Loop Spaces and Connections

AMS Central Section Meeting, Baylor University. October 16, 2009

David Ben-Zvi

University of Texas

Overview

Joint work with David Nadler (Northwestern)

Overview

Joint work with David Nadler (Northwestern)
“Loop Spaces and Connections”. Preliminary version: arXiv:0706.0322.

Overview

Joint work with David Nadler (Northwestern)
"Loop Spaces and Connections". Preliminary version: arXiv:0706.0322.

- Study relations between loop spaces and differential calculus, using derived algebraic geometry.

Overview

Joint work with David Nadler (Northwestern)
"Loop Spaces and Connections". Preliminary version: arXiv:0706.0322.

- Study relations between loop spaces and differential calculus, using derived algebraic geometry.
- Motivations from representation theory:

Overview

Joint work with David Nadler (Northwestern)
"Loop Spaces and Connections". Preliminary version: arXiv:0706.0322.

- Study relations between loop spaces and differential calculus, using derived algebraic geometry.
- Motivations from representation theory:
- New organizing principle for representations of Lie groups via topological field theory

Overview

Joint work with David Nadler (Northwestern)
"Loop Spaces and Connections". Preliminary version: arXiv:0706.0322.

- Study relations between loop spaces and differential calculus, using derived algebraic geometry.
- Motivations from representation theory:
- New organizing principle for representations of Lie groups via topological field theory
- See also arXiv:0805.0157 (with J. Francis) and arXiv:0904.1247.

Loop Spaces

X a topological space

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$
- Many fascinating structures, central in algebraic and differential topology.

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$
- Many fascinating structures, central in algebraic and differential topology.
- Composition of loops: LX is a family of “groups up to homotopy”, based loops $\Omega_x X$ for $x \in X$.

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$
- Many fascinating structures, central in algebraic and differential topology.
- Composition of loops: LX is a family of "groups up to homotopy", based loops $\Omega_x X$ for $x \in X$.
- Today: rotation of loops.

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$
- Many fascinating structures, central in algebraic and differential topology.
- Composition of loops: LX is a family of “groups up to homotopy”, based loops $\Omega_x X$ for $x \in X$.
- Today: rotation of loops.
  The circle group S^1 acts on LX by precomposition

Loop Spaces

X a topological space

- Definition: The free loop space of X is the space of maps from the circle to X, $LX = Map(S^1, X)$
- Many fascinating structures, central in algebraic and differential topology.
- Composition of loops: LX is a family of "groups up to homotopy", based loops $\Omega_x X$ for $x \in X$.
- Today: rotation of loops.
  The circle group S^1 acts on LX by precomposition
- Fixed points are constant loops $X \subset LX$.

Loop Spaces in Physics

- Central role in quantum field theory and string theory

Loop Spaces in Physics

- Central role in quantum field theory and string theory
- e.g. fields in the compactification (Kaluza-Klein reduction) of a field theory along a circle are loops in original fields

Loop Spaces in Physics

- Central role in quantum field theory and string theory
- e.g. fields in the compactification (Kaluza-Klein reduction) of a field theory along a circle are loops in original fields
- e.g. Topological A-model of strings (Gromov-Witten theory) of X symplectic: Morse-Floer theory of LX

Loop Spaces in Physics

- Central role in quantum field theory and string theory
- e.g. fields in the compactification (Kaluza-Klein reduction) of a field theory along a circle are loops in original fields
- e.g. Topological A-model of strings (Gromov-Witten theory) of X symplectic: Morse-Floer theory of LX
- Composition of loops: joining of strings on X.

Loop Spaces in Physics

- Central role in quantum field theory and string theory
- e.g. fields in the compactification (Kaluza-Klein reduction) of a field theory along a circle are loops in original fields
- e.g. Topological A-model of strings (Gromov-Witten theory) of X symplectic: Morse-Floer theory of LX
- Composition of loops: joining of strings on X.
- Rotation of loops key to E. Witten's work on Atiyah-Singer index theorem and elliptic genus.

Loop Spaces in Physics

- Central role in quantum field theory and string theory
- e.g. fields in the compactification (Kaluza-Klein reduction) of a field theory along a circle are loops in original fields
- e.g. Topological A-model of strings (Gromov-Witten theory) of X symplectic: Morse-Floer theory of LX
- Composition of loops: joining of strings on X.
- Rotation of loops key to E. Witten's work on Atiyah-Singer index theorem and elliptic genus.
- The main tool here is equivariant localization.

