T. Bridgeland - Flops & Fourier Transforms 11/20/02

<u>Theorem</u>: If Y_1, Y_2 are birationally equivalent smooth projective threefolds $/\mathbb{C}$ with nef canonical bundle (eg. CY trivial canonical bundle) then there is an equivalence of derived categories of coherent sheaves $D(Y_1) \simeq D(Y_2)$

Philosophy: aim to understand varieties via categories of sheaves on them (like studying rings via modules...)

$X \longrightarrow \mathrm{Coh}\, X$ abelian category $\longrightarrow D^b(\mathrm{Coh}\, X)$ triangulated
– determines X — determines birational class: captures basic geometry

A an abelian category
1). can identify A with full subcategory of $D(A)$ &
$\mathrm{Hom}^i_{D(A)}(M,N) := \mathrm{Hom}_{D(A)}(M, N[i]) = \mathrm{Ext}^i_A(M,N)$

X, Y smooth projective varieties $/\mathbb{C}$

<u>Theorem</u> (Bondal-Orlov) If $\pm K_X$ is ample then $D(X)$ determines X (explicitly reconstruct X!!)

(Bridgeland-Maciocia) <u>Theorem</u> If $\dim X \le 2$ then there are only fin many Y with $D(Y) \simeq D(X)$
But there are interesting examples of $D(X) \simeq D(Y)$

How do we construct them?

(uses projective) <u>Theorem</u> (Orlov) Any equivalence $D(Y) \xrightarrow{\Phi} D(X)$ is of the form

$$\begin{array}{ccc} & Y \times X & \\ \swarrow^{\pi_Y} & & \searrow^{\pi_X} \\ Y & & X \end{array}$$

$\Phi(-) = R\pi_{X*}(P \overset{L}{\otimes} \pi_Y^*(-))$ for some $P \in D(Y \times X)$
(any fully faithful exact functor is of this form.. not known for general exact functors)
"Fourier-Mukai transforms"

$n = \dim X = \dim Y$

<u>Theorem</u> (Bondal-Orlov, Bridgeland following Mukai):
Such a functor is an equivalence iff
a. $\mathrm{Hom}^i_{D(X)}(\Phi\mathcal{O}_{y_1}, \Phi\mathcal{O}_{y_2}) = 0$ unless $y_1 = y_2$, $0 \le i \le n$
b. $\mathrm{Hom}_{D(X)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y) = \mathbb{C}$
c. $\Phi\mathcal{O}_y \otimes \omega_X \simeq \Phi\mathcal{O}_y \quad \forall y \in Y$

- why? $\otimes \omega_X$ is Serre functor, purely categorical, $\Phi \mathcal{O}_y$ must be invariant since points are.

Suppose eg $\omega_X = \mathcal{O}_X$. Q: when is a matrix orthogonal? a, b: columns of matrix are orthonormal basis.. clearly necessary.. initially also need conditions that rows are orthonormal!: necessary if don't assume dimensions are equal!
square matrix: column condition sufficient (implies rows)
i.e. have to establish notion of dimension.
a, b $\Rightarrow \Phi$ fully faithful (like matrix being embedding)
$D(Y) \hookrightarrow D(X) \Rightarrow$ break up $D(X) = \langle \Phi D(Y), D(Y)^\perp \rangle$

c: $D(Y)^\perp$ right perp is also a left perp. if both right & left $\Rightarrow$ direct sum decomposition, know impossible on smooth proj variety
Automatic that inverse is the adjoint.
$\Rightarrow$ Y is a fine moduli space for the objects $\{\Phi \mathcal{O}_y \; y \in Y\}$
- since Φ is an integral kernel it must preserve families.. apply Φ^{-1} get family of points in Y $\Rightarrow$ map to Y.

i.e. S scheme $\mathcal{P} \in D(S \times X)$ s.t. $\mathcal{P}|_{\{s\} \times X} \in \{\Phi \mathcal{O}_y \; y \in Y\}$ for all $s \in S$ (derived restriction)
then $\exists$ map $S \xrightarrow{f} Y$ s.t. $\mathcal{P}|_{\{s\} \times X} = \Phi \mathcal{O}_{f(s)} \otimes \pi_X^* L$
L line bundle
or better $\mathcal{P} \simeq (f \times 1_X)^* \mathcal{P}$ ... careful about pullback of sheaves from S.
well really maybe not dg scheme to understand collisions of skyscrapers

