &= efe + fee + \frac{1}{2}h^2 e \\ &= (eef - eh) + (efe - he) + \frac{1}{2}(heh - he) \\ &= eef + efe - eh + \frac{1}{2}(eh^2 + eh) \\ &= eef + efe + \frac{1}{2}eh^2 = eC. \end{aligned}$$

The same calculation works for f and h.

$(\mathrm{Sym}^2\,\mathfrak{g})^{\mathfrak{g}}$ consists of invariant bilinear forms on $\mathfrak{g}$:

$$\langle [x,u], v\rangle + \langle u, [x,v]\rangle = 0.$$

$\mathfrak{g}$ is irreducible as a $\mathfrak{g}$-rep (i.e. $\mathfrak{g}$ is simple), then there is a unique (up to a scalar) invariant nondegenerate form.

Uniqueness:

$$\langle , \rangle : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$$

as $\mathfrak{g}$-representations. $\mathfrak{g}^*$ is irreducible, so this map is unique up to a scalar by Schur's lemma.

This form is easy to write for $\mathfrak{sl}_2$: for $x, y \in \mathfrak{sl}_2$ we have $\langle x, y\rangle = \mathrm{tr}(xy)$.

For example,

$$\langle e, f\rangle = 1, \quad \langle h, h\rangle = 2$$

and the other combinations are 0.

One can write $C \in U\mathfrak{g}_{\leq 2}$ ($\mathfrak{g}$ as above) as

$$C = \sum_i e_i e^i,$$

where e_i is a basis of $\mathfrak{g}$ and e^i is the dual basis.

If V is any representation of $\mathfrak{g}$, we have

$$\begin{array}{ccc} U\mathfrak{g} & \longrightarrow & \operatorname{End} V \\ \uparrow & \nearrow & \\ Z\mathfrak{g} & & \end{array}$$

So, we get a map $Z\mathfrak{g} \to \operatorname{End}_{\mathfrak{g}} V$, but for V irreducible Schur's lemma implies that $\operatorname{End}\mathfrak{g}V = \mathbf{C} \cdot \mathrm{id}$. So, we get a homomorphism

$$\chi_V : Z\mathfrak{g} \to \mathbf{C},$$

it is called the infinitesimal character.

For any $z \in Z\mathfrak{g}$ we have

$$z \cdot v = \chi(z) \cdot v \quad \forall v \in V.$$

E.g. for $V = V_n$: $e \cdot v_n = 0, h \cdot v_n = nv_n$ for a highest-weight vector $v_n \in V_n$. Then

$$\begin{aligned} C \cdot v_n &= efv_n + fev_n + \frac{1}{2}h^2 v_n \\ &= nv_n + \frac{1}{2}n^2 v_n = n(n/2+1)v_n = \left(\frac{1}{2}(n+1)^2 - \frac{1}{2}\right) v_n. \end{aligned}$$

So, C acts on V_n as $\frac{1}{2}(n+1)^2 - \frac{1}{2}$.

For V any finite-dimensional representation of $\mathfrak{sl}_2$,

$$V = \oplus V_n^{\oplus \mu_n},$$

but now each term $V_n^{\oplus \mu_n}$ is a C-eigenspace with an eigenvalue $\frac{1}{2}(n+1)^2 - \frac{1}{2}$.

Q: is there anything else in the center $Z\mathfrak{g}$?

Harish-Chandra isomorphism: $Z_2 = \mathbf{C}[\mathbf{C}]$.

Indeed, as a $\mathfrak{g}$-representation, $U\mathfrak{g}$ is just $\operatorname{Sym}\mathfrak{g}$. Need to calculate $(\operatorname{Sym}^n \mathfrak{g})^{\mathfrak{g}}$. Claim: it is $\mathbf{C}$ for even n and 0 for odd n.

One can just count 0 and 2 weight spaces in $\operatorname{Sym}^n \mathfrak{g}$, the difference is either 1 or 0.

In general,

Theorem (Harish-Chandra isomorphism)**.** $Z\mathfrak{g} \cong \mathbf{C}[\mathfrak{h}^*]^W$.

Here $\mathfrak{h}^*$ consists of all possible h-eigenvalues. For $\mathfrak{sl}_2$ $W = \mathbf{Z}/2$. It acts by reflection in $\lambda = -1 \in \mathfrak{h}^*$. I.e. it take λ to $-\lambda - 2$. Then

$$\mathbf{C}[\mathfrak{h}^*] = \mathbf{C}[C]$$

since

$$C = \frac{1}{2}((\lambda+1)^2 - 1)$$

gives a coordinate on the λ-plane after modding out by the $\mathbf{Z}/2$-action.

We saw that shift last week: $\lambda \mapsto -\lambda - 2$ is similar to the Serre duality on $\mathbf{CP}^1$: $\mathcal{O}(n) \mapsto \mathcal{O}(-n-2)$.

Classical analog of the Harish-Chandra isomorphism (for $\hbar = 0$), i.e. what is $(\operatorname{Sym}\mathfrak{g})^{\mathfrak{g}}$. This is the same as $(\operatorname{Sym}\mathfrak{g})^G$.

Observe, that $\operatorname{Sym}\mathfrak{g} = \mathbf{C}[\mathfrak{g}^*]$. One can identify $\mathfrak{g} \cong \mathfrak{g}^*$.

For a matrix we can associate the coefficients of the characteristic polynomial as GL_n-invariants. For SL_2 we only have the determinant: this is the classical version of the Casimir.

More generally, $\mathbf{C}[\mathfrak{gl}_n]^{GL_n} \cong \mathbf{C}[\mathfrak{h}]^{W=S_n}$. Here $\mathfrak{h}$ consists of diagonal matrices.

By conjugating one can transform

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mapsto \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$$

for any nonzero ϵ.

Date: 02/23

Harmonic analysis and the Casimir. Observe, that the Casimir $C(V_n) = \frac{1}{2}((n+1)^2 - 1)$ distinguishes irreps.

So, if V is any finite-dimensional representation, we can decompose

$$V = \oplus_{\lambda \in \mathbf{C}} V_\lambda$$

into C-eigenspaces. $V_\lambda = 0$ unless $\lambda = \frac{1}{2}((n+1)^2 - 1)$ in which case $V_\lambda = V_n^{\oplus \cdots}$.

Side note: let $V(t)$ be any finite-dimensional representation of $SL_2\mathbf{C}$ depending continuously on a parameter t. So, we get that any such family $V(t)$ is constant (up to an iso) as a representation.

Recall, that $\mathfrak{g}$ consists of left-invariant vector fields on G. These come from differentiating the *right* action of G on G.

Any $\alpha \in \mathrm{Diff}(G)^{G_l}$ is actually in G_r. Indeed, the map is given by

$$\alpha \mapsto \alpha(1) \in G.$$

For any $h \in G$ we have

$$\alpha(h) = \alpha(h \cdot 1) = h\alpha(1) = hg.$$

Now, $U\mathfrak{g}$ is the algebra generated by $\mathfrak{g}$ subject to the commutation relations. This is exactly left-invariant differential operators on G.

The center $Z\mathfrak{g}$ is invariant under left translations and commutes with $\mathfrak{g}$, which is generated by right translations, i.e. also right-invariant.

Proposition. *$Z\mathfrak{g}$ consists of $\mathcal{D}(G)^{G_l \times G_r}$, bi-invariant differential operators.*

E.g. if G is abelian (for example, $\mathbf{R}^n/\Lambda$). Then left-invariant = right-invariant = constant coefficient operators.

Casimir: if G is a Lie group with an invariant nondegenerate form on $\mathfrak{g}$, we defined $C \in Z\mathfrak{g}$ by $C = \sum e_i e^i$. So, we have a canonical biinvariant of order 2 on G: the Laplacian Δ.

If we look at any homogeneous space G/H, then $C = \Delta$ descends to a G-invariant second order differential operator on G/H. C preserves $C^\infty(G/H) = C^\infty(G)^H \subset C^\infty(G)$.

Take

$$C^\infty(G/H) \supset C_{fin}(G/H) = \oplus_{\lambda \in \mathbf{C}} \{f : Cf = \lambda f\}.$$

Decomposing $C_{fin}(G/H)$ as a representation of G is the same as eigenspace decomposition.

Basic problem: understand functions on G itself. We will look at $\mathbf{C}[G]$, polynomial functions.

We will consider

$$SL_2\mathbf{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad ad - bc = 1 \right\} \subset \mathrm{Mat}_{2\times 2} = \mathrm{End}(\mathbf{C}^2) = \mathbf{C}^2 \otimes (\mathbf{C}^2)^*.$$

Have two $SL_2\mathbf{C}$ actions on both $\mathbf{C}^2$ factors, these are left and right translations. So,

$$\begin{aligned} \mathbf{C}[SL_2\mathbf{C}] &= \mathbf{C}[a,b,c,d]/(ad-bc=1) \\ &= \oplus_n \mathrm{Sym}^n(\mathbf{C}^2 \otimes (\mathbf{C}^2)^*)/(ad-bc=1). \end{aligned}$$

Can decompose it into irreps using the Peter-Weyl theorem.

Theorem (Peter-Weyl)**.** *$\mathbf{C}[SL_2\mathbf{C}] \cong \oplus_{n\in\mathbf{Z}_+} V_n \otimes V_n^*$ as a representation of $SL_2\mathbf{C} \times SL_2\mathbf{C}$.*

Works more generally for reductive groups G: complexification of K compact Lie group.

We can write

$$\mathbf{C}[G] \cong L^2(K)_{fin} = \{f : k-\text{span of } f \text{ is finite-dimensional}\}.$$

For example, $K = S^1$, then $L^2(S^1) \supset \mathbf{C}[e^{2\pi i t}, e^{-2\pi i t}]$.

Proof. Matrix elements: let V be a rep of G. Then we have a canonical map

$$\begin{aligned} V \otimes V^* &\to \mathbf{C}[G] \\ v \otimes v^* &\mapsto f_{v,v^*}(g) = \langle v^*, g \cdot v \rangle. \end{aligned}$$

This is a map of $G \times G$-representations:

$$\begin{aligned} (h_1 \times h_2) f_{v,v^*}(g) &= \langle v^*, h_1^{-1} g h_2 \cdot v \rangle \\ &= \langle h_1 \cdot v^*, g \cdot (h_2 \cdot v) \rangle \\ &= f_{h_2 v, h_1 v^*}(g). \end{aligned}$$

If V is irreducible, then $V \otimes V^*$ is irreducible as a representation of $G \times G$. Can check

$$\mathrm{End}_{G\times G}(V \otimes V^*) = \mathrm{End}_G V \otimes \mathrm{End}_G V^*.$$

Then for an irrep V we have $V \otimes V^* \hookrightarrow \mathbf{C}[G]$.

Claim: this $V \otimes V^*$ is the entire V-isotypic component of $\mathbf{C}[G]$ as left G-representations.

$$\mathbf{C}[G] = \oplus_{V \text{ irrep}} V \otimes \mathrm{Hom}(V, \mathbf{C}[G]).$$

We want to make sure $\mathrm{Hom}(V, \mathbf{C}[G]) = V^*$.

Cheap argument for $SL_2\mathbf{C}$: V_n-isotypic component is precisely the $\frac{1}{2}((n+1)^2-1)$ eigenspace for C, which is the same for left and right G-actions. These chunks are preserved by

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1}. \end{aligned}$$

So, we know that $V^* \subset \mathrm{Hom}(V, \mathbf{C}[G])$, but using this symmetry we conclude that $V^* = \mathrm{Hom}(V, \mathbf{C}[G])$.

Argument for a finite G: we have a map

$$\begin{aligned} \mathbf{C}[G] &\xrightarrow{\mathrm{tr}} \mathbf{C} \\ f &\mapsto f(1) \end{aligned}$$

It gives a nondegenerate inner product $\langle f, g\rangle = \mathrm{tr}(f * g)$.

$f_{v,v^*}(1) = \langle v^*, v\rangle$, i.e. on $V \otimes V^* \hookrightarrow \mathbf{C}[G]$ the trace is just the pairing $V \otimes V^* \to \mathbf{C}$.

Third argument: $\oplus_{V \text{ irrep}} V \otimes V^* \hookrightarrow \mathbf{C}[G] \ni f$, which actually lies in a finite-dimensional G-invariant subspace W. Let $W = \mathrm{span}\{f_1, ..., f_n\}$. Then

$$g \cdot f_i = \sum M_{ij}(g) \cdot f_j.$$

$$\begin{aligned} f_i(g) &= (g^{-1} f_i)(1) \\ &= \sum M_{ij}(g^{-1}) f_j(1), \end{aligned}$$

i.e. f_i is a linear combination of $M_{ij}(g^{-1})$, matrix elements for the dual representation W^*. One can check that

$$f_{V,v,v^*} + f_{W,w,w^*} = f_{V\oplus W, v\oplus w, v^*\oplus w^*}.$$

The same works for tensor products:

$$f_{V,v,v^*} \cdot f_{W,w,w^*} = f_{V\otimes W, v\otimes w, v^*\otimes w^*}.$$

This gives a multiplication on $\mathbf{C}[G]$, not the convolution. □

We have

$$\begin{aligned} \mathbf{C}[G/H] &= \mathbf{C}[G]^H \\ &= (\oplus_V V \otimes V^*)^H \\ &= \oplus_V V \otimes (V^*)^H \end{aligned}$$

Let's look at $\mathbf{C}[x, y] = \mathbf{C}[G/N]$. Then

$$\mathbf{C}[G/N] = \oplus_{n\in\mathbf{Z}_+} V_n \otimes (V_n)^N = \oplus_{n\in\mathbf{Z}_+} V_n,$$

since V_n^N is just the space of highest-weight vectors, which is 1-dimensional in our case.

Another example:

$$\mathbf{C}[G/H] = \oplus_{n\in\mathbf{Z}_+} V_n \otimes (V_n)^{C^\times} = \oplus n \in \mathbf{Z}_+ V_{2n},$$

since $(V_n)^{\mathbf{C}^\times}$ is zero-dimensional for even-dimensional representations and 1-dimensional for odd-dimensional representations.

Consider the compact version of the above: $SU_2/T = S^2$.

$$\mathbf{C}[S^2] = \oplus V_n \otimes (V_n)^T = \oplus_l V_{2l}.$$

Date: 02/28

Last time we talked about the Peter-Weyl theorem (for $G = SL_2\mathbf{C}$):

$$\mathbf{C}[G] = \oplus_{V \text{ irrep}} V \otimes V^* = \oplus_{n\geq 0} V_n \otimes V_n.$$

For $G = SU_2$ one has

$$L^2(G) = \oplus_{V \text{ irrep}} \widehat{V \otimes} V^*.$$

Main example: $S^2 = SU_2/T$. One can think of $T = H \cap SU_2$. Alternatively, $S^2 = SO_3/SO_2$.