Equivariant Localization I

M: nice S^1-space.

Equivariant Localization I

M: nice S^1-space.

- Principle (Atiyah-Bott): can recover topological quantities on fixed points M^{S^1} from S^1-invariant quantities on M.

Equivariant Localization I

M: nice S^1-space.

- Principle (Atiyah-Bott): can recover topological quantities on fixed points M^{S^1} from S^1-invariant quantities on M.
- Equivariant cohomology measures S^1-invariant cohomology

Equivariant Localization I

M: nice S^1-space.

- Principle (Atiyah-Bott): can recover topological quantities on fixed points M^{S^1} from S^1-invariant quantities on M.
- Equivariant cohomology measures S^1-invariant cohomology
- Equivariant cohomology $H^*_{S^1}(M)$ is the cohomology of the corrected quotient space M/S^1

Equivariant Localization I

M: nice S^1-space.

- Principle (Atiyah-Bott): can recover topological quantities on fixed points M^{S^1} from S^1-invariant quantities on M.
- Equivariant cohomology measures S^1-invariant cohomology
- Equivariant cohomology $H^*_{S^1}(M)$ is the cohomology of the corrected quotient space M/S^1
- If action not free, pass to product with “harmless” space S^∞:

Equivariant Localization I

M: nice S^1-space.

- Principle (Atiyah-Bott): can recover topological quantities on fixed points M^{S^1} from S^1-invariant quantities on M.
- Equivariant cohomology measures S^1-invariant cohomology
- Equivariant cohomology $H^*_{S^1}(M)$ is the cohomology of the corrected quotient space M/S^1
- If action not free, pass to product with “harmless” space S^∞:
- **Definition:** Equivariant cohomology $H^*_{S^1}(X)$ is ordinary cohomology of the quotient $(M \times S^\infty)/S^1$.

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

- BS^1 is the classifying space of S^1, or “equivariant point”, $BS^1 = S^\infty/S^1 = \mathbb{CP}^\infty$.

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

- BS^1 is the classifying space of S^1, or “equivariant point”, $BS^1 = S^\infty/S^1 = \mathbb{CP}^\infty$.
- $H^*_{S^1}(pt) = \mathbb{Z}[u]$, polynomials in a generator $u \in H^2(\mathbb{CP}^\infty) \simeq \mathbb{Z}$.

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

- BS^1 is the classifying space of S^1, or “equivariant point”, $BS^1 = S^\infty/S^1 = \mathbb{CP}^\infty$.
- $H^*_{S^1}(pt) = \mathbb{Z}[u]$, polynomials in a generator $u \in H^2(\mathbb{CP}^\infty) \simeq \mathbb{Z}$.
- For any M, $H^*_{S^1}(M)$ is a module over $H^*_{S^1}(pt) = \mathbb{Z}[u]$.

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

- BS^1 is the classifying space of S^1, or "equivariant point", $BS^1 = S^\infty/S^1 = \mathbb{CP}^\infty$.
- $H^*_{S^1}(pt) = \mathbb{Z}[u]$, polynomials in a generator $u \in H^2(\mathbb{CP}^\infty) \simeq \mathbb{Z}$.
- For any M, $H^*_{S^1}(M)$ is a module over $H^*_{S^1}(pt) = \mathbb{Z}[u]$.
- **Theorem:** The equivariant cohomology $H^*_{S^1}(M)$, after inverting u, coincides with the cohomology of the fixed points $H^*(M^{S^1})$ tensored by $\mathbb{Z}[u, u^{-1}]$).

Equivariant Localization II

Equivariant cohomology of a point $H^*_{S^1}(pt) = H^*(BS^1)$:

- BS^1 is the classifying space of S^1, or “equivariant point”, $BS^1 = S^\infty / S^1 = \mathbb{CP}^\infty$.
- $H^*_{S^1}(pt) = \mathbb{Z}[u]$, polynomials in a generator $u \in H^2(\mathbb{CP}^\infty) \simeq \mathbb{Z}$.
- For any M, $H^*_{S^1}(M)$ is a module over $H^*_{S^1}(pt) = \mathbb{Z}[u]$.
- **Theorem:** The equivariant cohomology $H^*_{S^1}(M)$, after inverting u, coincides with the cohomology of the fixed points $H^*(M^{S^1})$ tensored by $\mathbb{Z}[u, u^{-1}]$).
- ...relate equivariant cohomology of loop space LX with cohomology of X...