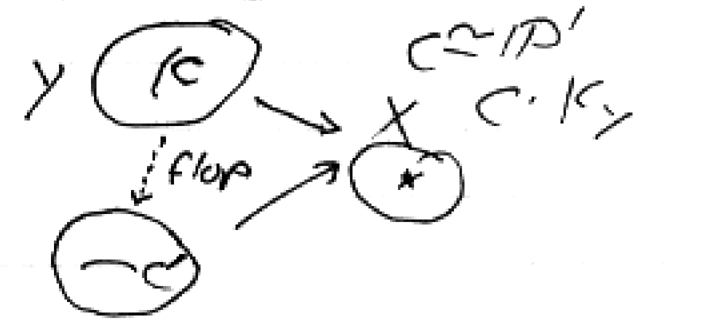
Examples 1. X an abelian variety, $Y = \hat{X}$ the dual variety
$\Rightarrow$ Y = moduli of deg 0 line bundles on X, $\{P_y : y \in Y\}$
- need to check $H^i(X, P_y) = 0 \;\forall i$ if $y \neq 0$
[Mukai:]

2. X a K3 surface, Y a fine 2-dim moduli space of stable sheaves on X, projective, then $D(Y) \xrightarrow{\sim} D(X)$.
Check that $Ext^i_X(P_{y_1}, P_{y_2}) = 0$ for $y_1 \neq y_2$
Hom vanishes by slope, Ext^2 by Serre duality
& Ext^1 since $\chi(P_{y_1}, P_{y_2}) = \chi(P_y, P_y) = 1-2+1 = 0$.
[Mukai]

<u>Minimal</u> <u>models</u> find good model in each birational class...
<u>Curves</u>: have <u>smooth</u> model
<u>Surfaces</u>: don't have unique smooth model due to blowups:
have (-1)-curves $K \cdot C = -1$, blow down
get minimal model X.
either: • X is $\mathbb{P}^2$ or ruled $\longleftrightarrow$ $C \cdot C = -1$ since $g(C)=0$
or • K_X is nef : $K_X \cdot C \geq 0$ $\forall$ curves $C \subset X$
– in second case minimal model is unique. (numerically effective)
<u>dim = 3</u> Mori theory, can still obtain a minimal model by a succession of divisorial contractions and flips (codim 2 surgeries..)
$\boxed{\text{Dichotomy}}$: X is Fano ($-K_X$ ample) or ruled or del Pezzo fibration
OR K_X nef. – very difficult.
Minimal model not unique in either case <u>&</u> must allow singularities

<u>Example</u> of flop (minimal model not unique)
$X = (xy - uv) \subset \mathbb{C}^4$, singular at origin

$\frac{x}{u} = \frac{v}{y} : X \dashrightarrow \mathbb{P}^1$ rational function, take closure of graph of this, get resolution single rational curve contracted
or use $\frac{x}{v} = \frac{u}{y}$



Theorem (Kollár) If Y_1, Y_2 smooth proj. threefolds with K_{Y_i} nef, then any birational equivalence factors as a finite chain of flops.
• $W \overset{f}{\dashrightarrow} Y$, $g \searrow X \swarrow f$ f,g contract only finitely many curves $C \cong \mathbb{P}^1$, $K \cdot C = 0$

- Can also assume D f-ample $\Rightarrow -\phi^{-1}(D)$ g-ample (positive curves $\leftrightarrow$ negative curves).

So want to show that for any flop get equivalence of derived categories: exhibit W as a moduli space of objects in $D(Y)$

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, "g"] & Y \arrow[dl, "f"] \\ X & \end{tikzcd}$$

There is an abelian subcategory $Per(Y/X) \subset D(Y)$

$E \in Per(Y/X) \iff$ a. $H^i(E)=0$ unless $i=0,-1$

b. $H^i(Rf_*(E))=0$ unless $i=0$

c. $Hom_Y(H^0(E), C)=0$ for all sheaves C on Y with $Rf_* C=0$

Write $D(Y) = D(X) \perp \langle \mathcal{O}_C(-1) \rangle$ — things which push down to zero

and tilt: instead of inclusion & restriction of open sets glue wrt pullback & pushforward

$Lf^*: D^-(X) \hookrightarrow D^-(Y)$, has orthogonal subcategory

shift one of these.