Therefore,

$$L^2(S^2) = \oplus_{n\geq 0} \widehat{(V_n \otimes} V_n)^T = \widehat{\oplus_{l\geq 0} V}_{2l}.$$

Instead of dealing with L^2 functions, one can study polynomials, then

$$\mathbf{C}[S^2] = \oplus_{l\geq 0} V_{2l}.$$

Here $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbf{R}^3$ is an algebraic variety. Then $\mathbf{R}[S^2]$ are real algebraic functions on $\mathbf{R}^3$, restricted to S^2. Then $\mathbf{C}[S^2] = \mathbf{R}[S^2] \otimes \mathbf{C}$.

$C = ef + fe + \frac{1}{2}h^2 \in U\mathfrak{sl}_2$. Want to write it in the $\mathfrak{su}_2$ coordinates:

$$\mathbf{i} = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} & i \\ -i & \end{pmatrix}.$$

Then

$$h = -i\mathbf{i}, \quad e = \frac{1}{2}(\mathbf{j} - i\mathbf{k}), \quad f = -\frac{1}{2}(\mathbf{j} + i\mathbf{k}).$$

Then

$$C = -\frac{1}{2}(\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2).$$

SU_2 acts on S^2, so $\mathfrak{su}_2 \to \operatorname{Vect} S^2$ and therefore $U\mathfrak{su}_2 \to \operatorname{Diff}(S^2)$. We get a map $Z\mathfrak{su}_2 \to$ invariant $\operatorname{Diff}(S^2)$. The only such operator (up to a scalar) is

$$C = \frac{1}{2}\Delta_{S^2}.$$

We have $C^\infty(S^2) \supset H = \oplus_{\lambda\in\mathbf{C}}\{f : \Delta_{S^2} f = \lambda f\}$. One can write

$$H = \oplus_{l\geq 0} H_l = \{f : \Delta_{S^2} f = l(l+1)f\},$$

here $l(l+1) = \frac{1}{4}((n+1)^2 - 1)$ and $n = 2l$.

$$H_l = \oplus_{m=-l}^{l}(H_l^m = \{\mathbf{i}f = imf\}).$$

The vector field corresponding to $\mathbf{i} \in \mathfrak{su}_2$ is $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \frac{\partial}{\partial \phi}$.

To prove that the irreps have multiplicity one in the decomposition, we can equivalently prove that $H_l^0 \subset H_l$ is one-dimensional. That means that we are looking for rotationally-symmetric f, such that

$$(\Delta - l(l+1))f = 0.$$

Separation of variables for $\Delta_{\mathbf{R}^3}$: take $f \in C^\infty(S^2)$, which is an eigenfunction for Δ_{S^2}, it extends uniquely to $\mathbf{R}^3 \backslash 0 \cong S^2 \times \mathbf{R}_+$ as a harmonic function $\Delta_{\mathbf{R}^3} \tilde{f} = 0$.

In fact, we have $\mathbf{C}[x,y,z] \cong \mathbf{C}[r^2] \otimes H$, where H consists of harmonic polynomials $f \in \mathbf{C}[x,y,z]$. When we restrict to the 2-sphere, we get $\mathbf{C}[S^2] \cong H$.

Algebraic version: $\mathbf{C}[\mathbf{C}^3] = \mathbf{C}[\mathfrak{g}^*]$ as a representation of $SL_2\mathbf{C}$.

But one has

$$\mathbf{C}[\mathfrak{g}^*] = \operatorname{Sym} \mathfrak{g} = \oplus_k \operatorname{Sym}^k(V_2).$$

Recall the Chevalley theorem: $\mathbf{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbf{C}[\mathfrak{h}]^W$.

That means, the conjugation-invariant function of a matrix is determined by its values on the diagonal.

For $SL_2\mathbf{C}$ we have

$$\mathbf{C}[\mathfrak{sl}_2\mathbf{C}]^{SL_2\mathbf{C}} = \mathbf{C}[a^2 + bc].$$

There are infinitely many trivial subrepresentations: $\mathbf{C}[\mathfrak{g}] \supset \mathbf{C}[\mathfrak{g}]^G$.

Geometrically, we have a map $\mathfrak{g} \to \operatorname{Spec} \mathbf{C}[\mathfrak{g}]^G = \mathfrak{g}//G \cong \mathfrak{h}/W$.

Over $0 \in \mathfrak{h}/W$ we have the nilpotent cone

$$\mathcal{N} = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \}.$$

It is $\mathbf{C}^\times$-invariant, so is called a cone. Any nonzero matrix is conjugate to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, there are two $SL_2\mathbf{C}$-orbits: the trivial orbit (zero matrix) and an open orbit.

For any other point in $\mathfrak{h}/W$, we get a smooth quadric.

The real slice $\mathfrak{so}_3 \subset \mathfrak{sl}_2$ intersects the nilpotent cone only at zero. We get the standard picture with concentric spheres.

What is $\mathbf{C}[\mathcal{N}]$? One can write

$$\mathcal{N} = SL_2\mathbf{C} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sqcup \{0\},$$

where $\{0\}$ is 0=dimensional, while the first orbit $\mathbf{O}$ is 2-dimensional. Hartogs' theorem implies that

$$\mathbf{C}[\mathcal{N}] = \mathbf{C}[\mathbf{O}].$$

Orbit-stabilizer theorem gives

$$\mathbf{O} \cong PSL_2\mathbf{C}/N,$$

the point being that a central element $\mathrm{diag}(\lambda, -\lambda)$ rescales the nilpotent matrix by λ^2, so $\lambda = \pm 1$ fixes it.

The map $PSL_2\mathbf{C}/N \to SL_2\mathbf{C}/B$ is a line bundle on $\mathbf{CP}^1$. A fiber over $l \in \mathbf{CP}^1$ is $l/\pm \cong l^2$. We conclude that $\mathbf{O}$ is the total space of $\mathcal{O}(-2) = T^*\mathbf{P}^1$ (minus the zero section).

$$\mathbf{C}[SL_2\mathbf{C}/N]^{\pm\,\mathrm{id}} = \oplus_n V_n^{\pm\,\mathrm{id}} = \oplus_l V_{2l} = H.$$

Note, that $\mathcal{N}$ is singular. Representation theory gives a way to resolve it (Springer resolution): let $\tilde{\mathcal{N}}$ be the space of $x \in \mathcal{N}$ and lines $l \in \mathbf{P}^1$, preserved by x.

One has maps $\tilde{\mathcal{N}} \to \mathcal{N}$ and $\tilde{\mathcal{N}} \to \mathbf{P}^1$ by forgetting one of the pieces. What is the fiber over $x \in \mathcal{N}$? If $x = 0$, l can be anything (i.e. $\mathbf{P}^1$). If $x \neq 0$, there is a single l, fixed by x (since x gives a flag).

Generic fiber looks like

$$\mathbf{O}_{\mathrm{diag}(\lambda,-\lambda)} = SL_2\mathbf{C}/H.$$

By Peter-Weyl we have

$$\mathbf{C}[\mathbf{O}_{\mathrm{diag}(\lambda,-\lambda)}] = \oplus_n V_n \otimes (V_n)^H = \oplus_l V_{2l}.$$

Theorem (Kostant, Wallach). $\mathbf{C}[\mathfrak{g}] \cong \mathbf{C}[\mathfrak{g}]^G \otimes H$.

How to pick $H \subset \mathbf{C}[\mathfrak{g}]$? Want to consider functions constant along the base $\mathbf{C}[fg]^G$.

Let V be a vector space with a G-action. To be continued next time.

Date: 03/01

Reference for the adjoint quotient: Chriss and Ginzburg, Representation Theory and Complex Geometry.

Theorem (Kostant-Wallach)**.** $\mathbf{C}[\mathfrak{g}] \cong \mathbf{C}[\mathfrak{g}]^G \otimes H$.

Note, that $\mathbf{C}[X \times Y] \cong \mathbf{C}[X] \otimes \mathbf{C}[Y]$.

Suppose V is a vector space (e.g. $\mathbf{C}^3$). Introduce $\mathcal{D}_V$, the algebra of constant coefficient differential operators on V (left-invariant differential operators on the abelian Lie group V). It has a basis consisting of monomials of the form $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$.

One can identify $\mathcal{D}_V = \operatorname{Sym} V \cong U(V)$. We aso have $\mathbf{C}[V] = \operatorname{Sym} V^*$. There is a non-degenerate pairing

$$\begin{aligned} \mathcal{D}_V \otimes \mathbf{C}[V] &\to \mathbf{C} \\ D \otimes P &\mapsto (D \cdot P)(0). \end{aligned}$$

If D and P have the same degree (i.e. both are homogeneous), then $D \cdot P$ is a constant $(DP)(0)$. If D and P have different degrees, then $(DP)(0) = 0$.

If G acts on V, then one has $\mathcal{D}_V^G \subset \mathcal{D}_V$, G-invariant constant coefficient operators.

Definition. *A function $P \in \mathbf{C}[V]$ is G-harmonic if $D \cdot P = 0$ for all $D \in \mathcal{D}_V^G$ with 0 constant term.*

Examples:

(1) If the action of G on V is trivial, then $\mathcal{D}_V^G = \mathcal{D}_V$, and harmonic polynomials are the same as constants.

(2) For the standard action of SO_3 on $\mathbf{C}^3$, one has $\mathcal{D}_{\mathbf{C}^3}^{SO_3} = \mathbf{C}[\Delta]$.

We want to compute

$$[\operatorname{Sym}(V = \mathbf{C}^3 = V_2 = \mathfrak{g})]^{SO_3},$$

let's consider its dual first:

$$[\operatorname{Sym} \mathfrak{g}^*]^G = \mathbf{C}[\mathfrak{g}]^G = \mathbf{C}[a^2 + bc].$$

Dualizing back, we get $\mathcal{D}_{\mathbf{C}^3}^{SO_3} = \mathbf{C}[\Delta]$. Then harmonic polynomials are indeed harmonic.

So,

$$\mathbf{C}[V] \cong \mathbf{C}[V]^G \otimes H,$$

where H consists of harmonic polynomials. This is true if $\mathbf{C}[V]$ is free as a $\mathbf{C}[V]^G$-module, i.e. $V \to V/G$ is flat.

In our case $\mathbf{C}[\mathfrak{g}]^G$ is a 1-dimensional polynomial ring, so free is the same as torsion-free.

We get

$$H = \oplus V_{2l} \subset \mathbf{C}[\mathfrak{g}],$$

where the inclusion is given by harmonic polynomials. Restriction to any fiber is an isomorphism.

This is the same as separation of variables: if f harmonic polynomial on $\mathbf{R}^3$, then $f \mapsto f|_{S^2}$ is an isomorphism $H \to \mathbf{C}[S^2]$.

Equivalently, if f is an eigenfunction of Δ_{S^2} on S^2, then there is a unique extension $\tilde{f}$ to $\mathbf{R}^3$ as a harmonic polynomial.

Characters. Let V be a finite-dimensional representation of G. One has a map

$$V \otimes V^* \to \mathbf{C}[G]$$

given by matrix elements.

But one also has $\operatorname{End} V \cong V \otimes V^*$. It has an element $\mathrm{id}_V \in \operatorname{End} V$, which maps precisely to the character χ_V.

In bases e_i, e^i of V and V^* we have

$$\mathrm{id}_V = \sum e_i \otimes e^i.$$

Then

$$\chi_V(g) = \sum_i \langle e^i, g \cdot e_i \rangle = \operatorname{tr}_V g.$$

For example, $\chi_V(1) = \dim V$, which makes sense only if V is finite-dimensional.

If V is irreducible, then $\mathbf{C}\,\mathrm{id}_V \subset V \otimes V^*$ is precisely the diagonal G-invariants. This is easily seen from Schur's lemma: $(\operatorname{End} V)^G = \operatorname{End}_G V = \mathbf{C}\,\mathrm{id}$.

Even if V is not irreducible, then id_V is invariant under diagonal G, so $\chi_V \in \mathbf{C}[G]^G$, i.e. it is a class function.

$$\begin{aligned} \mathbf{C}[G]^G &\cong \bigoplus_{V \text{ fd irreps}} (V \otimes V^*)^G \\ &= \bigoplus_{V \text{ fd irreps}} \mathbf{C}\chi_V, \end{aligned}$$

so class functions are spanned by characters, which are linearly independent. In other words,

$$\chi : \{\text{irreps}\} \leftrightarrow \{\text{class functions}\}.$$

Consider χ_V restricted to $H \cong \mathbf{C}^\times \subset SL_2\mathbf{C}$. Elements of H are $\operatorname{diag}(q, q^{-1})$.

As a H-representation,

$$V \cong \bigoplus_{n\in\mathbf{Z}} [V]_n,$$

where

$$\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \cdot v = q^n v$$

for any element $v \in [V]_n$.

Then

$$\chi_V \left(\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \right) = \sum_{n\in\mathbf{Z}} \dim[V]_n q^n.$$

This is symmetric under $q \leftrightarrow q^{-1}$. So, it gives a function on H/W.

Theorem (Chevalley restriction)**.** $\mathbf{C}[G]^G \to \mathbf{C}[H]^W$.

So, a character is completely determined by a list of numbers.

Weyl character formula:

$$\chi_{V_n}(q) = q^{-n} + q^{-n+2} + ... + q^{n-2} + q^n.$$

One has

$$\frac{1}{1-q^{-2}} = 1 + q^{-2} + q^{-4} + q^{-6} + ...$$

Then we can write

$$\begin{aligned} \chi_{V_n}(q) &= q^n \left(\frac{1}{1-q^{-2}}) \right) - q^{-n-2} \left(\frac{1}{1-q^{-2}} \right) \\ &= \frac{q^n - q^{-n-2}}{1-q^{-2}} \\ &= \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}, \end{aligned}$$

this is what the usual Weyl character formula gives.

$$\chi_{V_n}(q) = \frac{\sum_{w\in W} (-1)^{\operatorname{sign} w} q^{w(n+1)}}{\sum_{w\in W} (-1)^{\operatorname{sign} w} q^{-w}}.$$

Verma modules. A collection of infinite-dimensional of $\mathfrak{g}$ (or, equivalently, a module for $U\mathfrak{g}$).

Recall the Borel

$$\mathfrak{b} = \operatorname{span}(e, h) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Consider a 1-dimensional representation of $\mathfrak{b}$ $\mathbf{C}_\lambda = \mathbf{C} v_\lambda$ for some $\lambda \in \mathbf{C}$. The relations are

$$e v_\lambda = 0, \quad h v_\lambda = \lambda v_\lambda.$$

This is a representation of B iff λ is an integer. Define the Verma module to be

$$M_\lambda = \operatorname{Ind}_{U\mathfrak{b}}^{U\mathfrak{g}} \mathbf{C}_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbf{C}_\lambda = U\mathfrak{g} \otimes v_\lambda / \sim,$$

where $xb \otimes v_\lambda \sim x \otimes (bv_\lambda)$ for any $x \in U\mathfrak{g}, b \in U\mathfrak{b}$.

So,

$$M_\lambda = U\mathfrak{g}/U\mathfrak{g}\langle e, h - \lambda\rangle.$$

As a vector space

$$M_\lambda \cong U\mathfrak{n}_- v_\lambda = \mathbf{C}[f] \cdot v_\lambda,$$

this follows from the PBW theorem.

In fact, this is true as a $U\mathfrak{n}_-$-module.