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:
- **Theorem**(J. Jones) The Hochschild homology of cochains on X is $H^*(LX)$, and the cyclic homology pf cochains on X is $H^*_{S^1}(LX)$.

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:
- **Theorem**(J. Jones) The Hochschild homology of cochains on X is $H^*(LX)$, and the cyclic homology pf cochains on X is $H^*_{S^1}(LX)$.
- In hands of algebraic geometers, this leads to a new kind of “loop space”:

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:
- **Theorem**(J. Jones) The Hochschild homology of cochains on X is $H^*(LX)$, and the cyclic homology pf cochains on X is $H^*_{S^1}(LX)$.
- In hands of algebraic geometers, this leads to a new kind of “loop space”:
- replace cochains by any ring, e.g. functions on a variety..

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:
- **Theorem**(J. Jones) The Hochschild homology of cochains on X is $H^*(LX)$, and the cyclic homology pf cochains on X is $H^*_{S^1}(LX)$.
- In hands of algebraic geometers, this leads to a new kind of “loop space”:
- replace cochains by any ring, e.g. functions on a variety..and interpret its Hochschild homology as functions on some space

An Algebraic Take on Loop Spaces

- The cohomology of loop spaces can be accessed algebraically:
- **Theorem**(J. Jones) The Hochschild homology of cochains on X is $H^*(LX)$, and the cyclic homology pf cochains on X is $H^*_{S^1}(LX)$.
- In hands of algebraic geometers, this leads to a new kind of “loop space”:
- replace cochains by any ring, e.g. functions on a variety..and interpret its Hochschild homology as functions on some space
- Result: “derived loop spaces”

Why Derived Loop Spaces?

- Natural analog of loop space in derived algebraic geometry:

Why Derived Loop Spaces?

- Natural analog of loop space in derived algebraic geometry: a melding of algebraic geometry and homotopy theory being developed by J. Lurie, B. Toën, G. Vezzosi,..

Why Derived Loop Spaces?

- Natural analog of loop space in derived algebraic geometry: a melding of algebraic geometry and homotopy theory being developed by J. Lurie, B. Toën, G. Vezzosi,..
- Provides geometric intuition to algebraic theory of Hochschild and cyclic homology

Why Derived Loop Spaces?

- Natural analog of loop space in derived algebraic geometry: a melding of algebraic geometry and homotopy theory being developed by J. Lurie, B. Toën, G. Vezzosi,..
- Provides geometric intuition to algebraic theory of Hochschild and cyclic homology - important in deformation theory, noncommutative geometry, topological field theory, representation theory..

Why Derived Loop Spaces?

- Natural analog of loop space in derived algebraic geometry: a melding of algebraic geometry and homotopy theory being developed by J. Lurie, B. Toën, G. Vezzosi,..
- Provides geometric intuition to algebraic theory of Hochschild and cyclic homology - important in deformation theory, noncommutative geometry, topological field theory, representation theory..
- derived loop picture explains mysterious structures - e.g. Deligne conjecture

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!- construct topological field theories in algebra, derived loops play role of compactification

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!- construct topological field theories in algebra, derived loops play role of compactification
- Key example: mirror symmetry

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!- construct topological field theories in algebra, derived loops play role of compactification
- Key example: mirror symmetry
  Derived loops in complex manifolds are B-model "mirror" to ordinary loops in symplectic manifolds in A-model

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!- construct topological field theories in algebra, derived loops play role of compactification
- Key example: mirror symmetry
  Derived loops in complex manifolds are B-model "mirror" to ordinary loops in symplectic manifolds in A-model
- Same in geometric Langlands program (BZ-Nadler):

More Motivation

- Motivation from topological QFT: can recognize hidden loop symmetries!- construct topological field theories in algebra, derived loops play role of compactification
- Key example: mirror symmetry
  Derived loops in complex manifolds are B-model "mirror" to ordinary loops in symplectic manifolds in A-model
- Same in geometric Langlands program (BZ-Nadler):
- Derived loops in flag manifolds on "Galois side" are Langlands dual to loop groups on "automorphic side"

Context

- Our results hold for algebraic varieties or stacks..

Context

- Our results hold for algebraic varieties or stacks.. over any ring...