Def A perverse point sheaf is an object $E \in Per(Y/X)$ numerically equivalent to $\mathcal{O}_y$, s.t. there is a surjection $\mathcal{O}_Y \twoheadrightarrow E$ in $Per(Y/X)$

Theorem The flop W is the fine moduli space for perverse point sheaves on Y, & the universal object defines an equivalence $D(W) \to D(Y)$

Van den Bergh: coherent sheaf of algebras (noncomm.) so that W is moduli of modules. (two by two matrix algebra)

Need $Y \to X$ crepant of rel dimension 1

$$\begin{tikzcd} & W \times_X Y \arrow[dl] \arrow[dr] & \\ W \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

different types of flops -- for simplest flops $W \times_X Y$ is smooth eg in toy example of quadric ..

Look at

$$\begin{tikzcd} & W \times_X Y \arrow[dl] \arrow[dr] & \\ W & & Y \end{tikzcd}$$

pullback & pushdown is equivalence (Bondal-Orlov) some times

~~Reid's~~ [illegible]

Bondal Orlov proved in case of normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ or $\mathcal{O}\oplus\mathcal{O}(-2)$ or $\mathcal{O}(1)\oplus\mathcal{O}(-3)$ (reduces to first two by "Reid's Pagoda")

Compute in this case images $\Phi(\mathcal{O}_w)$: $H^{-1}(\Phi(\mathcal{O}_w)) = \mathcal{O}_C(-1)$

$H^0(\Phi(\mathcal{O}_w)) = \mathcal{O}_C$

- guess categorical meaning of this answer.

$f: Y \to X$ morphism with $f^*\omega_X = \omega_Y$

then $Lf^*, Rf_*, f^!$ adjoint compare f^* & $f^!$ using ω

(coincide on perfect complexes, ~~up to a shift~~ but not on all...)

Important fact : X has rational singularities, ie $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ (& Y smooth).

$$\Rightarrow \quad Rf_* \circ Lf^* = id_X$$

(Elkik: canonical sings are rational)

So $Lf^* : D(X) \to D(Y)$ is fully faithful

problem: Rf_* doesn't take perfect to perfect....

Should consider unbounded complexes (on left).

Whenever have fully faithful functor with adjoint in triangulated category

$$f^* \circ f_* E \longrightarrow E \longrightarrow C = \text{cone} \qquad \Rightarrow \qquad f_* f^* f_* E \xrightarrow{\sim} E, \quad \boxed{f^* C = 0}$$

[ω_Y gives same functor on perfect complexes - but f_* doesn't give perfect $\Rightarrow$ pass to unbounded but then f^* doesn't exist ....]

Any rational sings: $Rf_*\mathcal{O}_Y = \mathcal{O}_X$

$D^-(X) \xrightarrow{f^*} D^-(Y)$ $\quad \mathcal{C} = \{ C \in D^-(Y) : f_*(C) = 0 \}$

Then every $E \in D^-(Y)$ sits in a unique triangle

$A \to E \to C$ $\quad A \in \mathcal{A} = D^-(X)$, $C \in \mathcal{C}$ $\quad D(X) = D(Y)/\mathcal{C}$ semi-orthogonal

no homs between $\mathcal{A}, \mathcal{C}$.