We would like to say that its character is

$$q^\lambda \frac{1}{1 - q^{-2}},$$

but this makes sense only if $\lambda \in \mathbf{Z}$.

If $\lambda = n \in \mathbf{Z}$, then M_n is a representation of $\mathfrak{g}$ and of group H. Then $\operatorname{diag}(q, q^{-1})$ acts by q^k on k-th eigenspace. And as such it has character

$$\frac{q^n}{1 - q^{-2}}.$$

Date: 03/27

Let $(\mathfrak{g}, K)$ be as follows: $\mathfrak{g}$ is a Lie algebra, K Lie group; $\mathfrak{k} = \operatorname{Lie} K \subset \mathfrak{g}$ and Ad : action of K on $\mathfrak{g}$.

Definition. *A $(\mathfrak{g}, K)$-module is a vector space V with actions of K and $\mathfrak{g}$, which agree on $\mathfrak{k}$. Moreover, $k(x \cdot v) = \operatorname{Ad} k(x) k \cdot v$ for $k \in K$ and $x \in \mathfrak{g}$.*

So, it is a representation of $\mathfrak{g}$, such that the action of $\mathfrak{k} \subset \mathfrak{g}$ integrates.

Harish-Chandra: equivalence between theories of admissible representations of G and admissible $(\mathfrak{g}, K)$-modules, where $\mathfrak{g} = \operatorname{Lie} G$ and K is the maximal compact subgroup.

Here admissible means that every irreducible representation of K appears with a finite multiplicity. This includes irreducible unitary representations.

Want to study $G = SL_2\mathbf{R}$, then $K = SO(2)$, the group of rotations

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The Lie algebra of G is ${}_2\mathbf{R}$, but any representation of ${}_2\mathbf{R}$ on a complex vector space extends to the complexification $\mathfrak{g} =_2 \mathbf{C}$.

Want to classify modules M for $\mathfrak{g} =_2 \mathbf{C} = \operatorname{span}\{\mathbf{e}, \mathbf{h}, \mathbf{f}\}$, such that h eigenvalues are all integers. Therefore, $M = \oplus_{n \in \mathbf{Z}} M_n$, where each M_n is finite-dimensional. Moreover, we can split $M = M^{even} \oplus M^{odd}$ and the $\mathfrak{g}$-action preserves the summands separately.

Let's use the Casimir: $C = 2ef + 2fe + h^2$. If M is irreducible, C acts by a scalar. We do not assume irreducibility, but let's assume C does act by a scalar.

If M, N are two representations in which C acts by a different scalar, then $\operatorname{Hom}_{\mathfrak{g}}(M, N) = 0$ (moreover, there are no extensions).

Fix M, so that C acts on M by $\lambda^2 - 1$. E.g. if we have a highest-weight vector v_n and $hv_n = nv_n$, $ev_n = 0$, i.e. $v_n \in M_n$.

Then $C = 2ef + 2fe + h^2$ acts by

$$Cv_n = 2efv_n + h^2 v_n = (n^2 + 2n)v_n.$$

In this case $\lambda = n + 1$.

Let's take $v \in M_n$, i.e. $hv = nv$. We also have

$$\begin{aligned} \lambda^2 v &= (C+1)v \\ &= (2ef + 2fe + h^2 + 1)v \\ &= (4ef + h^2 - 2h + 1)v \\ &= 4ef \cdot v + (n^2 - 2n + 1)v. \end{aligned}$$

So,

$$ef \cdot v = \frac{1}{4}\left((n-1)^2 - \lambda^2\right) v.$$

Similarly,

$$fe \cdot v = \frac{1}{4}\left((n+1)^2 - \lambda^2\right).$$

Therefore, ef is invertible unless $\lambda = \pm(n-1)$ and fe is invertible unless $\lambda = \pm(n+1)$.

- $\lambda \notin \mathbf{Z}$, then e and f are invertible on M. We only need to specify one of the vector spaces in the even chain and one in the odd chain. Hence,

  $$\mathrm{Mod}_{(\mathfrak{g},K)}^{C+1=\lambda^2} \cong \mathrm{Vect} \oplus \mathrm{Vect}\,.$$

  We have 2 irreducible representations for each $\lambda \notin \mathbf{Z}$: the one with all even vector spaces being 1-dimensional and the one with all odd vector space being 1-dimensional.

  These are called the principal series representations (or "nonunitary principal series").
- $\lambda \in \mathbf{Z}\backslash 0$, and λ and n are odd (the same for even). Again we have principal series.
- λ is odd and n is even. We have 3 chunks: $(-\infty, -\lambda - 1)$, $(-\lambda+1, \lambda-1)$ and $(\lambda+1, \infty)$. All the maps inside the chunks are isomorphisms. So, we need to specify 3 vector spaces: A, B, C and the maps

  $$C \overset{r,s}{\leftrightarrows} A \overset{p,q}{\leftrightarrows} B.$$

  We have $rs = sr = 0$ and $qp = pq = 0$.

  Recall

  Definition. *A quiver is a directed graph.*

  Definition. *A representation of a quiver is the data of a vector space for each vertex and a morphism for each edge.*

  So, we get

  Theorem. *Category of admissible even representations of* $({}_2, K)$ *with* λ *an odd integer is equivalent to the category of representations of the quiver above with relations.*

  We can ask about irreducible representations. There are three irreps: $(\mathbf{C}00)$, $(0\mathbf{C}0)$ and $(00\mathbf{C})$.

  For example, $(0\mathbf{C}0)$ corresponds to the finite-dimensional irrep. $(\mathbf{C}00)$ is a highest-weight Verma module. $(00\mathbf{C})$ is a lowest-weight dual Verma module.

But the representations are not necessarily semisimple: consider ($\mathbf{CC}0$) with the map $A \to C$ nonzero. This is again a Verma module, it has a subrepresentation ($\mathbf{C}00$) and a quotient ($0\mathbf{C}0$), but it is not a direct sum.

The representations corresponding to ($\mathbf{C}00$) and ($00\mathbf{C}$) are called discrete series representations.

- $\lambda = 0$. In this case there are only two chunks: $(-\infty, -1)$ and $(1, \infty)$. We get an odd representation parametrized by the quiver $C \leftrightarrow B$ with $pq = qp = 0$. These are called "limits of discrete series".

Principal series.

$$SL_2\mathbf{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$. It acts on $\mathbf{CP}^1$ by Möbius transformations preserving $\mathbf{RP}^1 = \mathbf{R} \cup \infty$. Moreover, it maps the upper half-plane to itself.

Let's make a change of coordinates:

$$z \mapsto \frac{z - i}{z + i}.$$

This transforms $SL_2\mathbf{R}$ into $SU(1,1)$, which consists of Möbius transformations preserving the disc.

$$SU(1,1) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 - |\beta|^2 = 1$.

We'll replace e, h, f by

$$E = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

In particular, the exponential of H coincides with the usual rotation matrix.

Let's consider the Borel subgroup

$$B_{\mathbf{R}} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

where $a, b \in \mathbf{R}$.

Let's decompose $B_{\mathbf{R}} = MAN$, where $M = \pm\,\mathrm{id}$,

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

for $a > 0$, and

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Consider $\mathbf{C}_{b,s}$, a 1-dimensional representation of $B_{\mathbf{R}}/N$. b is either trivial or the sign representation of $\mathbf{Z}/2$. $s \in \mathbf{C}$ gives a representation of $\mathbf{R}_+$:

$$a \mapsto a^s.$$

Any $g \in B_{\mathbf{R}}$ acts on $\mathbf{C}_{b,s} = \mathbf{C}v_{b,s}$ as

$$\begin{aligned} g \cdot v_{b,s} &= (man)v_{bs} \\ &= (-1)^{b(m)} a^s v_{b,s}. \end{aligned}$$

Then $V_{b,s} = \mathrm{Ind}_{B_{\mathbf{R}}}^{SL_2\mathbf{R}} \mathbf{C}_{b,s}$, i.e.

$$V_{b,s} = \{F : SL_2\mathbf{R} \to \mathbf{C} \text{ smooth} | \quad F(gman) = (-1)^{b(m)} a^s F(g)\}.$$

Date: 03/29

Principal series. $G = SL_2\mathbf{R} \supset B$ the Borel subgroup. We used the Langlands decomposition: $B = MAN$.

Irreps of $B/N = MA = \mathbf{R}^\times$ (which is abelian) are 1-dimensional and denoted by $\mathbf{C}_{b,s}$: parametrized by $s \in \mathbf{C}$ and a sign.

$$V_{b,s} = \operatorname{Ind}_B^{SL_2\mathbf{R}} \mathbf{C}_{b,s},$$

the smooth induction.

$$SL_2\mathbf{R}/N = \mathbf{R}^2\backslash 0.$$

$$SL_2\mathbf{R}/AN = S^1.$$

Finally,

$$SL_2\mathbf{R}/B = \mathbf{RP}^1.$$

Iwasawa decomposition: $G = KAN$, where $K = SO_2$ is the maximal compact. This is exactly the Gram-Schmidt orthogonalization, since this is just $G = SO_2 B^+_{\mathbf{R}}$.

$$\begin{array}{ccccc} G/AN & \xrightarrow{\sim} & K & \xrightarrow{\sim} & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ G/B & \xrightarrow{\sim} & K/M & \xrightarrow{\sim} & \mathbf{RP}^1 \end{array}$$

Fix $s \in \mathbf{C}$. Then C^∞-functions on $\mathbf{R}^2\backslash 0$ which are $s-$homogeneous, i.e. $f(ax) = a^s f(x)$, are isomorphic to $C^\infty(S^1)$, which is the same as $C^\infty(K)$.

So, $V_{b,s}$ consists of functions $\tilde{F} \in C^\infty(K)$, such that $\tilde{F}(km) = (-1)^{b(m)}\tilde{F}(k)$.

Spherical principal series consist of the representations with $b(m) = 1$.

Recall, an s-form on $\mathbf{R}$ is an expression $f(x)dx^s$. Functions act as

$$\begin{aligned} g \cdot (f(x)dx^s) &= f(g^{-1}x)(d(g^{-1}x))^s \\ &= f(g^{-1}x)((g^{-1})')^s dx^s. \end{aligned}$$

s-density is an expression of the form $f(x)|dx|^s$. Similarly,

$$g \cdot (f(x)|dx|^s) = f(g^{-1}x)|(g^{-1})'|^s|dx|^s.$$

We have a line bundle $\mathbf{R}^\times \to \mathbf{R}^2\backslash 0 \to \mathbf{RP}^1$. This is similar to the tautological line bundle $\mathcal{O}(-1)$ on $\mathbf{CP}^1$: $\mathbf{C}^\times \to \mathbf{C}^2\backslash 0 \to \mathbf{CP}^1$.

The cotangent bundle is $\mathcal{O}(-2)$. Therefore, sections of $\mathcal{O}(-1)$ are $\frac{1}{2}$-forms.

Linear functions on $\mathbf{R}^2\backslash 0$ are the same as sections of $\mathcal{O}(1)$, i.e. $-\frac{1}{2}$-forms.

$V_{b,s}$ is the space of C^∞-sections of $\mathcal{O}(1)^\int$, where $\int : \mathbf{R}^\times \to \mathbf{C}^\times$.

Proposition. *$V_{b,s}$ consists of $-s/2$-densities on $\mathbf{RP}^1$ for b the trivial representation and $-s/2$-forms for b the sign representation.*

Proof. Let $\mathbf{RP}^1 = G_\mathbf{R}/B_\mathbf{R}$ and consider

$$N_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \subset G.$$

On the Lie algebra level, we have $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}$, so there is an open dense subset of G/B, which is isomorphic to N_-.

Similarly, we have $N_-B \subset G$ an open dense subset.

Take $F \in V_{b,s}$ and restrict it to $N_- \cong \mathbf{R}$, get $f \in C^\infty(\mathbf{R})$. Concretely,

$$f(x) = F\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right).$$

Take

$$G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It acts as

$$g^{-1} \cdot x = g^{-1} \begin{pmatrix} 1 \\ x \end{pmatrix} = \frac{ax - c}{-bx + d}.$$

The derivative is

$$(g^{-1})' = \frac{1}{(-bx + d)^2}.$$

It acts on a function $f \in C^\infty(\mathbf{R})$ as

$$
\begin{aligned}
(g \cdot f)(x) &= F\left(g^{-1}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) \\
&= F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) \\
&= F\begin{pmatrix} d-bx & -b \\ -c+ax & a \end{pmatrix} \\
&= F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax-c}{-bx+d} & 1 \end{pmatrix} \cdot \begin{pmatrix} d-bx & -b \\ 0 & \frac{1}{d-bx} \end{pmatrix}\right) \\
&= f\left(\frac{ax-c}{-bx+d}\right) \cdot (-1)^{b(d-bc)}|d-bx|^s \\
&= f(g^{-1}x)(-1)^{b(d-bx)}|(g^{-1})'|^{-s/2}.
\end{aligned}
$$

□

What is the action of e, f, h? $f \in \operatorname{Lie} N_-$ act as $-\frac{d}{dx}$.

$$(\exp(tf) \cdot f)(x) = f(x-t).$$

$$
\begin{aligned}
(\exp(te)f)(x) &= f\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix}\right) \\
&= f\left(\frac{x}{-tx+1}\right)(-1)^{b(-tx+1)}|-tx+1|^s.
\end{aligned}
$$

One computes

$$
\begin{aligned}
e &= x^2\frac{d}{dx} - sx \\
h &= 2x\frac{d}{dx} - s
\end{aligned}
$$

The Casimir is $C = 2ef + 2fe + h^2 = s^2 + 2s$. So,

$$C + 1 = \lambda^2, \quad \lambda = s + 1.$$

The basis for the Fourier modes is

$$f_n = e^{2\pi in\theta}(d\theta)^{s/2}.$$

On the real line $x = -\tan\theta$, so

$$f_n = e^{-2\pi in \arctan x}(1+x^2)^{s/2}|dx|^{-s/2}.$$

The $SU(1,1)$ generators H, X, Y act naturally on f_n. For example, $Hf_n = nf_n$.

Lemma. *$Xf_n = \frac{-n-2}{2}f_{n+2}$. Similarly, $Yf_n = \frac{n+s}{2}f_{n-2}$.*

When $s \notin \mathbf{Z}$, we get the spherical representation and the odd one.

For $s = -1$ (i.e. $-s/2 = 1/2$, we are looking at half-forms), the even representation is irreducible, while the odd representation breaks as a direct sum of two.

For $n = s > -1$ we have the representation which is represented by a quiver $\mathbf{C} \to \mathbf{C} \leftarrow \mathbf{C}$. It has a finite-dimensional subrepresentation: the full representation consists of C^∞ sections of $\mathcal{O}(n)$; the subset corresponds for polynomial sections of $\mathcal{O}(n)$, i.e. homogeneous polynomials of degree n, i.e. V_n.

For $s < -1$ we have the quiver $\mathbf{C} \leftarrow \mathbf{C} \to \mathbf{C}$ and so the finite-dimensional irreducible representation is a quotient.

Theorem (Casselman's subrepresentation theorem)**.** *Every irreducible Harish-Chandra module appears as a subrepreentation of a principal series.*

Date: 04/03

Principal series representations. When is a principal series representation unitary?