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....
- but basic ideas can be illustrated with ordinary manifolds, to which we stick from now on..

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....
- but basic ideas can be illustrated with ordinary manifolds, to which we stick from now on..
- result: funny abstract way to think about calculus on manifolds..

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....
- but basic ideas can be illustrated with ordinary manifolds, to which we stick from now on..
- result: funny abstract way to think about calculus on manifolds.. advantage: general theory allows to import manifold intuition to many exotic settings

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....
- but basic ideas can be illustrated with ordinary manifolds, to which we stick from now on..
- result: funny abstract way to think about calculus on manifolds.. advantage: general theory allows to import manifold intuition to many exotic settings
- Basic idea: get a different notion of loop space by changing POV on S^1:

Context

- Our results hold for algebraic varieties or stacks.. over any ring... or over the “brave new rings” of stable homotopy theory....
- but basic ideas can be illustrated with ordinary manifolds, to which we stick from now on..
- result: funny abstract way to think about calculus on manifolds.. advantage: general theory allows to import manifold intuition to many exotic settings
- Basic idea: get a different notion of loop space by changing POV on S^1: replace S^1 by its combinatorial or algebraic avatars from homotopy theory.

Simplicial approach

First approach - treat the circle combinatorially:

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:
- Maps from two points to manifold M gives $M \times M$

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:
- Maps from two points to manifold M gives $M \times M$
- One arc says the points are equal:

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:
- Maps from two points to manifold M gives $M \times M$
- One arc says the points are equal: cuts out diagonal $\Delta \in M \times M$

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:
- Maps from two points to manifold M gives $M \times M$
- One arc says the points are equal: cuts out diagonal $\Delta \in M \times M$
- Other arc says they're equal again!

Simplicial approach

First approach - treat the circle combinatorially:

- Model S^1 by two points joined by two arcs or arrows
- Interpret points as generators, arcs as imposing relations:
- Maps from two points to manifold M gives $M \times M$
- One arc says the points are equal: cuts out diagonal $\Delta \in M \times M$
- Other arc says they're equal again!
  ..so consider self-intersection $\Delta \cap \Delta$ in $M \times M$..

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0 - need to encode to get derived loop space

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0 - need to encode to get derived loop space
- Perturb: excess intersection encoded by the normal bundle to $M = \Delta$ in $M \times M$..

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0 - need to encode to get derived loop space
- Perturb: excess intersection encoded by the normal bundle to $M = \Delta$ in $M \times M$..which is the tangent bundle TM.

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0 - need to encode to get derived loop space
- Perturb: excess intersection encoded by the normal bundle to $M = \Delta$ in $M \times M$..which is the tangent bundle TM.
  Or rather "$-TM$", virtual bundle..

Simplicial approach II

- $\Delta \cap \Delta$ very nontransverse intersection..expected dimension 0 - need to encode to get derived loop space
- Perturb: excess intersection encoded by the normal bundle to $M = \Delta$ in $M \times M$..which is the tangent bundle TM.
  Or rather "$-TM$", virtual bundle..
- Result: $\mathcal{L}M$ is the odd tangent bundle, i.e., TM considered as a supermanifold.

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating “corrected” functions on $\Delta \cap \Delta$:

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating “corrected” functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating “corrected” functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds - impose equations of both submanifolds

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating “corrected” functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds - impose equations of both submanifolds
.. but nontransversality $\Rightarrow$ *Tor functors* or derived tensor products

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating "corrected" functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds - impose equations of both submanifolds
  .. but nontransversality $\Rightarrow$ *Tor functors* or derived tensor products
- Definition Hochschild homology of ring R is derived tensor product $Tor_{R \otimes R}(R, R)$

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating "corrected" functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds - impose equations of both submanifolds
  .. but nontransversality $\Rightarrow$ *Tor functors* or derived tensor products
- Definition Hochschild homology of ring R is derived tensor product $Tor_{R \otimes R}(R, R)$ - if $R = Fun(M)$, this is corrected $Fun(\Delta \cap \Delta)$

Functions on $\mathcal{L}M$

- Make $\mathcal{L}M$ concrete by calculating “corrected” functions on $\Delta \cap \Delta$:
- Functions on intersection: *tensor products* of functions on the two submanifolds - impose equations of both submanifolds
  .. but nontransversality $\Rightarrow$ *Tor functors* or derived tensor products
- Definition Hochschild homology of ring R is derived tensor product $Tor_{R\otimes R}(R, R)$ - if $R = Fun(M)$, this is corrected $Fun(\Delta \cap \Delta)$
- Calculation (Hochschild-Kostant-Rosenberg): For $R = Fun(M)$, get $\Omega^\bullet(M)$, exterior algebra of differential forms..