Beilinson: if have one more adjoint can glue t-structures on $\mathcal{A}, \mathcal{C}$ to t-structure on big category

Perverse world

$$\underset{\text{open}}{U} \overset{i}{\hookrightarrow} Z \overset{j}{\hookleftarrow} \overset{\text{closed}}{T}$$

$$D(U) \xleftarrow{j^*} D(Z) \xhookleftarrow{i_*} D(T) \qquad D(U) \simeq D(Z)/D(T)$$

embedding as thick subcategory

$$\left(\longleftrightarrow \; D(X) \xleftarrow{f_*} D(Y) \hookleftarrow \mathcal{C} \; \right)$$

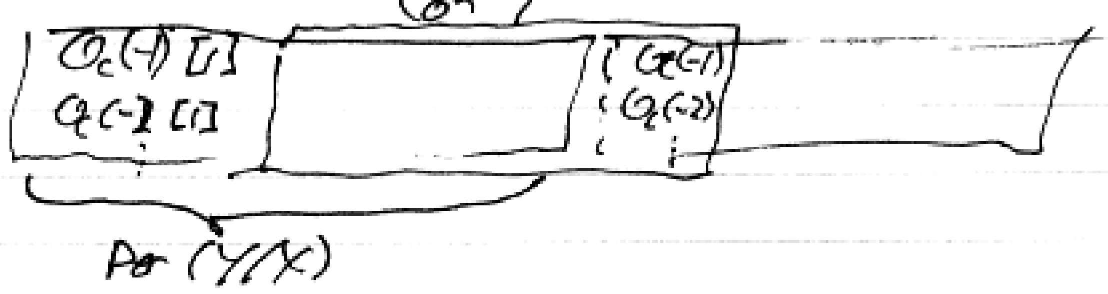
Whenever have two adjoints (equivalent on either side)

$$D(X) \leftrightarrows D(Y) \leftrightarrows \mathcal{C}$$

$\Rightarrow$ get t-structures from gluing any two t-structures on $D(X)$, $\mathcal{C}$.

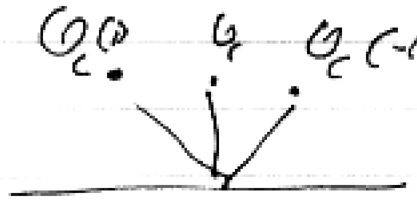
Don't need crepant, only rational singularities so $D(X) = D(Y)/\mathcal{C}$.

Coherent perverse sheaves $Per(Y/X)$ are here by shifting t-structure on quotient by 1.



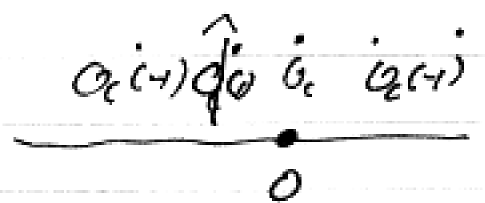
Douglas picture: abelian category gets tilted as we move around moduli space of CFTs.

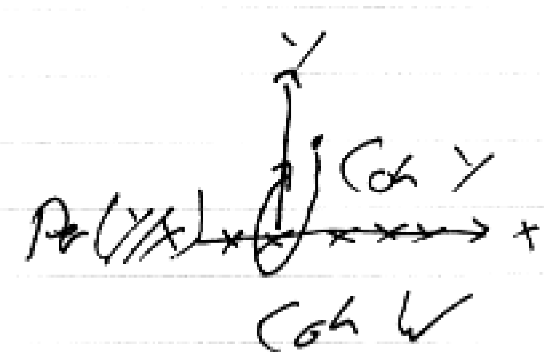
"Flop corresponds to an elementary monodromy" (Picard-Lefschetz-type transformation on the derived category)
- reflection wrt spherical object



Write $Z(\mathcal{O}_C(-k)) = y - kx$ coords on $\mathbb{R}^2$

$y > 0$





as y crosses axis get

Objects $\mathcal{O}_C(-k)$ are all spherical!

$$\mathrm{Ext}^i_Y(\mathcal{O}_C(-k), \mathcal{O}_C(-k)) = \begin{cases} \mathbb{C} & i = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

- get monodromy around them tells you where to tilt.

T. Bridgeland Office Hours

$$\begin{array}{l} X \\ \downarrow \\ Y \end{array}$$ crepant resolution (rel dim 1) $\Rightarrow$ can define $\mathrm{Per}(X/Y)$

Gorenstein

Nakajima: monadic construction of sheaves on $\mathbb{C}^2/\Gamma \to \mathbb{C}/\Gamma$

Drinfeld: interpret as perverse sheaves.