Suppose $H \subset G$ and $(V, \langle , \rangle)$ is a unitary representation of H, i.e. $H \to U(V) \subset GL(V)$. Can we get a unitary structure on Ind_H^G?

E.g. $V = \mathbf{C}$, $\mathrm{Ind}_H^G \mathbf{C} = C^\infty(G/H)$.

$$\langle f, \rangle = \int f \cdot g |dvol|$$

.

We have

$$\int : Dens(X) \to \mathbf{C}.$$

Here densities $Dens(X)$ are sections of $| \wedge^{top} T^*X|$.

Define $L^2(X)$ the space of 1/2-densities on X, it has a natural pairing since

$$\int f(x)|dx|^{1/2} \cdot g(x)|dx|^{1/2} = \int f(x)g(x)|dx|.$$

For $X = G/H$ we take $T^*_{1H}G/H = (\mathfrak{g}/\mathfrak{h})^*$. Note, that T^*G/H is a G-equivariant vector bundle and so is determined by a representation of $H = Stab(1H)$ on the fiber.

Densities on G/H are sections of an equivariant vector bundle on G/H attached to a representation δ of H:

$$\begin{aligned} h \mapsto & |\operatorname{Det}_{(\mathfrak{g}/\mathfrak{h})^*}(h)| \\ &= \frac{|\operatorname{Det}_{\mathfrak{h}} h|}{|\operatorname{Det}_{\mathfrak{g}} h|} \\ &= \frac{\delta_G(h)}{\delta_H(h)} \in \mathbf{R}_+ \end{aligned}$$

In general, take the adjoint representation of G on $\mathfrak{g}$, then G has a one-dimensional representation on $\wedge^{top}\mathfrak{g}^*$, get a canonical character $\delta_G : G \to \mathbf{R}_+$ called the modular character.

Proposition. *G has a biinvariant volume form (density) iff δ_G is trivial.*

Proposition. *Semisimple Lie groups (e.g. SL_2) are unimodular, i.e. $\delta_G = 1$.*

The reason being that $\mathfrak{g}$ has a nondegenerate invariant bilinear form, and so

$$\begin{array}{ccc} G & \longrightarrow & GL(\mathfrak{g}) \\ \downarrow & \nearrow & \\ O(\mathfrak{g}, \langle , \rangle) & & \end{array}$$

Note, that $\langle , \rangle$ can be indefinite (for noncompact groups).

$$\delta_{G/B} = \frac{\delta_G}{\delta_B} = \frac{1}{\delta_B}.$$

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

action on (e, h) is $(a^2 e, h)$. The modular character is

$$B \to \mathbf{R}_+$$

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^{-2}.$$

In other words, $L^2(\mathbf{RP}^1)$, the space of half-densities on $\mathbf{RP}^1$, is $\mathrm{Ind}_B^G(\delta_{G/B}^{1/2})$, which is the same as $V_{b,s}$ for b the trivial representation and $s = -1$.

Normalized/unitary induction:

$$U\,\mathrm{Ind}_H^G(V, \langle , \rangle) = \mathrm{Ind}_H^G(V \otimes \delta_{G/H}^{1/2}),$$

which is the space of smooth section of $\mathcal{V} \otimes Dens^{1/2}$, i.e. half-densities valued in $\mathcal{V}$. Concretely, it consists of

$$\{F : G \to V | \quad F(gh) = F(g)\delta^{1/2}(h)\pi_V(h)\}.$$

Now let's add a condition

$$\int_{G/H} \langle F, F \rangle < \infty.$$

Inner product of F, G, two half-densities in $\mathcal{V}$:

$$(F, G) = \int_{G/H} \langle F, \overline{G} \rangle,$$

where $\langle , \rangle$ is the Hermitian inner product on $\mathcal{V}$.

Recall the principal series representations were defined by $V_s + \mathrm{Ind}_B^G \mathbf{C}_s$.

When is $\mathbf{C}_s$ a unitary representation of B? It is just

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^s (-1)^{\mathrm{sign}} \in U(1).$$

This is true if s is purely imaginary, i.e. $\Re s = 0$.

Corollary. *V_s is unitary if $\Re s = -1$.*

The line $\Re s = -1$ consists of the so-called unitary principal series.

For example, the point $s = -1$ corresponds to the action of $SL_2\mathbf{R}$ on $L^2(\mathbf{RP}^1)$.

In general, we have

$$\overline{V}_s^* \xrightarrow{\sim} V_{-2-s},$$

i.e. we have a natural pairing $V_{-2-s} \otimes V_2 \to \mathbf{C}$.

Indeed, we have

$$\mathcal{V}_s \otimes \mathcal{V}_{-2-s} \xrightarrow{\sim} \mathcal{V}_{-2} \xrightarrow{\sim} Dens \xrightarrow{\int} \mathbf{C}.$$

For $s \notin \mathbf{Z}$, the Harish-Chandra modules V_s and V_{-2-s} are isomorphic. Recall, that we had a formula for the Casimir:

$$C + 1 = (s+1)^2,$$

which is the same for V_s and V_{-2-s}.

The Casimir $C \in Z$ gives a function on the parameter space for principal series since it acts by a scalar on each V_s (away from integral points).

The function $C = (s+1)^2 - 1$ is shifted $\mathbf{Z}/2 = W$-invariant.

Question: can we construct an isomorphism $V_s \xrightarrow{\sim} V_{-2-s}$?

$\mathrm{Ind}_B^G \mathbf{C}_s$, where s was a character of

$$H_{\mathbf{R}} = \begin{pmatrix} * & \\ & * \end{pmatrix} \cong \mathbf{R}^\times \cong \mathbf{R}_+ \times \mathbf{Z}_2.$$

So, we get a functor from representations of H to representation of $SL_2\mathbf{R}$ called the parabolic induction. This is not the usual induction, instead we first go through representations of B (where $H \cong B/N$) and then use the usual induction.

$$W = N_G(H)/H = \{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\} \quad \text{mod diagonal matrices.}$$

Let's call the nontrivial element by $w \in W$.

W acts on H:

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \overset{w}{\mapsto} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix},$$

so it acts on characters of H:

$$\mathbf{C}_s \to \mathbf{C}_{-s}.$$

So, W acts on representation of H, but W also acts on B: the nontrivial element w maps B to the opposite Borel subgroup B_-.

So, we get an isomorphism

$$\operatorname{Ind}_B^G \mathbf{C}_s \xrightarrow{\sim} \operatorname{Ind}_{B_-}^G \mathbf{C}_{-s}$$
$$f \mapsto \tilde{f}(g) = f(gw).$$

Define

$$I(f) = \int_N \tilde{f}(g \cdot n)dn = \int_N f(gnw)dn.$$

$$\begin{aligned}(I(g \cdot f))(h) &= \int_N f(g^{-1}hnw)dn \\ &= \int_N f(g^{-1}(gnw))dn \\ &= (g \cdot I(f))(h)\end{aligned}$$

So, $I : \operatorname{Fun}(G) \to \operatorname{Fun}(G)$ commutes with the group action, i.e. is a map of G-representations. It is also called an intertwiner.

Date: 04/05

Last time we considered $V_s = \operatorname{Ind}_B^G \mathbf{C}_s$ and we had a symmetry $s \to -s-2$.

$$\begin{aligned}
I(f)\left(g\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\right) &= \int_N f\left(g\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} w\right) dn \\
&= \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & b+n \\ & 1\end{pmatrix} w\right) dn \\
&= \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} w\right) dn \\
&= \int_N f\left(g\begin{pmatrix} 1 & a^2 n \\ & 1\end{pmatrix} w \begin{pmatrix} a^{-1} & 1 \\ & a\end{pmatrix}\right) dn \\
&= |a|^{-s}\int_N f\left(g\begin{pmatrix} 1 & a^2 n \\ & 1\end{pmatrix} w\right) dn \\
&= |a|^{-s}\int_N f\left(g\begin{pmatrix} 1 & n' \\ & 1\end{pmatrix} w\right) \frac{dn'}{|a|^2}
\end{aligned}$$

So, I of V_s is precisely V_{-s-2}.

Proposition. *I converges for $\Re s \gg 1$ and has an analytic continuation to $\mathbf{C}$ with poles at negative integers.*

For s not integer both V_s and V_{-s-2} are irreducible, so I is an isomorphism.

For $s \notin \mathbf{Z}$ we had a non-degenerate Hermitian pairing between V_s and V_{-s-2}, while the latter is now identified with V_s, so we have a non-degenerate Hermitian inner product on V_s. However, it is not positive-definite. Otherwise, V_s would be unitary.

Theorem. *The Hermitian pairing is unitary iff $-2 < s < 0$.*

These are called the complementary series.

Let $\mathbf{V}$ be an irreducible representation and V the corresponding $(\mathfrak{g}, K)$-module.

Theorem (Unitarizability)**.** *$\mathbf{V}$ is a unitary representation iff V has $\langle , \rangle$ is a positive-definite invariant inner product.*

$$\begin{aligned}
\langle hv, w\rangle &= -\langle v, hw\rangle \\
\langle ev, w\rangle &= -\langle v, ew\rangle \\
\langle fv, w\rangle &= -\langle v, fw\rangle.
\end{aligned}$$

Let's use a different basis using an isomorphism $SL_2\mathbf{R} \cong SU(1,1)$. Then we have

$$H = H^*, \quad X^* = -Y.$$

Let $s+1 = \lambda \notin \mathbf{Z}$. Then the corresponding representation has a basis

$$v_{2n} = X^n v_0, \quad n \in \mathbf{Z},$$

where v_0 is a vector in the weight 0 subspace.

Define

$$Yv_n = c_{n-2}v_{n-2}, \quad ||v_n||^2 = a_n^2.$$

We know

$$XYv_n - YXv_n = nv_n = (c_{n-2} - c_n)v_n.$$

We also have

$$a_{n+2}^2 = \langle Xv_n, v_{n+2}\rangle = -\langle v_n, Yv_{n+2}\rangle = -\bar{c}_n a_n^2,$$

so c_n is real and negative.

$$(C+1)v_n = \lambda^2 v_n = (s+1)^2 v_n.$$

So,

$$\lambda^2 v_n = v_n(1 + n^2 + 2c_{n-2} + 2c_n).$$

Therefore,

$$\lambda^2 - n^2 - 1 = 4c_n + 2n.$$

We get

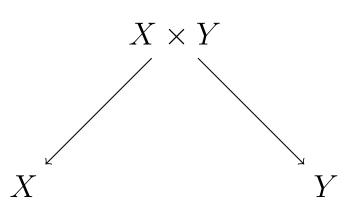
$$4c_n = \lambda^2 - (n+1)^2 < 0$$

for every $n \in 2\mathbf{Z}$.

Corollary. *V_s is unitarizable only if $\lambda \in i\mathbf{R}$ (i.e. $s \in i\mathbf{R} - 1$) or $-1 < \lambda < 1$ ($-2 < s < 0$).*

Function theories. X and Y manifolds (e.g. finite sets).

What are the maps $\mathrm{Fun}(X) \to \mathrm{Fun}(Y)$? If both are finite sets, the maps are given by matrices.



Take $K(x,y) \in \mathrm{Fun}(X \times Y)$.

A function $f \in \mathrm{Fun}(x)$ is mapped to $K * f \in \mathrm{Fun}(Y)$ by

$$K * f(y) = \int_X K(x,y) f(x) dx.$$

This gives an isomorphism

$$\mathrm{Fun}(X \times Y) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Fun}(X), \mathrm{Fun}(Y)).$$

Recall the Riesz representation theorem: maps $L^2(X) \to L^2(\mathrm{pt})$ are representable by $f \in L^2(X \times \mathrm{pt})$.

Similarly, there is the Schwartz representation theorem

$$\mathrm{Hom}(C^\infty(X), C^\infty(Y)) = \mathit{Dist}(X \times Y).$$

Here we need distributions, since for example the identity function is the δ-function supported on the diagonal.

Now consider the flag manifold $G/B = \mathbf{RP}^1$.

$$\begin{array}{ccc} & G/B \times G/B & \\ \swarrow & & \searrow \\ G/B & & G/B \end{array}$$

$C_s^\infty(G/B) \to C_s^\infty(G/B)$, a G-map.

Lemma. *Intertwiners (i.e. G-invariant maps) are given by G-invariant integral kernels (G acts diagonally on $G/B \times G/B$).*

$K(x,y) = K(gx, gy)$.

G-orbits in $G/B \times G/K$ is

$$G\backslash G/B \times G/K = B\backslash G/K.$$

Recall the Bruhat decomposition $B\backslash G/B$.

$$\begin{array}{ccc} & \mathbf{RP}^1 \times \mathbf{RP}^1 & \\ \swarrow{\scriptstyle \pi_1} & & {\scriptstyle \pi_2}\searrow \\ \mathbf{RP}^1 & & \mathbf{RP}^1 \end{array}$$

There are only two G-orbits in the product: the diagonal Δ and its complement $\mathcal{H} = \mathbf{RP}^1 \times \mathbf{RP}^1 \backslash \Delta$. Let $\pi_{1,2} : \mathcal{H} \to \mathbf{RP}^1$

We have the Radon transform:

$$I(f) = \int_{\pi_2} \pi_1^* f.$$

That means
$$I(f)(l') = \int_{l \neq l'} f(l) dl.$$
This is precisely the intertwiner we wrote down before.
Elements $g \in SL_2\mathbf{R}$ fall into 3 types:

- Hyperbolic: $|\operatorname{tr} g| > 2$. Equivalently, it is conjugate to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ for $a \in \mathbf{R}$. It has 2 fixed pints on S^1.
- Parabolic: $|\operatorname{tr} g| = 2$, conjugate to $\pm \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$. Have a single fixed point on S^1. These are translations on the upper half-plane.
- Elliptic: $|\operatorname{tr} g| < 2$, conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. These are rotations of the disk, so there is a single fixed point.

Date: 04/10

Last time we had parabolic elements which were unipotent and hyperbolic and elliptic elements which were semisimple, i.e. diagonalizable over $\overline{\mathbf{R}} = \mathbf{C}$ (they were also regular, i.e. distinct eigenvalues). Hyperbolic elements are split: diagonalizable already over $\mathbf{R}$.

Centralizer of hyperbolic elements is $Z_G(g)^0 \cong \mathbf{R}_+$. Centralizer of elliptic elements is $Z_G(g) \cong S^1$. These are both real forms of $\mathbf{C}^\times$. So, these are two tori of $SL_2\mathbf{R}$.

In $SL_2\mathbf{R}$ there are 2 conjugacy classes of maximal tori: split and compact.

Recall, that for a finite group the number of conjugacy classes coincides with the number of irreps. For finite abelian group: the set of conjugacy classes is in bijection with irreps of $\hat{G}$.

Harish-Chandra: enumerate representations of G using tori.

Representations of G come in series, one for each conjugacy class of tori in G. Series is the same as irreps of T modulo W_T. Here $W_T = N(T)/T$.

Irreps of a torus are simple: for S^1 the dual group is $\mathbf{Z}$ (Fourier series), for $\mathbf{R}^+$ it is $\mathbf{R}$ (Fourier transform).