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle:

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*:

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*: Replace S^1 by the ring $H^*(S^1)$, its (real) cohomology

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*: Replace S^1 by the ring $H^*(S^1)$, its (real) cohomology
- $H^*(S^1, \mathbf{R}) = \mathbf{R}[\epsilon]/\epsilon^2$, with odd (degree 1) coordinate ϵ..

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*: Replace S^1 by the ring $H^*(S^1)$, its (real) cohomology
- $H^*(S^1, \mathbf{R}) = \mathbf{R}[\epsilon]/\epsilon^2$, with odd (degree 1) coordinate ϵ.. i.e. functions on an odd infinitesimal ray, denoted $\mathbf{R}^{0|1}$.

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*: Replace S^1 by the ring $H^*(S^1)$, its (real) cohomology
- $H^*(S^1, \mathbf{R}) = \mathbf{R}[\epsilon]/\epsilon^2$, with odd (degree 1) coordinate ϵ.. i.e. functions on an odd infinitesimal ray, denoted $\mathbf{R}^{0|1}$.
- Recall: tangent vectors are paths $\mathbf{R} \to M$ modulo ϵ^2 terms..

Cohomological approach

So $Fun(\mathcal{L}M) = \Omega^\bullet(M)$ are functions on odd tangent bundle: linear functions Ω^1 anticommute.

- Can recover answer by treating S^1 *cohomologically*: Replace S^1 by the ring $H^*(S^1)$, its (real) cohomology
- $H^*(S^1, \mathbf{R}) = \mathbf{R}[\epsilon]/\epsilon^2$, with odd (degree 1) coordinate ϵ.. i.e. functions on an odd infinitesimal ray, denoted $\mathbf{R}^{0|1}$.
- Recall: tangent vectors are paths $\mathbf{R} \to M$ modulo ϵ^2 terms..
- So maps $\mathbf{R}^{0|1} \to M$ are the odd tangent bundle $\mathcal{L}M$!

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..i.e., an odd vector field, a derivation of functions of degree 1

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..i.e., an odd vector field, a derivation of functions of degree 1
- Associativity of action: translating twice gives zero..

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..i.e., an odd vector field, a derivation of functions of degree 1
- Associativity of action: translating twice gives zero..
- Case of $\mathcal{L}M$: find a derivation of $\Omega^\bullet(M)$ of degree 1 with square zero...

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..i.e., an odd vector field, a derivation of functions of degree 1
- Associativity of action: translating twice gives zero..
- Case of $\mathcal{L}M$: find a derivation of $\Omega^\bullet(M)$ of degree 1 with square zero...
- ...we've recovered the the de Rham differential d!

Derived loop rotation

S^1 acts on $\mathcal{L}M$ through its homotopical avatars - i.e., action of odd translation $\mathbf{R}^{0|1}$

- An action of $\mathbf{R}^{0|1}$ gives an odd translation, up to first order..i.e., an odd vector field, a derivation of functions of degree 1
- Associativity of action: translating twice gives zero..
- Case of $\mathcal{L}M$: find a derivation of $\Omega^\bullet(M)$ of degree 1 with square zero...
- ...we've recovered the the de Rham differential d!
(in its guise as the Connes differential on Hochschild homology)

Cyclic homology

Consider S^1 equivariant cohomology of derived loops:

Cyclic homology

Consider S^1 equivariant cohomology of derived loops: imposing cohomological invariance under d..

Cyclic homology

Consider S^1 equivariant cohomology of derived loops: imposing cohomological invariance under d..
- goes by the name Cyclic Homology (A. Connes, B. Feigin-B. Tsygan)

Cyclic homology

Consider S^1 equivariant cohomology of derived loops: imposing cohomological invariance under d..
- goes by the name Cyclic Homology (A. Connes, B. Feigin-B. Tsygan)

- **Theorem:** The cyclic homology $H^*_{S^1}(\mathcal{L}M)$ made periodic - inverting $u \in H^*(\mathbb{C}P^\infty)$, coincides with the de Rham cohomology of M tensored by $\mathbb{Z}[u, u^{-1}]$.