In ADHM have nondegeneracy ensuring one gets a vector bundle from monad — without it get coherent (middle perversity) perverse sheaves — look on $\mathbb{P}^2$ complexes in $D^b(\mathbb{P}^2)$ s.t.

1. outside fin set are vector bundle in deg 0
2. at points set H^1 torsion, ~~deg~~

& Dual complex $R\mathrm{Hom}(-, \mathcal{O})$ get object of same type

- eg any finite subscheme its ideal is OK or vector bundle + map to skyscraper $\Rightarrow$ dual lies in this category

Not abelian category...

Student of Abramovich: Bridgeland functor is given by pullback & pushforward from fiber product $W \times_X Y$ even when not smooth

any smooth W

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "\text{birational}"'] & & Y \end{tikzcd}$$

(proper birational equiv of smooth varieties) is always $\pi_* \mathcal{O} = \mathcal{O}$ gives same ~~equiv~~ functor of derived categories

- in fact should rather take the graph of $X \dashrightarrow Y$ & use it for ~~[illegible]~~ functor on derived categories.
- this gives equivalence, while smooth resolution W doesn't give equivalence in general!

$$\begin{tikzcd} W \arrow[dr, "g"'] & & Y' \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

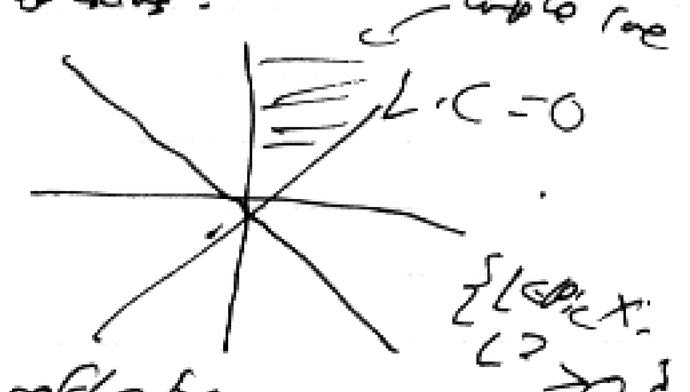
$\mathrm{Per}(Y/X) \subset D(Y)$

If apply construction in opposite order don't get inverse to our functor: composition is twist functor by spherical object assoc to

$$\begin{tikzcd} Y \arrow[d, "f", "\text{rel dim } 1"'] \\ X \end{tikzcd}$$

- inverse to $P_2 \times P_1^*$ is $P_1 \times P_2^!$ rather: have to twist by canonical bundle.

G/B flag variety with Schubert correspondences $\rightarrow$
functors on perverse sheaves, all give equivalences,
but $()_*$, $()_!$ interchanged — naively from Weyl group
but action on derived category gives braiding.

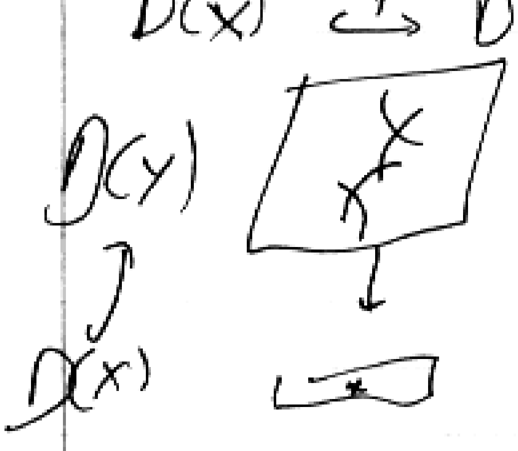


Flops live in chamber structure.
Surface case $\{L^2 > 0\}$ divisors

Coherent sheaves on X correspond to 1 chamber
$C^2 = -2$ curve Picard-Lefschetz reflections
— "Hecke algebra for Borcherds algebra"