Similarly, it works for finite groups of Lie type, e.g. $SL_2\mathbf{F}_p$.

Principal series were parametrized by characters of the split torus $\mathbf{Z}/2 \times \mathbf{R}_+$. We had a symmetry $s \to -s-2$ which reflects the action of the Weyl group.

Discrete series correspond to characters of a compact torus.

For $SL_2\mathbf{R}$ we have a coincidence: compact torus T is the maximal compact subgroup $SO_2\mathbf{R} \subset SL_2\mathbf{R}$.

Use induction: consider sections of a line bundle on $SL_2\mathbf{R}/T = SL_2\mathbf{R}/K$, it is a symmetric space.

Consider the action of $SL_2\mathbf{R}$ on $\mathbf{CP}^1$. There are 3 orbits: upper half-plane $\mathbf{H}$, lower half-plane $\mathbf{H}^-$ and the equator $\mathbf{RP}^1$. We saw that $\mathbf{RP}^1 = SL_2\mathbf{R}/B_{\mathbf{R}}$. Similarly, $\mathbf{H} = SL_2\mathbf{R}/K$.

Let $\mathbf{C}_n$ be the following representation of T: $\theta \mapsto e^{2\pi i n\theta}$. The induction is

$$\operatorname{Ind}_T^{SL_2\mathbf{R}} \mathbf{C}_n = \{F \in C^\infty(SL_2\mathbf{R}) : \quad F(gt) = \chi_n(t)F(g)\}.$$

We have a natural circle bundle $PSL_2\mathbf{R} \to \mathbf{H}$. It turns out the be the unit tangent bundle to $\mathbf{H}$. Then $SL_2\mathbf{R} \to \mathbf{H}$ is a square root of that.

Then the character χ_n gives rise to $n/2$ forms on $\mathbf{H}$, i.e. $(T\mathbf{H})^{-n/2}$. This is a holomorphic line bundle.

We can define the discrete series as

$$D_n = \mathrm{HolInd}_T^{SL_2\mathbf{R}}\,\mathbf{C}_n,$$

where HolInd is the holomorphic induction:

$$\mathrm{HolInd}_T^{SL_2\mathbf{R}}\,\mathbf{C}_n = \{f(z)dz^{n/2}: \quad f \text{ holomorphic on } \mathbf{H}\}.$$

An element $g \in SL_2\mathbf{R}$ acts as

$$\begin{aligned} f(z)dz^{n/2} &\mapsto f(g^{-1}z)(d(g^{-1}(z)))^{n/2} \\ &= f\left(\frac{az-c}{-bz+d}\right)((-bz+d)^2)^{n/2}dz^{n/2} \\ &= f(g^{-1}z)\cdot(-bz+d)^n dz^{n/2}. \end{aligned}$$

Similarly, could also consider

$$\overline{D}_n = \{\text{antiholomorphic } f(z)dz^{n/2} \text{ on } \mathbf{H}\} = \{\text{holomorphic } f(z)dz^{n/2} \text{ on } \mathbf{H}^-\}.$$

Let's look at the corresponding Harish-Chandra modules. What are the K-types of D_n?

Let's work on the disk instead of the upper half-plane.

$$f(z) = \sum_{m\geq 0} a_m z^m.$$

An element

$$g = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \in K$$

acts on a monomial $z^m dz^{n/2}$ in the following manner:

$$\begin{aligned} z^m dz^{n/2} &\mapsto (g^{-1}\cdot z)^m((g^{-1})')^{n/2}dz^{n/2} \\ &= e^{-2mi\theta}z^m(e^{-2i\theta})^{n/2}dz^{n/2} \end{aligned}$$

$$= e^{-(2m+n)i\theta}z^m dz^{n/2}.$$

We have a series of weights $-n, -n-2, -n-4, \ldots$, the corresponding basis is $dz^{n/2}, zdz^{n/2}, z^2dz^{n/2}, \ldots$

There is a natural unitary structure on these representations. $\mathbf{H}$ has a hyperbolic (Poincaré) metric: the corresponding volume form is

$$\frac{dx\wedge dy}{y^2} = \frac{dzd\overline{z}}{1-|z|^2}.$$

The measure is invariant under $SL_2\mathbf{R}$.

$$\begin{aligned}\langle f,g\rangle &= \int_D f\overline{g}(1-|z|^2)^{n/2-1}|dz||d\overline{z}| \\ &= \int_D \frac{f(z)dz^{n/2}\overline{g}(z)d\overline{z}^{n/2}}{\frac{dz^{n/2}d\overline{z}^{n/2}}{(1-|z|^2)^{n/2}}}\frac{|dzd\overline{z}|}{1-|z|^2}.\end{aligned}$$

$$D_n = \{f(z)dz^{n/2} : \quad ||f||^2 < \infty\}.$$

Recall, that a tempered representation is a unitary representation that appears (weakly) in $L^2(G)$. Equivalently (Langlands classification), it comes from induction of unitary characters of a torus.

Theorem (Harish-Chandra)**.** *An irreducible representation V appears in $L^2(G)$ iff V is a discrete series representation.*

Recall the Peter-Weyl theorem for compact groups:

$$L^2(G) = \oplus_{\text{irreps}} V \otimes V^*.$$

We would like to find a similar decomposition for $\mathbf{H} = G/K$:

$$L^2(G/K) = \oplus_{\text{irreps}} V \otimes (V^*)^K.$$

Our case $K = SO_2$. K-fixed vectors: spherical vectors (weight 0).

Recall, that an irreducible Harish-Chandra module M with $M^K \neq 0$ has $M^K \cong \mathbf{C}$.

Date: 04/12

Harmonic analysis on H. Here $\mathbf{H} = SL_2\mathbf{R}/SO_2$.

Recall, we had $S^2 = SO-3/SO_2$ and we proved

$$L^2(S^2) = \widehat{\oplus_{\text{even irreps}}} V.$$

Spherical representation V of $SL_2\mathbf{C}/SU_2$ is the one such that $V^{SO_2} \neq 0$, i.e. 0 weight space is nontrivial.

Now, let $SL_2\mathbf{R}$ act on

$$L^2(\mathbf{H}) = \int_{s\in i\mathbf{R}-1} V_s d\mu.$$

Let $\Delta = -\frac{1}{4}C = -\frac{1}{4}(H^2 + 2XY + 2YX)$.

So, on V_s we have $(C+1) = (s+1)^2$, so

$$\Delta = -\frac{1}{4}(s^2+2s) = -\frac{s}{2}\left(1+\frac{s}{2}\right).$$

For $s \in i\mathbf{R}-1$ we have $\Delta \geq \frac{1}{4}$.

For $s \in [-2,0]$ we have $\Delta \in [0, \frac{1}{4}]$. Finally, for $s \in \mathbf{Z}\backslash\{-1\}$ Δ is negative.

A matrix $g \in SL_2\mathbf{R}$ can be written as ($G = KAN$ decomposition)

$$g = \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then it acts on the upper half-plane as $g \cdot i = x + iy$.

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta}.$$

On the upper half-plane we have

$$\Delta_{\mathbf{H}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

the Laplace-Beltrami operator on $\mathbf{H}$. It is a positive self-adjoint operator on $L^2(\mathbf{H})$.

Note, that $L^2(\mathbf{H})_{\Delta = -\frac{s}{2}(1+\frac{s}{2})} = 0$ unless s is between $[-2,0]$ or in $i\mathbf{R}-1$.

$$L^2(\mathbf{H})_{\Delta=\lambda} = \{f : \quad f = \lambda f\}$$

is $SL_2\mathbf{R}$ invariant. For $\lambda \geq \frac{1}{4}$ we have an irreducible representation V_s. For $0 \leq \lambda \leq \frac{1}{4}$ we don't get anything, so there are no complementary series. The spectrum is continuous $[\frac{1}{4}, \infty)$.

The reason complementary series do not appear is that they are not tempered, i.e. they don't appear in $L^2(SL_2\mathbf{R})$.

Poisson transform. Take a function $f \in C(S^1)$, one can extend it uniquely to a harmonic function on the disc:

$$\begin{aligned} Pf &\in \mathrm{Fun}(\overline{D}) \\ Pf|_{S^1} &= f \\ \Delta Pf &= 0 \end{aligned}$$

Explicitly,

$$Pf(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt,$$

where

$$P_r(\theta) = \Re\left(\frac{1+z}{1-z}\right) = \sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}$$

is the Poisson kernel.

In terms of coordinates on the upper half-plane,

$$Pf(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_y(x-t)f(t)dt, \quad P_y(x) = \frac{y}{x^2+y^2}.$$

The defining property of the Poisson transform is that it is $SL_2\mathbf{R}$-invariant.

We know that $S^1 = SL_2\mathbf{R}/B = G/B$ and $\mathbf{H} = SL_2\mathbf{R}/SO_2 = G/K$. G-invariant functions $S^1 \to \mathbf{H}$ are given by

$$\mathrm{Fun}_G(G/B \times G/K) = \mathrm{Fun}(G\backslash G/B \times G/K) = \mathrm{Fun}(B\backslash G/K).$$

But this is a point, so there is a unique such transform. One has to check that the inverse transform $\mathbf{H} \to S^1$ is just a restriction to the boundary.

Hardy spaces H^p: $f \in L^p(\mathbf{R})$ such that Pf is holomorphic.

Why do we get something harmonic? $\mathrm{Fun}(G/B)$ is $C(SL_2\mathbf{R}/B)$, and the representation is in the principal series with $s = 0$, so $\Delta = 0$. In particular, it is automatically in $C^\infty(\mathbf{H})$.

There is a variant of the Poisson transform for any eigenvalue called the s-Poisson transform.

$$V_s \cong C(S^1) \to C^\infty(\mathbf{H})_{\Delta = -\frac{s}{2}(1+\frac{s}{2})}.$$

$$Pf(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_{1-s}(x-t)f(t)dt, \quad P_{1-s} = \left(\frac{y}{x^2+y^2}\right)^{1-s}.$$

$f_s(x+iy) = y^{s/2}$ is an eigenfunction

$$\Delta_{\mathbf{H}} f_s = \frac{s}{2}\left(1 - \frac{s}{2}\right) f_s.$$

Any G-translate of f_s is also an eigenfunction.

$$\phi_s(z) = \int_K f_s(k \cdot z) dk$$

is a K-invariant eigenfunction of the Laplacian called the zonal spherical function. So, it is a Poisson transform of a spherical vector $V_s^K \cong \mathbf{C}$.

Spherical function

$$\phi_s(g) = \langle v_0^*, g v_0 \rangle.$$

It is K-biinvariant, i.e. a function on $K \backslash G / K$. This is just $\mathbf{R}_{\geq 0}$. Let t be a coordinate on the ray, then the Laplacian takes the form

$$\Delta = -\partial_t^2 + \coth t \partial_t.$$

The eigenvalue equation is called the Legendre equation.

Date: 04/17

What are the G-symmetries of $L^2(G/K)$? These are the same as functions on $G\backslash(G/K \times G/K) = K\backslash G/K$. These are called Hecke algebras.

Consider the group algebra of G: the underlying vector space is $C_c(G)$. The multiplication is given by convolution:

$$\begin{array}{ccccc} & & G \times G & & \\ & \swarrow^{\pi_1} & \downarrow^{\mu} & \searrow^{\pi_2} & \\ G & & G & & G \end{array}$$

The convolution is then

$$(f * h)(g) = \int_{g_1 g_2 = g} f(g_1)h(g_2)dg.$$

Proposition. *For $K \subset G$ a closed subgroup the Hecke algebra $\mathcal{H}(G,K) := C_c(K\backslash G/K) =^K C_c(G)^K$ is a subalgebra under $*$.*

Proof. Consider the following diagram:

$$\begin{array}{ccccc} & & K\backslash G \times_K G/K & & \\ & \swarrow^{\pi_1} & \downarrow^{\mu} & \searrow^{\pi_2} & \\ K\backslash G/K & & K\backslash G/K & & K\backslash G/K \end{array}$$

The multiplication here is well-defined, so the Hecke algebra is indeed a subalgebra. ☐

One can think of a group algebra as functions on $G \times G$ invariant under diagonal G.

On $G \times G$ we have a matrix multiplication given by the diagram

$$\begin{array}{ccccc} & & G \times G \times G & & \\ & \swarrow^{\pi_{12}} & \downarrow^{\pi_{13}} & \searrow^{\pi_{23}} & \\ G \times G & & G \times G & & G \times G \end{array}$$

Then the group algebra is $C_c(G) = C_c(G \times G)^G$. Similarly,

$$C_c(G\backslash[G/K \times G/K]) \cong C_c(K\backslash G/K).$$

Hecke algebras are in general noncommutative.

In out case $G/K = \mathbf{H}$. Then $K\backslash G/K = SO_2\backslash\mathbf{H} = \mathbf{R}_+$. This can also be seen from the Cartan decomposition: $G = KAK$.

Suppose $K = K(r) \in C_c(K\backslash G/K)$ and $f \in L^2(\mathbf{H})$. Then

$$(K * f)(z) = \int_r \int_{|z-w|=r} K(r)f(w)dw.$$

Theorem. *$C_c(K\backslash G/K)$ is commutative. In this case (G, K) form a so-called Gelfand pair.*

Proof. Gelfand trick: transpose $G \to G$ is an antiautomorphism, i.e. $(gh)^t = h^t g^t$.

This induces an antiautomorphism of $C_c(G)$. So, we also get an antiautomorphism of the subalgebra $C_c(K\backslash G/K)$. Double cosets $K\backslash G/K$ are represented by symmetric matrices $f, h \in C_c(K\backslash G/K)$. Then

$$f * h = (f * h)^t = h^t * f^t = h * f.$$

□

Therefore $Aut_G(L^2(\mathbf{H}))$ are commutative. Decompose into irreducibles:

$$V = \oplus V_i^{\oplus n_i}.$$

By Schur's Lemma $\operatorname{Aut}_G V_i = \mathbf{C}^\times$ and $\operatorname{Aut}_G V_i^{\oplus n_i} \cong GL_{n_i}$. So, commutativity implies that n_i is either 0 or 1.

Decompose $L^2(G/K)$ as a module over $C_c(K\backslash G/K)$.

If A is the closure of $C_c(K\backslash G/K)$ in $B(L^2(G/K))$, it becomes a commutative C^*-algebra.

Theorem (Gelfand-Naimark)**.** *Commutative C^*-algebras are equivalent to locally-compact Hausdorff spaces.*

For X a space, one has $A = C_c(X)$. For A a C^*-algebra, the space X consists of (continuous $*$-)homomorphisms $A \to \mathbf{C}$, i.e. maximal ideals of A or unitary irreps of A.

Each $x \in X = \operatorname{Spec} \overline{C_c(K\backslash G/K)}$ corresponds to $\phi_x \in L^\infty(K\backslash G/K)$. In fact, the functions are real analytic. These are precisely zonal spherical functions, i.e. matrix elements corresponding to spherical vectors in unitary irreps of G.