Cyclic homology

Consider S^1 equivariant cohomology of derived loops: imposing cohomological invariance under d..
- goes by the name Cyclic Homology (A. Connes, B. Feigin-B. Tsygan)

- **Theorem:** The cyclic homology $H^*_{S^1}(\mathcal{L}M)$ made periodic - inverting $u \in H^*(\mathbb{C}P^\infty)$, coincides with the de Rham cohomology of M tensored by $\mathbb{Z}[u, u^{-1}]$.
- Without localizing: keeping careful track of $\mathbf{Z}$-grading recovers Hodge filtration on de Rham cohomology (BZ-Nadler).

Cyclic homology

Consider S^1 equivariant cohomology of derived loops: imposing cohomological invariance under d..
- goes by the name Cyclic Homology (A. Connes, B. Feigin-B. Tsygan)

- **Theorem:** The cyclic homology $H^*_{S^1}(\mathcal{L}M)$ made periodic - inverting $u \in H^*(\mathbb{C}P^\infty)$, coincides with the de Rham cohomology of M tensored by $\mathbb{Z}[u, u^{-1}]$.
- Without localizing: keeping careful track of **Z**-grading recovers Hodge filtration on de Rham cohomology (BZ-Nadler).
- Get interpretation of calculus and de Rham theory in algebraic, singular, brave new and even noncommutative settings!

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

- Consider S^1-equivariant vector bundles V on $\mathcal{L}M$:

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

- Consider S^1-equivariant vector bundles V on $\mathcal{L}M$: data of isomorphisms $i_\theta : \theta^* V \to V$ for $\theta \in S^1$, satisfying:

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

- Consider S^1-equivariant vector bundles V on $\mathcal{L}M$: data of isomorphisms $i_\theta : \theta^* V \to V$ for $\theta \in S^1$, satisfying:
- continuity in θ,

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

- Consider S^1-equivariant vector bundles V on $\mathcal{L}M$: data of isomorphisms $i_\theta : \theta^* V \to V$ for $\theta \in S^1$, satisfying:
- continuity in θ,
- i_1 is the identity, and

Equivariant vector bundles

BZ-Nadler: Categorify the above story - replace functions by vector bundles and cohomology by K-theory or the derived category..
(motivation from boundary conditions or D-branes in physics, geometric representation theory)

- Consider S^1-equivariant vector bundles V on $\mathcal{L}M$: data of isomorphisms $i_\theta : \theta^* V \to V$ for $\theta \in S^1$, satisfying:
- continuity in θ,
- i_1 is the identity, and
- associativity.

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M,

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^\bullet = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^\bullet = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.
- Equivariance along $\mathbf{R}^{0|1}$ lifts the de Rham differential to sections:

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^\bullet = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.
- Equivariance along $\mathbf{R}^{0|1}$ lifts the de Rham differential to sections:
$E \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \cdots \xrightarrow{\nabla} \Omega^n(E)$

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^\bullet = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.
- Equivariance along $\mathbf{R}^{0|1}$ lifts the de Rham differential to sections:
  $E \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \cdots \xrightarrow{\nabla} \Omega^n(E)$
- Leibniz rule: compatibility with action on functions

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^\bullet = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.
- Equivariance along $\mathbf{R}^{0|1}$ lifts the de Rham differential to sections:
$E \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \cdots \xrightarrow{\nabla} \Omega^n(E)$
- Leibniz rule: compatibility with action on functions
- Associativity forces $\nabla^2 = 0$..

Vector bundles on loops

- Bundle on $\mathcal{L}M$: look like pullback of bundle E from M, i.e., sections $E \otimes \Omega^{\bullet} = E \oplus \Omega^1(E) \oplus \cdots \oplus \Omega^n(E)$.
- Equivariance along $\mathbf{R}^{0|1}$ lifts the de Rham differential to sections:
$E \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \cdots \xrightarrow{\nabla} \Omega^n(E)$
- Leibniz rule: compatibility with action on functions
- Associativity forces $\nabla^2 = 0$..
- S^1 equivariant bundles on $\mathcal{L}M$ give vector bundles with flat connection on M.