-2 ~~surfaces~~ curves. $\mathcal{O}_C(-1)$ gives spherical objects
Each chamber should correspond to some perverse
category in $D(X)$ — usual action corresponds
to ample cone...
-2 curve $\Rightarrow$ equivalence of derived category,
acts on K-group as reflection, on derived category
get braiding. If all curves intersect in A_n chain
get A_n braid group.
Groupoid of all autoequivalences have give action
on $D(X)$, automorphisms of a single one,
Isom classes of manifolds $\longrightarrow$ t-structures on given Δ-cat / autoequivalences

(other manifolds $\longleftrightarrow$ simple objects in heart
of t-structures... with some duality conditions)
All these coming from flops act by identity on
$D(X) \xrightarrow{f^*} D(Y)$



autoequivalences preserving $D(X)$ are braid
group (A_n type here) interchanging curves

$D(X) \hookrightarrow D(Y) \twoheadrightarrow \mathcal{C}$ quotient: $D6$
Category of modules for some algebra get
braid ~~category~~ actions (Seidel-Thomas?) on derived categories
of special fibre, its formal nbhd.

$H^2(\mathcal{O}^*)$ -- not alg group with Lie algebra $H^2(\mathcal{O})$
- doesn't exist in algebrogeometric sense.
generic pt of $\mathcal{O}^*$-gerbe $\to$ K^*-gerbe,
can look in étale topology, + generically, some power
is trivial - comes from gerbe over roots of unity
Complex analytically $\to$ get only torsion parts, not
continuous objects... can't see in étale world (unless
p-adic...
Analytically: sheaf of categories with no global
sections from analytic exponentiation, coherent category
will be supported on curves, no vector bundles...
Pf. look at Determinant line bundle, will live on n times the gerbe

---

Mutations on $\mathbb{P}^2$ eg act on set of exceptional
collections $\Rightarrow$ on set of t-structures give braid
group action, but not autoequivalences of derived category,
BUT if take $K3 \to\to \mathbb{P}^2$ branched cover
these exceptionals pull back to spherical objects
give twist functors, which do act (braid g) by
autoequivalences

T. Bridgeland – McKay correspondence 11/26/02

M smooth q-proj dim n, $G \subset \operatorname{Aut} M$ finite

~~Gonzalez-Sprinberg/Verdier~~ :

Nakamura's G-Hilb: describe resolution $Y \xrightarrow{f} X$ via moduli of equivariant sheaves

<u>Def</u> A G-cluster = G-invariant 0-dim subscheme $Z \subset M$ s.t. $H^0(M, \mathcal{O}_Z) \simeq$ regular rep

$G \subset \operatorname{Aut} M$ finite subgroup s.t. ω_M loc triv as G-sheaf
$Y \subset G\text{-Hilb}(M)$ crepant resolution of $X = M/G$ providing $\dim Y \times_X Y \le n+1$. In this case $\Rightarrow$ equivalence
$D(Y) \xrightarrow{\sim} D^G(M)$
($X = M/G$ always Gorenstein)

- Condition always holds if $\dim M = 3$

$\Phi : D(Y) \to D^G(M)$ given by

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[dr, "f"'] & & M \arrow[dl, "\pi"] \\ & M/G & \end{tikzcd}$$

$\Phi(-) = Rq_* \circ p^*(-)$ $\quad \Phi(\mathcal{O}_y) = \mathcal{O}_{Z_y}$
$Z \subset Y \times M$ universal cluster subscheme.

$G\text{-Hilb}(M) \subset \operatorname{Hilb}^{|G|}(M)$,
$\searrow$ Hilbert-Chow
M/G

Need to check: • Y is smooth • $G\text{-Ext}^i_M(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}) = 0$
for $y_1 \neq y_2$ • $G\text{-Hom}_M(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_y}) = \mathbb{C}$

Ext condition doesn't hold in general – eg in dim = 4 !
special to dim = 3.