Theorem (Harish-Chandra)**.** *$L^2(\mathbf{H})$ has spectrum coinciding with unitary principal series. That is, characters of $C_c(K\backslash G/K)$ are given by $s \in i\mathbf{R} - 1$.*

Suppose V is a representation of G. Given $K \subset G$, what acts on V^K?

Proposition. *$C_c(K\backslash G/K)$ preserves V^K.*

Proof. For simplicity, assume G, K are finite. Then

$$\mathcal{H} = C(K\backslash G/K) = C(G\backslash[G/K \times G/K]) = \mathrm{End}_G(C(G/K)).$$

The space of functions $C(G/K)$ is the same as $\mathrm{Ind}_K^G \mathbf{C}$, where K acts trivially on $\mathbf{C}$. Then

$$\mathrm{Hom}_G(\mathrm{Ind}_K^G \mathbf{C}, V) = \mathrm{Hom}_K(\mathbf{C}, V) = V^K.$$

Therefore $\mathcal{H}$ acts by precomposition on $\mathrm{Hom}_G(C(G/K), V) = V^K$. □

If K is big, then $K\backslash G/K$ is small, then $\mathcal{H}$ is simple (e.g. commutative).

If K is small, many representations will have K-invariants.

Theorem. *Irreducible representations of G with nontrivial K-invariants is in bijection with irreducible representations of the Hecke algebra.*

More generally, there is an equivalence of categories between representations generated by K-invariants and representations of the Hecke algebra.

Discrete subgroups of $SL_2\mathbf{R}$. Let's look at discrete subgroups $\Gamma \subset SL_2\mathbf{R}$.

If X is a Riemann surface of $g > 1$. Then the universal cover of X is $\mathbf{H}$, i.e. $X = \Gamma\backslash\mathbf{H}$.

$PSL_2\mathbf{R}$ is the unit tangent bundle to $\mathbf{H}$, so $\Gamma\backslash PSL_2\mathbf{R}$ is the unit tangent bundle to X.

One also has many arithmetic subgroups. For example, one has $SL_2\mathbf{Z} \subset SL_2\mathbf{R}$. More generally, we have

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad c \equiv 0 \mod N\}.$$

$H_\Gamma = L^2(\Gamma\backslash G)$ is a representation of $SL_2\mathbf{R}$. We get discrete series, unitary principal series, complementary series.

Date: 04/19

Last time we had several examples of discrete subgroups $\Gamma \subset SL_2\mathbf{R}$:

- Fundamental groups of compact Riemann surfaces of genus $g > 1$ (via the uniformization theorem). These groups are cocompact, i.e. $\Gamma\backslash SL_2\mathbf{R}$ is compact.
- Arithmetic subgroups: $\Gamma(N)$ and $\Gamma_0(N)$. These are not cocompact.

Consider the action of G on $L^2(\Gamma\backslash G)$, this is a unitary representation. If Γ is cocompact, the decomposition into irreducibles is a direct sum of:

- Principal series: $\Delta > \frac{1}{4}$
- Complementary series: $0 \leq \Delta \leq \frac{1}{4}$
- Discrete series: $\Delta = -\frac{s}{2}\left(1 + \frac{s}{2}\right)$ for $s \in \mathbf{Z}$.

Assume $-\mathrm{id} \in \Gamma$: for example, fundamental groups of Riemann surfaces are subgroups of $PSL_2\mathbf{R}$, so their lift to $SL_2\mathbf{R}$ contains $-\mathrm{id}$. In this case only even representations occur.

Proposition. *The multiplicity of V_s ($s \notin \mathbf{Z}$) is the dimension of $L^2(\Gamma\backslash G/K)_{\Delta=-\frac{s}{2}(1+\frac{s}{2})}$.*

Note, that $\Gamma\backslash G/K = \Gamma\backslash\mathbf{H}$. For Riemann surfaces this is just $X = \Gamma\backslash\mathbf{H}$.

One can write

$$X = \Gamma\backslash G/K = G\backslash[G/K \times G/\Gamma].$$

A function on X is the same as an intertwiner for functions on G/K to functions on G/Γ.

Note, that $L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash G)^K$. So, we are looking for spherical vectors in $L^2(\Gamma\backslash G)$.

The principal series representations $(V_s)^K = \mathbf{C}v_0$. Now fix Δ eigenvalue to be $-\frac{s}{2}(1+\frac{s}{2})$ for $s \notin \mathbf{Z}$.

Since V_s contains a one-dimensional weight space for any even weight, instead of considering K-invariants, one can look at even eigenvalues.

Lifting of weight k. Consider C^∞ functions on $\mathbf{H}$ as a G-representation, such that for any function f

$$(g \cdot f)(z) = \frac{1}{(cz+d)^k} f(g^{-1}z), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is the same as looking at $f(z)dz^{k/2}$.

One can map this space to C^∞ functions on G, with G acting on the left.

Recall the Iwasawa decomposition:

$$SL_2\mathbf{R} \ni g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = NAK.$$

The map is

$$\begin{aligned} f \mapsto F(g) = (g \cdot f)(i) &= \frac{1}{(ci+d)^k} f(g^{-1} \cdot i) \\ &= e^{ik\theta} y^{k/2} f(x+iy). \end{aligned}$$

Note, that F has a property

$$F(g \cdot r(\theta)) = F(g) e^{ik\theta},$$

where $r(\theta)$ is the rotation matrix.

The inverse map is given by

$$f(x+iy) = y^{-k/2} F \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}.$$

Here we are using that

$$\begin{aligned} NA &\cong \mathbf{H} \\ dz y^{-1/2} &\leftrightarrow dz \end{aligned}$$

Denote $L^2(\Gamma\backslash G/K, k)$ the space of functions on $\Gamma\backslash G$, which have eigenvalue k with respect to K. Equivalently, these are $k/2$-forms on $\Gamma\backslash G/K$.

Lemma. *The space* $L^2(\Gamma\backslash G/K, k)$ *is identified with*

$$\{f \in C^\infty(G/K) : \quad f(z) = \frac{1}{(cz+d)^k} f(\gamma^{-1} z)\},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

For k even, the number of appearances of V_s in $L^2(\Gamma\backslash G)$ is the dimension of $L^2(\Gamma\backslash G/K, k)_{\Delta = -\frac{s}{2}(1+\frac{s}{2})}$.

Reference: Lang, $SL_2\mathbf{R}$, section VI.4.

Let's write in our coordinates x, y, θ vector fields on G coming from the right action in a basis of $\mathfrak{sl}_2\mathbf{R} \otimes \mathbf{C} = \mathfrak{sl}_2\mathbf{C} = \mathfrak{su}(1,1) \otimes \mathbf{C}$.

Recall we had the following basis of $\mathfrak{su}(1,1)$:

$$H = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

We like this basis since $\exp(i\theta H) = r(\theta)$ is the rotation matrix.

On our weight k lifts F, $r_H F = kF$.

The vector field corresponding to the right action of x is

$$\begin{aligned} r_X &= -iye^{-2i\theta}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) + \frac{i}{2}e^{-2i\theta}\frac{\partial}{\partial\theta} \\ &= -2iye^{-2i\theta}\frac{\partial}{\partial\bar{z}} + \frac{i}{2}e^{-2i\theta}\frac{\partial}{\partial\theta}. \end{aligned}$$

Then

$$\begin{aligned} r_X F &= r_x(e^{ik\theta}y^{k/2}f(x+iy)) \\ &= -iy^{k/2+1}e^{2i(k/2-1)\theta}\frac{\partial f}{\partial\bar{z}}. \end{aligned}$$

So,

Corollary. *f is holomorphic iff $r_X F = 0$.*

So, any such F is a highest weight vector for the Lie algebra action of $\mathfrak{g}$ on $C^\infty(G)$.

When we looked at the discrete series $D_n^\pm$, we defined them to be holomorphic sections on $\mathbf{H}$ of homogeneous line bundle $(dz)^{n/2}$.

Let's take $v_{-n} \in (D_n^-)^*$. Then the matrix element gives a function on the group:

$$\begin{aligned} D_n^- &\to C^\infty(G) \\ v &\mapsto \langle v, g\cdot v_{-n}\rangle. \end{aligned}$$

Here we map to functions on G of weight n. Observe, that $Xv_{-n} = 0$, i.e. v_{-n} is a highest weight vector. Therefore, we moreover land in functions on G, which are fixed by the vector field r_H. But then this space is isomorphic to holomorphic $n/2$-forms $fdz^{n/2}$ on $\mathbf{H}$.

Theorem (Gelfand-Graev, Piatetski-Shapiro)**.** *For Γ cocompact, $L^2(\Gamma\backslash G)$ is a direct sum of unitary principal and complementary series V_s with multiplicity* $\dim L^2(\Gamma\backslash\mathbf{H})_{\Delta=-\frac{s}{2}(1+\frac{s}{2})}$ *and discrete series $D_n^\pm$ with (equal) multiplicity $h^0(\Gamma\backslash\mathbf{H}, \omega^{n/2})$, i.e. the dimension of holomorphic $n/2$-forms on $X = \Gamma\backslash G$.*

Modular forms. Now Γ is an arithmetic subgroup, e.g. $\Gamma = SL_2\mathbf{Z}, \Gamma_0(N), \Gamma(N)$.

- $SL_2\mathbf{Z}\backslash SL_2\mathbf{R}$ is the space of unimodular lattices $\Lambda \subset \mathbf{R}^2$.
- $\Gamma(N)\backslash SL_2\mathbf{R}$ is the space of unimodular lattices and a trivialization $\Lambda \mod N \cong (\mathbf{Z}/N)^{\oplus 2}$.
- $\Gamma(N)\backslash SL_2\mathbf{R}$ is the space of unimodular lattices and a line in Λ mod N.

$SL_2\mathbf{Z}$ is generated by $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The fundamental domain is noncompact, want to compactify by adding a cusp.

Γ naturally acts on $\mathbf{P}^1_{\mathbf{Z}} = SL_2\mathbf{Z}/B_{\mathbf{Z}} = SL_2\mathbf{Q}/B_{\mathbf{Q}} = \mathbf{Q} \cup \infty$ with finitely many orbits, called cusps of Γ.

Modular curves $\Gamma\backslash\mathbf{H} \subset \Gamma\backslash(\mathbf{H} \cup \mathbf{P}^1_{\mathbf{Q}}) = X(N)$. The points that we add are called cusps.

Definition. *A holomorphic) modular form of weight k for Γ is a holomorphic section $f \in H^0(X(\Gamma), \omega^k)$.*

That is, f is a holomorphic function on $\mathbf{H}$, such that

$$f(\gamma z) = (cz + d)^k f(z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Moreover, it is holomorphic at cusps.

Observe, that $SL_2\mathbf{Z} \supset \langle \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \rangle \cong \mathbf{Z}$. So, any $SL_2\mathbf{Z}$-modular form f obeys $f(z) = f(z + 1)$ and so one can write it in terms of $q = e^{2\pi i z}$. In terms of q the cusp is at $q = 0$. We can write

$$f(q) = \sum_{n \in \mathbf{Z}} a_n q^n.$$

If it is holomorphic at the cusp $q = 0$, $a_n = 0$ for $n < 0$. a_0 turns out to be important.

Date: 04/24

Last time we looked at $\Gamma \subset SL_2\mathbf{Z} \subset SL_2\mathbf{R}$. We compactified

$$\Gamma\backslash SL_2\mathbf{R}/SO_2 = \Gamma\backslash\mathbf{H} \subset \Gamma\backslash\mathbf{H} \cup \mathbf{P}^1_{\mathbf{Q}}.$$

Γ acts on $\mathbf{P}^1_{\mathbf{Q}}$ with finitely many orbits, "cusps" of Γ. Assume that the cusp $\gamma \in \mathbf{P}^1_{\mathbf{Q}}$ is $\gamma = \infty$ (via conjugation by $SL_2\mathbf{Z}$).

$$\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty).$$

It consists of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

where $a, a^{-1} \in \mathbf{Z}$ and so $a = \pm 1$.

$[\Gamma : \Gamma_\infty]$ is finite, so Γ_∞ contains a matrix of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Any Γ-invariant function of $z \in \mathbf{H}$ is a function of $q = e^{2\pi i z/n}$. Can expand

$$f(q) = \sum a_k q^k.$$

f is a holomorphic modular form iff $a_k = 0$ for $k < 0$.

Definition. *f is a cusp form if f vanishes at the cusp (i.e. $a_0 = 0$).*

$f(x + iy)$: for x bounded

$$f \sim a_1 e^{-2\pi y/n} + a_2 e^{-4\pi y/n} + \ldots$$

so a cusp form is exponentially decreasing in y.

So, if f is a cusp form,

$$F(g) = f(z) y^{k/2} e^{ik\theta} \in L^2(\Gamma\backslash G).$$

Cusp forms count appearances of discrete series representation D_k in $L^2(\Gamma\backslash G)$.

Maass forms: $f \in C^\infty(\Gamma\backslash\mathbf{H})$, which is a Δ-eigenfunction of eigenvalue $-\frac{s}{2}(1 + \frac{s}{2})$ of moderate growth at cusps. Cusp form if f rapidly decreasing iff $F(g) \in L^2(\Gamma\backslash G)$, so they count appearance of V_s in $L^2(\Gamma\backslash G)$.

Horocycle correspondence. Consider $G/K = \mathbf{H}$. A horocycle is a circle through a point, which is tangent to the circle at infinity.

The space

$$H\{x \in \mathbf{H}, \quad F\text{horocycle}| \quad x \in F\}$$

has a map to $H \to G/K = \mathbf{H}$ by forgetting the horocycle. The horocycle is given uniquely by specifying a tangent line, so the fibers are projectivized tangent spaces. Moreover, one has $H = G/M$, where

$$M = \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}.$$

On the other hand, we can forget the point, and so we have a map to the space of horocycles Ξ. It is isomorphic to G/MN.

Finally, we can forget the horocycle and remember the intersection with infinity; this gives a map $\Xi \to \mathbf{RP}^1 = G/B = G/MAN$.

The Poisson transform is the push-pull (integral) transform from $\mathbf{RP}^1$ to $\mathbf{H}$.

The space of functions on the space of horocycles Ξ is called the universal (spherical) principal series repersentation $L^2(G/MN)$. It has a left action of G and a right action of A.