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules:

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules: stars of geometric analysis (as systems of algebraic differential equations)

Flat connections and loops

- Can replace vector bundles by suitable sheaves:
singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same,
$\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules:
stars of geometric analysis (as systems of algebraic differential equations) and geometric representation theory (as representations of Lie algebras)

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules: stars of geometric analysis (as systems of algebraic differential equations) and geometric representation theory (as representations of Lie algebras)
- **Theorem:** The category of S^1-equivariant sheaves on $\mathcal{L}M$ is equivalent to the category of $\mathcal{D}$-modules on M

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules: stars of geometric analysis (as systems of algebraic differential equations) and geometric representation theory (as representations of Lie algebras)
- **Theorem:** The category of S^1-equivariant sheaves on $\mathcal{L}M$ is equivalent to the category of $\mathcal{D}$-modules on M
- - more precisely, need to invert $u \in H^*(S^1)$, or keep track of gradings/Hodge filtrations everywhere

Flat connections and loops

- Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
- Flat connections on sheaves defined same, $\nabla : E \to \Omega^1(E)$ satisfying Leibniz with $\nabla^2 = 0$.
- These go by name $\mathcal{D}$-modules: stars of geometric analysis (as systems of algebraic differential equations) and geometric representation theory (as representations of Lie algebras)
- **Theorem:** The category of S^1-equivariant sheaves on $\mathcal{L}M$ is equivalent to the category of $\mathcal{D}$-modules on M
- - more precisely, need to invert $u \in H^*(S^1)$, or keep track of gradings/Hodge filtrations everywhere (and work with derived categories..)

Dropping flatness

Question: What is loop space analog of arbitrary connections??

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

- Solution: weaken notion of S^1-equivariance - drop associativity..

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

- Solution: weaken notion of S^1-equivariance - drop associativity..
- “Nonassociative” action of a group G: bunch of maps labeled by G

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

- Solution: weaken notion of S^1-equivariance - drop associativity..
- “Nonassociative” action of a group G: bunch of maps labeled by G - i.e., action of free group $F(G)$ on underlying set

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

- Solution: weaken notion of S^1-equivariance - drop associativity..
- "Nonassociative" action of a group G: bunch of maps labeled by G - i.e., action of free group $F(G)$ on underlying set
- Our setting: want version of free group preserving continuity and unit..

Dropping flatness

Question: What is loop space analog of arbitrary connections??
Need to lose condition $\nabla^2 = 0$..

- Solution: weaken notion of S^1-equivariance - drop associativity..
- "Nonassociative" action of a group G: bunch of maps labeled by G - i.e., action of free group $F(G)$ on underlying set
- Our setting: want version of free group preserving continuity and unit..
- Solution: X pointed space, $F(X) = \Omega\Sigma X$, based loops on the suspension (James construction)

Connections via loops

- So $F(S^1) = \Omega S^2$...

Connections via loops

- So $F(S^1) = \Omega S^2$...
- For a group G, we have canonical map $F(G) \to G$..

Connections via loops

- So $F(S^1) = \Omega S^2$...
- For a group G, we have canonical map $F(G) \to G$..
- So ΩS^2 acts on $\mathcal{L}M$ via $F(S^1) \to S^1$, can consider equivariant bundles:

Connections via loops

- So $F(S^1) = \Omega S^2$...
- For a group G, we have canonical map $F(G) \to G$..
- So ΩS^2 acts on $\mathcal{L}M$ via $F(S^1) \to S^1$, can consider equivariant bundles:
- **Theorem** ΩS^2-equivariant vector bundles on $\mathcal{L}M$ are canonically identified with arbitrary bundles with connection on M.

Connections via loops

- So $F(S^1) = \Omega S^2$...
- For a group G, we have canonical map $F(G) \to G$..
- So ΩS^2 acts on $\mathcal{L}M$ via $F(S^1) \to S^1$, can consider equivariant bundles:
- **Theorem** ΩS^2-equivariant vector bundles on $\mathcal{L}M$ are canonically identified with arbitrary bundles with connection on M.
- But where is curvature? and what's the deal with ΩS^2??

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.
- $S^2 = \mathbb{C}P^1 \subset BS^1 = \mathbb{C}P^\infty$.

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.
- $S^2 = \mathbb{C}P^1 \subset BS^1 = \mathbb{C}P^\infty$.
- The restriction of the S^1-bundle $S^\infty \to BS^1$ to S^2 is the Hopf fibration $S^3 \to S^2$...