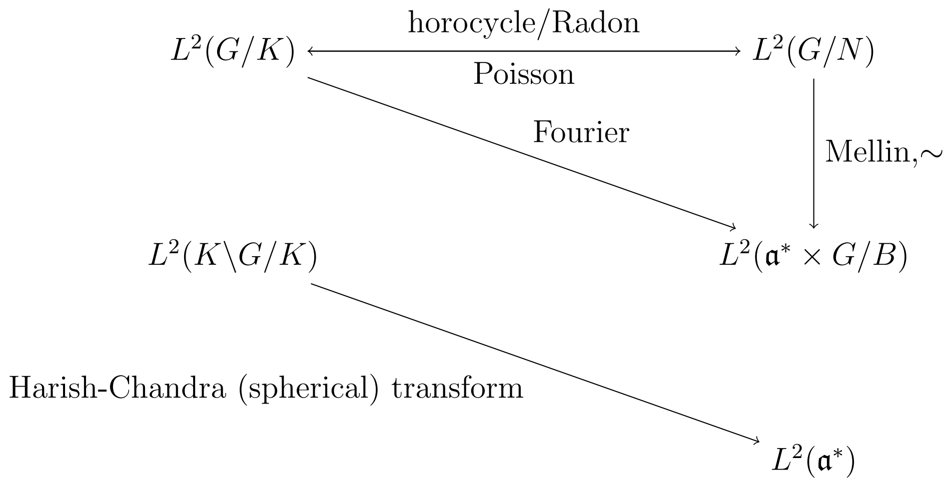
Mellin transform is the Fourier transform on $\mathbf{R}_+$:

$$f \mapsto \hat{f}(s) = \int_0^\infty f(t) t^s \frac{dt}{t}.$$

So, for $A \cong \mathbf{R}_+$, Mellin transform gives an isomorphism $L^2(A) \cong L^2(\mathfrak{a}^*)$.

$$\begin{aligned} L^2(G/N) &\cong L^2(\mathfrak{a}^* \times G/B) \\ &= L^2(\mathfrak{a}^*, L^2(G/B)) \end{aligned}$$

A picture of different transforms:



Fourier transform $L^2(G/K) \to L^2(\mathfrak{a}^* \times G/B)$ is the usual Fourier transform, where y^s play the role of exponents. It depends on the choice of the KAN decomposition, and so let N be the stabilizer of a cusp. Given $l \in G/B$ (cusp) and $s \in \mathfrak{a}^*$, we get a function "y_l^s".

$$f \in L^2(G/K) \mapsto \mathbf{F}(f)s, l) = \int_{\mathbf{H}} f(z) y_l^s dz.$$

The Harish-Chandra transform is the Fourier transform for K-invariant functions. Recall, that the zonal spherical functions are

$$\phi_s(z) = \int_K y_l^s dl.$$

The transform is given by

$$HC : f \mapsto \tilde{f}(s) = \int_{\mathbf{H}} f(z)\phi_s(z) dz.$$

Reference: Gelfand-Grindkin-Graev.

Find a finite-dimensional representation V of G and a vector $v \in V$ with stabilizer K. $G \cdot v \cong G/K \subset V$.

For $\lambda \neq 0$ consider $G \cdot (\lambda \cdot v) \cong G/K$ and take the limit $\lambda \to 0$.

More precisely, consider $[v] \in \mathbf{P}(V)$ and look at $G \cdot [v] \subset \mathbf{P}(V)$. Take the closure $\overline{G \cdot [v]} \subset \mathbf{P}(V)$ and the preimage $\overline{G \cdot v} \subset V$.

Date: 04/26

Spherical harmonics. Recall, that $L^2(S^2 = SU_2/K) = \widehat{\oplus_l V_{2l}}$, even finite-dimensional representations of SU_2 or $SL_2\mathbf{C}$. This space is identified with the space H of harmonic polynomials on $\mathbf{R}^3$: $\{p \mid \Delta p = 0\}$.

$$H \otimes \mathbf{C}[\mathbf{R}^3]^{SU_2} \cong \mathbf{C}[\mathbf{R}^3].$$

Now recall the picture of the adjoint quotient: $\mathfrak{sl}_2\mathbf{C} \to \mathfrak{h}/W$.

For $\mathfrak{sl}_2\mathbf{R}$ we have the map

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto -a^2 - bc$$

The fiber over the origin is $N = \{a^2 + bc = 0\} \cong \{\mathbf{R}^2 \backslash 0\}/\pm 1 \cup 0$. Representatives are

$$\begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad \begin{pmatrix} & -1 \\ & \end{pmatrix}.$$

For $a^2 + bc < 0$ the fiber is $\mathbf{H} \cup \mathbf{H}^-$, i.e. $G/K \cup G/K$, this is the elliptic conjugacy class. Representatives are

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

For $a^2 + bc > 0$ the fiber is a hyperboloid (split/hyperbolic conjugacy class). This is $G/MA \cong \mathbf{RP}^1 \times \mathbf{RP}^1 \backslash \Delta$.

The representative is

$$\begin{pmatrix} a & \\ & -a \end{pmatrix}.$$

Similar to the transform we had in the SU_2 case, here we have a transform

$$C^\infty(G/H) \to C^\infty(G/MN).$$

Eisenstein series. Last time we looked at cusp forms: forms on $\Gamma \backslash \mathbf{H}$, which vanish at cusps.

Definition. *Automorphic form for* Γ*:* $\mathcal{A}(\Gamma \backslash G)$*, which are smooth,* K*-finite and* Z*-finite (here* $Z = Z(U\mathfrak{g}) = \mathbf{C}[C]$*) of moderate growth near every cusp:* $|f(g)| \leq Cy(g)^N$*.*

Let's fix a cusp, i.e. a Γ-orbit on $\mathbf{P}^1_{\mathbf{Z}}$. The stabilizer is $\Gamma_\infty = \Gamma \cap B$, where B is the stabilizer of $l \in \mathbf{RP}^1$ in $SL_2\mathbf{R}$. Explicitly,

$$\Gamma_\infty = \left\{ \begin{pmatrix} \pm 1 & n \\ & \pm 1 \end{pmatrix} \right\} \subset MN.$$

Note, that $N\Gamma_\infty \backslash G = MN \backslash G = (\mathbf{R}^2 \backslash 0)/\pm 1$.

Consider the map $\Gamma_\infty\backslash G \to \Gamma\backslash G$. The fibers are Γ/Γ_∞, i.e. a Γ-orbit of the cusp in $\mathbf{P}^1_{\mathbf{Z}}$. In the $SL_2(\mathbf{Z})$ case this is just $\mathbf{P}^1_{\mathbf{Z}}$.

$$\begin{array}{ccc} & \Gamma_\infty\backslash G & \\ \swarrow & & \searrow \\ \Gamma\backslash G & & MN\backslash G = N\Gamma_\infty\backslash G \end{array}$$

Now we have the following integral transforms

$$\mathcal{A}(\Gamma\backslash G) \underset{C}{\overset{E}{\leftrightarrows}} \mathcal{A}(N\Gamma_\infty\backslash G),$$

where E is the so-called Eisenstein series and C is the constant term.

$$C(f)(g) = \int_{\Gamma_\infty|backslashN} f(n \cdot g)dn.$$

For example, for modular forms one gets

$$C(\sum a_n q^n) = a_0.$$

One can also define the Whittaker functional

$$W(f)(g) = \int_{\Gamma_\infty\backslash N} f(ng)e^{-i\pi n}dn = a_1.$$

The Eisenstein operator is

$$E(\phi)(g) = \sum_{\Gamma_\infty\backslash\Gamma} \phi(\gamma g) = \sum_{(c,d)\in\mathbf{Z}^2\backslash 0 \text{ rel prime}} \phi(\gamma_{c,d} g).$$

One can think of the Eisenstein operator as $E = E(f,s)$, where $f \in C^\infty(G/B)$ and $s \in \mathbf{C}$ via the Mellin transform: $G/MN \leftrightarrows \mathfrak{a}^* \times G/B$.

Theorem. *$E(f,s)$ is absolutely convergent for $\Re s \gg 0$. Moreover, $E(f,s)$ is holomorphic in s and has an analytic continuation as a meromorphic function.*

If we decompose $g = nyk$ via the $G = NAK$ decomposition, then

$$E(f,s) = \sum_{(c,d)=1} y(\gamma g)^s f(K(\gamma g)).$$

Using

$$\Im(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z) = \frac{\Im z}{|cz+d|^2},$$

one obtains

$$E(1,s) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}.$$

Instead of summing over $\mathbf{P}^1_{\mathbf{Z}}$, one can sum over $\mathbf{Z}^2\backslash 0$:

$$\tilde{E} = \sum_{c,d\in\mathbf{Z}^2\backslash 0} \frac{y^s}{|cz+d|^{2s}} = \sum_{n>0}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}\frac{1}{n^{2s}} = \zeta(2s)E(f,s).$$

If we take $\chi_k = e^{2\pi ik}$, one can compute

$$\tilde{E}(\chi_k, s) = \sum_{\mathbf{Z}^2\backslash 0} \frac{y^s}{|cz+d|^{2s}}\frac{1}{(cz+d)^k}.$$

At $s=0$ one gets

$$\sum_{\mathbf{Z}^2\backslash 0} \frac{1}{(cz+d)^k}.$$

Date: 05/01

Eisenstein series.

$$C^\infty(\Gamma\backslash G) \underset{E}{\overset{C}{\leftrightarrows}} C^\infty(MN\backslash G).$$

C, E are adjoint operators. Therefore, $\Im(E) \perp \mathrm{Ker}(C)$. Here $\mathrm{Ker}(C)$ consists of cusp forms.

Indeed,

$$\begin{aligned}
\int_{\Gamma\backslash G} E(f,s)\phi(g)dg &= \int_{\Gamma\backslash G}\left(\sum_{\Gamma_\infty\backslash\Gamma} f_s(\gamma g)\right)\phi(g)dg \\
&= \int_{\Gamma_\infty\backslash G} f_s(g)\phi(g)dg \\
&= \int_{N\gamma\backslash G} f_s(g)\left(\int_{\Gamma_\infty\backslash N}\phi(ng)dn\right)dg
\end{aligned}$$

We have a decomposition

$$L^2(\Gamma\backslash G) \cong \mathbf{C} \oplus L^2_{cusp} \oplus L^2_{cont},$$

where L^2_{cusp} contains modular and Maass forms, and L^2_{cont} is the image of Eisenstein series.

What is the constant term of Eisenstein series?

$SL_2\mathbf{Z}$ case:

$$\Gamma_\infty\backslash\Gamma = \mathbf{P}^1_{\mathbf{Z}} = \left\{\begin{pmatrix} 1 & \\ & 1\end{pmatrix} \cup \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\cdot\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}\right\}/B.$$

Let's compute the composition CE for a function f on $MN\backslash G$:

$$\begin{aligned}
CE(f)(g) &= \int_{\Gamma_\infty\backslash N} E(f)\left(\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} g\right)dn \\
&= \int_{\gamma_\infty\backslash N}\sum_{\mathbf{P}^1_{\mathbf{Z}}} f\left(\gamma\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} g\right)dn \\
&= f(g) + \int_{\Gamma\backslash N}\sum_{\alpha\in\mathbf{Z}} f\left(w_0\begin{pmatrix} 1 & n+\alpha \\ & 1\end{pmatrix} g\right)dn \\
&= f(g) + \int_N f\left(w_0\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} g\right)dn \\
&= f(g) + I(f)(g).
\end{aligned}$$

Here I is the intertwiner $I : V_s \to V_{-2-s}$.

Corollary (Functional equation)**.** $E(f,s) = E(If, -s-2)$.

$SL_2(\mathbf{Q}_p)$

$\mathcal{O} = \mathbf{Z}_p$ is the ring of p-adic integers. Its fraction field is $\mathcal{K} = \mathbf{Q}_p$, p-adic numbers. Here $\mathbf{Z}_p$ is the completion of $\mathbf{Z}$ at the prime ideal (p).

These are Taylor (Laurent) series in p. So,

$$\mathbf{Q}_p = \{\sum_{i=-N}^{\infty} a_i p^i\},$$

where $a_i = 0, ..., p-1$.

$\mathbf{Z}_p$ is the closed unit disk in $\mathbf{Q}_p$.

Definition. *A function $f : \mathbf{Q}_p \to \mathbf{C}$ is smooth if it is continuous for the discrete topology on $\mathbf{C}$.*

Equivalently, f is locally-constant: $\forall x \in \mathbf{Q}_p, \forall a \; f(x) = f(x + p^N a)$ for $N \gg 0$.

On $\mathbf{Z}_p$ a smooth function is the same as a $p^N\mathbf{Z}_p$-invariant function for $N \gg 0$.

This space is denoted by $C^\infty(\mathbf{Q}_p)$.

Consider the chain

$$\mathbf{Q}_p/\mathbf{Z}_p \subset \mathbf{Q}/\mathbf{Z} \subset \mathbf{R}/\mathbf{Z}.$$

Here $\mathbf{R}/\mathbf{Z}$ is a circle, $\mathbf{Q}/\mathbf{Z}$ consists of torsion and $\mathbf{Q}_p/\mathbf{Z}_p$ consists of p-torsion.

$$GL_1(\mathbf{Q}_p) = \mathbf{Q}_p^\times = \langle p \rangle \times \mathbf{Z}_p^\times \cong \mathbf{Z} \times \mathbf{Z}_p^\times.$$

A smooth character is a continuous map $GL_1(\mathbf{Q}_p) \to \mathbf{C}^\times \subset \mathbf{C}$.

Unramified characters: the ones that factor through $\mathbf{Q}_p^\times/\mathbf{Z}_p^\times$:

$$\mathbf{Q}_p^\times \to \mathbf{Q}_p^\times/\mathbf{Z}_p^\times \to \mathbf{C}^\times.$$

Since $\mathbf{Q}_p^\times/\mathbf{Z}_p^\times \cong \mathbf{Z}$ these are parametrized by $s \in \mathbf{C}^\times$. Explicitly,

$$p^{-N}(a_0 + a_1 p + ...) \mapsto s^{-N}.$$

$$G = SL_2\mathbf{Q}_p \supset SL_2\mathbf{Z}_p = K.$$

It turns out K is the maximal compact subgroup.

There is a unique Haar measure on G with $\mathrm{vol}(K) = 1$.

Consider a subgroup $K_n \subset G$:

$$K_N = \{g \in SL_2\mathbf{Z}_p : g \cong \mathrm{id}(p^N)\}.$$

These subgroups are compact and open.

Definition. *A smooth representation of G is $\pi : G \to GL(V)$, which is continuous in the discrete topology on $GL(V)$.*

Equivalently, all matrix elements are smooth. In other words, $\forall v \in V$ there is an N, such that $K_N \cdot v = v$. One can write

$$V = \cup_N V^{K_N}.$$

V^{K_N} is a module for $C_c^\infty(K_N \backslash G / K_N)$ known as the Hecke algebra of level N.

The group algebra in the p-adic case is $C_c^\infty(G)$ (sometimes called "the" Hecke algebra). If V is a smooth representation, it is a $C_c^\infty(G)$-module.

For H compact subgroup in G, the Hecke algebra $\mathcal{H}(G,H) = C_c^\infty(H \backslash G / H)$ acts on V^H.

If V is irreducible and $V^{K_n} \neq 0$ for some N. Then $V = GV^{K_N}$. So, V is completely determined by V^{K_N} as a $\mathcal{H}(G, K_N)$-module.

Definition. *An irreducible representation V is unramified if $V^K \neq 0$, where $K = SL_2\mathbf{Z}_p$.*

For reducible representations V should be generated by V^K.