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.
- $S^2 = \mathbb{C}P^1 \subset BS^1 = \mathbb{C}P^\infty$.
- The restriction of the S^1-bundle $S^\infty \to BS^1$ to S^2 is the Hopf fibration $S^3 \to S^2$...
- Our "freeing" S^1 spaces means restricting to $S^2 \subset BS^1$.

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.
- $S^2 = \mathbb{C}P^1 \subset BS^1 = \mathbb{C}P^\infty$.
- The restriction of the S^1-bundle $S^\infty \to BS^1$ to S^2 is the Hopf fibration $S^3 \to S^2$...
- Our "freeing" S^1 spaces means restricting to $S^2 \subset BS^1$.
- The map $\Omega S^2 \to S^1$ is loops Ω applied to $S^2 \to BS^1$.

Freeing the circle

- Recall that $BS^1 = \mathbb{C}P^\infty$, and all S^1-equivariant gadgets "live" over the "equivariant point" BS^1.
- $S^2 = \mathbb{C}P^1 \subset BS^1 = \mathbb{C}P^\infty$.
- The restriction of the S^1-bundle $S^\infty \to BS^1$ to S^2 is the Hopf fibration $S^3 \to S^2$...
- Our "freeing" S^1 spaces means restricting to $S^2 \subset BS^1$.
- The map $\Omega S^2 \to S^1$ is loops Ω applied to $S^2 \to BS^1$.
- The "kernel" of $\Omega S^2 \to S^1$ is the looped Hopf fibration $\Omega S^3 \to \Omega S^2$.

The birth of curvature

- Note that $\Omega S^3 = \Omega \Sigma S^2 = F(S^2)$, the free group on S^2:

The birth of curvature

- Note that $\Omega S^3 = \Omega \Sigma S^2 = F(S^2)$, the free group on S^2:
- So we have "resolved" S^1 by free groups,
  $F(S^2) \to F(S^1) \to S^1$.

The birth of curvature

- Note that $\Omega S^3 = \Omega\Sigma S^2 = F(S^2)$, the free group on S^2:
- So we have “resolved” S^1 by free groups,
  $F(S^2) \to F(S^1) \to S^1$.
  (Truly “resolve” if we keep track of gradings/Hodge filtrations..)

The birth of curvature

- Note that $\Omega S^3 = \Omega\Sigma S^2 = F(S^2)$, the free group on S^2:
- So we have "resolved" S^1 by free groups,
  $F(S^2) \to F(S^1) \to S^1$.
  (Truly "resolve" if we keep track of gradings/Hodge filtrations..)
- An action of $F(S^2)$ on a bundle on $\mathcal{L}M$ is simply an endomorphism of degree two...i.e., an element of $\Omega^2(EndE)$.

The birth of curvature

- Note that $\Omega S^3 = \Omega\Sigma S^2 = F(S^2)$, the free group on S^2:
- So we have “resolved” S^1 by free groups,
  $F(S^2) \to F(S^1) \to S^1$.
  (Truly “resolve” if we keep track of gradings/Hodge filtrations..)
- An action of $F(S^2)$ on a bundle on $\mathcal{L}M$ is simply an endomorphism of degree two...i.e., an element of $\Omega^2(EndE)$.
- **Theorem** Identifying $\Omega S^2 = F(S^1)$-equivariant bundles on $\mathcal{L}M$ with connections (E, ∇) on M, the action of $\Omega S^3 = F(S^2)$ on E is precisely the curvature of ∇.

The birth of curvature

- Note that $\Omega S^3 = \Omega \Sigma S^2 = F(S^2)$, the free group on S^2:
- So we have “resolved” S^1 by free groups, $F(S^2) \to F(S^1) \to S^1$. (Truly “resolve” if we keep track of gradings/Hodge filtrations..)
- An action of $F(S^2)$ on a bundle on $\mathcal{L}M$ is simply an endomorphism of degree two...i.e., an element of $\Omega^2(EndE)$.
- **Theorem** Identifying $\Omega S^2 = F(S^1)$-equivariant bundles on $\mathcal{L}M$ with connections (E, ∇) on M, the action of $\Omega S^3 = F(S^2)$ on E is precisely the curvature of ∇.
- Flatness of ∇ is equivalent to triviality of the $F(S^2)$ action.

The End

Thus the Hopf fibration may be considered the universal source of curvature.

The End

Thus the Hopf fibration may be considered the universal source of curvature.

Thank you for listening!