Unramified principal series.

$$G \supset G \supset T.$$

Here

$$T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid \quad a \in \mathbf{Q}_p^\times \right\} \cong \mathbf{Z} \times \mathbf{Z}_p^\times.$$

If we pick $s \in \mathbf{C}^\times$, then we have a character

$$\begin{array}{ccc} T & \xrightarrow{\chi_s} & \mathbf{C} \\ \downarrow & \nearrow & \\ \mathbf{Z} = T(\mathcal{K})/T(\mathcal{O}) & & \end{array}$$

$$V_s = \operatorname{Ind}_B^G \chi_s = \{ f \in C^\infty(G) | \quad f(ntf) = \chi_s(t) f(g), n \in N(\mathcal{K}), t \in T(\mathcal{K}) \}.$$

Date: 05/03

Tree for $SL_2\mathbf{Q}_p$. This is an analog of $\mathbf{H}$.

Let $E = \mathbf{Q}_p^2$.

Definition. *$L \subset E$ is a lattice if L is a finitely-generated $\mathbf{Z}_p$-submodule, generating E, i.e. $E = L \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.*

Note, that $L \cong \mathbf{Z}_p^2$ as a $\mathbf{Z}_p$-module.

Parametrize lattices up to homothety (rescaling by $\mathbf{Q}_p^\times$).

Let's make a collection of lattices into a graph: vertices are lattices up to homothety; there is an edge between L_1 and L_2 if $L_1 \subset L_2$ if $L_2/L_1 \cong \mathbf{Z}/p\mathbf{Z}$.

Note, that the relation is actually symmetric (up to scale): $L_2 \supset L_1 \supset pL_2$. This is a $(p+1)$-regular tree. To see that it is a tree, use a distance function $d(L_1, L_2)$ is the length of L_1/L_2 (arrange $L_2 \subset L_1$ by rescaling).

$$L_1/L_2 \cong \mathbf{Z}/p^{d(l_1,l_2)\mathbf{Z}}.$$

The space of all lattices carries a transitive action of $GL_2\mathbf{Q}_p$, which factors through $PGL_2\mathbf{Q}_p$ since we mod out by rescalings. The stabilizer of the basepoint is $PGL_2\mathbf{Z}_p$. So, the tree is in bijection with $PGL_2\mathbf{Q}_p/PGL_2\mathbf{Z}_p$.

Recall, that $PGL_2\mathbf{Z}_p$ is the maximal compact subgroup of $PGL_2\mathbf{Q}_p$; so, it is an analog of $SL_2\mathbf{R}/SO_2 \cong \mathbf{H}$.

On the other hand, there are two orbits of $SL_2\mathbf{Q}_p \to PGL_2\mathbf{Q}_p$: $SL_2\mathbf{Q}_p$ does not connect lattices of distance 1.

Let's call X the ends of the tree. It is a collection of infinite paths (without backtracking) modulo an equivalence relation that they eventually agree.

Ends are in bijection with $\varprojlim \mathbf{P}^1(\mathbf{Z}/p^n\mathbf{Z}) = \mathbf{P}^1(\mathbf{Z}_p)$. This is also isomorphic to $\mathbf{P}^1(\mathbf{Q}_p)$. In particular, it carries an action of $PGL_2\mathbf{Q}_p$ since $\mathbf{P}^1(\mathbf{Q}_p) \cong PGL_2\mathbf{Q}_p/B$.

The torus $T(\mathbf{Q}_p)$ consists of matrices preserving the decomposition $E = E_1 \oplus E_2$. So, it preserves the standard path of lattices $\langle e_1, p^n e_2\rangle$. Moreover, T preserves distance to l orbits.

Now let's look at $N(\mathbf{Q}_p) \cong \mathbf{Q}_p$. It preserves a unique $\mathbf{Q}_p$-line $W \subset E$. So, it preserves a unique end of X. Moreover, it preserves a relative distance to the end: distance from p to W is a $\mathbf{Z}$-torsor.

N is normalized by T. T acts on the collection of horocycles (N-orbits) through $T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \cong \mathbf{Z}$.

Recall, that in the $\mathbf{H}$ case we had a space of horocylces $SL_2\mathbf{R}/MN$. Principal series were functions with fixed homogeneity with respect to $\mathbf{R}_+ = A$.

Horocycles in X are $PGL_2\mathbf{Q}_p/N(\mathbf{Q}_p)T(\mathbf{Z}_p)$. This space has a right action of $T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \cong \mathbf{Z}$. The quotient is $PGL_2\mathbf{Q}_p/B = \mathbf{P}^1(\mathbf{Q}_p)$.

Unramified characters are parametrized by

$$s \in \mathbf{C}^\times = \mathrm{Hom}(T(\mathbf{Q}_p)/T(\mathbf{Z}_p), \mathbf{C}^\times).$$

The principal series are

$$V_s = \mathrm{Ind}_B^G \mathbf{C}_s = \{f : G \to \mathbf{C} | \quad f(gtn) = \chi_s(t) f(g)\}$$
$$= \{\text{smooth functions } f \text{ on } SL_2\mathbf{Q}_p/N(\mathbf{Q}_p)T(\mathbf{Z}_p) \text{ homogeneous of degree } s \text{ wrt } \mathbf{Z} = T(\mathbf{Q}_p)/T$$

Finally, $PGL_2\mathbf{Z}_p$ preserves the distance from L_0. So,

$$PGL_2\mathbf{Z}_p \backslash PGL_2\mathbf{Q}_p/PGL_2\mathbf{Z}_p \cong \mathbf{N}.$$

Iwahori subgroup: $I \subset PGL_2\mathbf{Q}_p$, stabilizer of an edge. Explicitly,

$$I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c \equiv 0 \mod p.$$

Hecke algebras: fix K_N compact subgroup $PGL_2\mathbf{Q}_p$. $C_c^\infty(K_n \backslash PGL_2\mathbf{Q}_p/K_N)$ acts on V^{K_N}.

Spherical hecke algebra is

$$H_{sph} = C_c^\infty(K\backslash G/K),$$

where $K = PGL_2(\mathbf{Z}_p)$ is the maximal compact in $G = PGL_2\mathbf{Q}_p$. The algebra is spanned by T_{p^n}: constant function on $d(L_0, L) = n$. T_{p^n} corresponds to the double coset of

$$K \begin{pmatrix} 1 & \\ & p^n \end{pmatrix} K.$$

$$(T_{p^n} * f)(x) = \sum_{d(x,y)=n} f(y),$$

where $f \in C^\infty(X) = C^\infty(G/K)$.

$$T_p T_{p^n} = T_{p^{n+1}} + pT_{p^n-1}.$$

One can write

$$\sum_0^\infty T_{p^n}\lambda^n = \frac{1}{1 - T_p\lambda + p\lambda^2}.$$

(G, K) is a Gelfand pair, so $\mathcal{H}(G, K)$ is commutative.

Theorem (Satake isomorphism)**.** $\mathcal{H}_{sph} = \mathbf{C}[T_p] \cong \mathrm{Rep}\, SL_2\mathbf{C}$.

Date: 05/08

Last time we looked at the Satake isomorphism. Let $G = PSL_2\mathbf{Q}_p$ and $K = PSL_2\mathbf{Z}_p$ is the maximal compact subgroups. Then the spherical Hecke algebra is

$$H_p^{sph} = C_c^\infty(K\backslash G/K),$$

moreover, it is commutative.

Theorem. *Unramified G-reps are equivalent to modules over H^{sph} by taking K-invariants.*

In particular, $G-$irreps go to irreps of H^{sph}. But H^{sph} is commutative, so we get something 1-dimensional.

Theorem (Satake isomorphism)**.** $H_p^{sph} \cong \mathbf{C}[T_p]$.

So, the irreps are classified by λ, i.e. let $T_p v_\lambda = \lambda v_\lambda$, then v_λ spans $V_\lambda = \mathbf{C}v_\lambda$.

Idea of the proof of Satake isomorphism: realize all representations in unramified principal series V_s for $s \in \mathbf{C}^\times$, the character of $T(\mathcal{K})/T(\mathcal{O}) \cong \mathbf{Z} \to \mathbf{C}^\times$.

V_s^K is one-dimensional. We have a map

$$\begin{aligned} H^{sph} &\to \mathbf{C}[\mathbf{C}^\times] \\ T &\mapsto T(s) = T\text{-eigenvalue on } V_s^K. \end{aligned}$$

This is not an isomorphism, since we have intertwiners between s and $1/s$. So, get

$$H^{sph} \cong \mathbf{C}[\mathbf{C}^\times]^{\mathbf{Z}_2} \cong \operatorname{Rep} SL_2\mathbf{C}.$$

Similarly,

$$H^{sph}_{SL_2\mathbf{Q}_p} \cong \operatorname{Rep} PSL_2\mathbf{C}.$$

In general,

Theorem (Satake isomorphism)**.** $C_c^\infty(G(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)/G(\mathbf{Z}_p)) \cong \operatorname{Rep} {}^LG_{\mathbf{C}}$.

Here ${}^LG_{\mathbf{C}}$ is the Langlands dual group to the *complex* group $G_{\mathbf{C}}$.

For real groups: any irreudcible admissible representation is realized inside principal series.

For p-adic groups: most are not inside of principal series. These are called supercuspidals.

How it all relates to modular forms. What are the symmetries of $C^\infty(\Gamma\backslash\mathbf{H})$?

- It has an action of $\Delta_{\mathbf{H}}$, Casimir of $\mathfrak{sl}_2$. $\Gamma\backslash SL_2\mathbf{R}$ has an $SL_2\mathbf{R}$-action and so the center $\mathbf{C}[\Delta_{\mathbf{H}}] = Z(U\mathfrak{g})$ acts on the quotient $\Gamma\backslash SL_2\mathbf{R}/SO_2$.
- Action of $C_c^\infty(SO_2\backslash SL_2\mathbf{R}/SO_2)$ on $C^\infty(\Gamma\backslash SL_2\mathbf{R}/SO_2)$ again coming from the $SL_2\mathbf{R}$ action on $\Gamma\backslash SL_2\mathbf{R}$.

There are many more symmetries if Γ is arithmetic: $SL_2\mathbf{Z}, \Gamma(N), \Gamma_0(N)$. Recall, the following identifications:

- $SL_2\mathbf{Z}\backslash SL_2\mathbf{R}$ parametrizes unimodular lattices $\Lambda \subset \mathbf{R}^2$.s
- $SL_2\mathbf{Z}\backslash SL_2\mathbf{R}/SO_2$ parametrizes unimodular lattices up to rotation (moduli space of elliptic curves).
- $\Gamma(N)\backslash SL_2\mathbf{R}/SO_2$ parametrizes unimodular lattices up to rotation and a trivialization $\Lambda \equiv (\mathbf{Z}/N\mathbf{Z})^2 \mod N$.

For any p not dividing N we have correspondences

$$\Gamma\backslash\mathbf{H} \longleftarrow \mathrm{Hecke}_p \longrightarrow \Gamma\backslash\mathbf{H}$$

Here

$$\mathrm{Hecke}_p = \{\Lambda_1 \subset \Lambda_2, \quad \Lambda_2/\Lambda_1 \cong \mathbf{Z}/p\mathbf{Z}\}.$$

So, we get an operator T_p acting on $C^\infty(\Gamma\backslash\mathbf{H})$. These all commute for different p.

So, we get an action of

$$\bigotimes_{p\nmid N} H_p^{sph} \cong \mathbf{C}[T_p, ...].$$

Let Γ be an arithmetic subgroup.

Theorem (Strong approximation theorem)**.**

$$\Gamma\backslash SL_2\mathbf{R}/SO_2 \cong SL_2\mathbf{Q}\backslash {\prod_p}' SL_2\mathbf{Q}_p \times SL_2\mathbf{R}/\prod_{p\nmid N} SL_2\mathbf{Z}_p \times \prod_{p\mid N} K_p \times SO_2.$$

Here $\prod'$ means that all but finitely many factors are in $SL_2\mathbf{Z}_p$. $K_p \subset SL_2\mathbf{Z}_p$ are some subgroups of $SL_2\mathbf{Z}_p$.

One can also write it as

$$SL_2\mathbf{Q}\backslash {\prod_{p\nmid N}}' X_p \times \mathbf{H} \times \prod_{p\mid N} SL_2\mathbf{Q}_p/K_p.$$

For $\Gamma = SL_2\mathbf{Z}$ one can write it as

$$SL_2\mathbf{Q}\backslash SL_2\mathbf{A}/SL_2\mathcal{O}_\mathbf{A}.$$

Here

$$\mathbf{A} = \prod{}' \mathbf{Q}_p \times \mathbf{R}$$

is the adeles. The ring of integers is $\mathcal{O}_\mathbf{A} = \prod_p \mathbf{Z}_p$. Note, that $\mathbf{Z} \hookrightarrow \mathcal{O}_\mathbf{A} \times \mathbf{R}$. In particular, $\mathbf{Z} \hookrightarrow \mathcal{O}_\mathbf{A}$ is dense:

$$\mathcal{O}_\mathbf{A} = \prod \mathbf{Z}_p = \hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/N\mathbf{Z}.$$

So, we see that $C^\infty(\Gamma\backslash\mathbf{H})$ has symmetries coming from $C^\infty(SL_2\mathbf{Q}\backslash \prod' SL_2\mathbf{Q}_p \times SL_2\mathbf{R})$, which is a representation of $SL_2\mathbf{Q}_p$ and $SL_2\mathbf{R}$. Take an irreducible subrep $V^{irrep} \cong \otimes_{p,\infty} V_p$.

Theorem. *V_p is unramified for all but finitely many p.*

Recall that $C^\infty(\Gamma\backslash\mathbf{H})_{\Delta_\mathbf{H} = -\frac{s}{2}(1+\frac{s}{2})}$ gave us principal series in $\Gamma\backslash SL_2\mathbf{R}$.

$H^0(\Gamma\backslash\mathbf{H}, \omega^k)$ gave us discrete series D_k in $\Gamma\backslash SL_2\mathbf{R}$.

We can look at special elements (eigenforms) $f \in \otimes_p V_p^{SL_2\mathbf{Z}_p}$, which are diagonal for the action of the Hecke algebra:

$$T_p f = a_p f.$$

Recall, that we can write

$$f = \sum_{n=0}^{\infty} a_n q^n.$$

A cusp form has $a_0 = 0$. Furthermore, we can normalize $a_1 = 1$.

Claim: coefficients of the Fourier expansion of the eigenform coincide with the eigenvalues of the Hecke operators.

Remark: this is true for $SL_2\mathbf{Z}$, for other Γ one can get Fourier coefficients from the eigenvalues.

Question: which a_p occur?

Equivalently, which representations of $\prod SL_2\mathbf{Q}_p \times SL_2\mathbf{R}$ occur in the quotient?

Langlands conjecture: the answer is given in terms of Galois representations $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to PSL_2\mathbf{C} = {}^L G_\mathbf{C}$.

Given ρ, a 2-dimensional representation of $\mathrm{Gal}_\mathbf{Q}$, get a list of numbers $\lambda \in \mathbf{C}$, in fact appear as semisimple conjugacy classes in $PSL_2\mathbf{C}$.

For all but finitely many p get $\rho(\mathrm{Frob}_p) \in PSL_2\mathbf{C}$, a well-defined conjugacy class.

Langlands: representation theory of $SL_2\mathbf{A}$ acting on $L^2(SL_2\mathbf{Q}\backslash SL_2\mathbf{A})$ coincides with 2d representations of $\mathrm{Gal}_\mathbf{Q}$.