Key ingredient: The Intersection Theorem (Hochster - Roberts)
Let (A, m) be a local $\mathbb{C}$-algebra of dimension d.
Suppose that $0 \to M_s \to M_{s-1} \to \cdots \to M_1 \to M_0 \to 0$ is a nonexact complex of f.g. free A-modules, with each homology module $H_i(M_\cdot)$ of finite length $/A$. <u>Then $s \ge d$.</u>
Moreover if $s = d$ & $H_0(M_\cdot) = A/m \Rightarrow$
$H_i(M_\cdot) = 0 \quad \forall i \neq 0$ & A is regular

(part 1) $\Rightarrow$ ~~X~~ a scheme of finite type $/\mathbb{C}$, $E \in D^b(Coh_Y)$
then homological dimension $(E) \geq$ codim$(\bigcup$ support $H^i(E))$

Y Cohen-Macaulay – follows from depth considerations... highly nontrivial in general. Conjectured by Serre
Don't need to assume perfect complex – otherwise hom dimension is infinite!
Pf: tensor with maximal Cohen-Macaulay module, then get things of depth d...

(part 2) $\Rightarrow$ ($0 \neq E \in D^b(Coh_Y)$ is supported at a closed point in $Y \ni y$ & has hom dim $\leq n = \dim Y$, and $H^0(E) = \mathcal{O}_y$
$\Rightarrow$ Y is smooth at y (more precisely bottom cohomology should be $\mathcal{O}_y$.) .

Generalization of Serre criterion: structure sheaf of point has finite hom dimension $\Leftrightarrow$ scheme smooth at that point – this is generalization to complexes.

$$\begin{array}{c}
 \\
G\text{-Hilb} = Y
\end{array}
\quad
\begin{tikzcd}
 & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y \arrow[dr, "f"'] & & M \circlearrowleft G \arrow[dl] \\
 & X = M/G &
\end{tikzcd}$$

p — flat map (universal subscheme)

$\Phi(\ ) = Rq_* p^*(-)$

$\Phi : D(Y) \to D^G(M)$ has a left adjoint by Grothendieck duality, $\Psi : D^G(M) \to D(Y)$

$Z \subset Y \times M$, $Y \leftarrow \ \to M$

$\Phi(-) = R\pi_{M*}(\mathcal{O}_Z \otimes \pi_Y^*(-))$

$\Psi(-) = [R\pi_{Y*}(\mathcal{O}_Z^\vee \otimes \pi_M^* \omega_M [n] \otimes \pi_M^*(-))]^G$

(need that q_* has a right adjoint by Grothendieck duality)

Composition $\Psi \circ \Phi$ is of the form $R\pi_{2*}(Q \otimes \pi_1^*(-))$ for some $Q \in D(Y \times Y)$ – ie functor is given by kernel.
$$\begin{tikzcd}
 & Y \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y & & Y
\end{tikzcd}$$

$\Psi \Phi \mathcal{O}_y = Q|_{\{y\} \times Y}$ derived restriction.
(Want to prove $Q =$ structure sheaf of diagonal.)

$Hom^i_{D(Y \times Y)}(Q, \mathcal{O}_{(y_1, y_2)})$ restriction of Q to (y_1, y_2)
$= Hom^i_{D(Y)}(\Psi \Phi \mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = G\text{-}Ext^i_M(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}})$

ie have object Q whose cohomology parametrizes
G-Ext groups of the $\mathcal{O}_{Z_y}$: ...

If $y_1 \neq y_2 \Rightarrow G\text{Hom}_M(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}) = 0$
(no Homs between distinct clusters, even if coincide set-theoretically
- because each $\mathcal{O}_{Z_{y_i}}$ is regular representation, so
$1 \to 1$, so get isomorphism)
and so $G\text{-Ext}^n_M(\mathcal{O}_{Z_{y_2}}, \mathcal{O}_{Z_{y_1}}) = 0$ (Serre duality)
... canonical bundle trivial on orbit! $\therefore \omega_M$ locally
trivial as G-sheaf ...

So $Q|_{Y\times Y - \Delta}$ has homological dim $\leq n-2$:
derived restriction to points vanishes ~~except~~ between $0-n$
& away from diagonal $0, n$ vanish also.
But $G\text{-Ext}^q(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}) = 0$ unless $f(y_1) = f(y_2)$
ie unless $\text{Supp}(Z_{y_1}) = \text{Supp}(Z_{y_2})$.

So $\text{Supp}\, Q \subset Y \times_X Y$ which has codim $\geq n-1$ by assumption.

This contradicts Intersection theorem unless $\text{Supp}\, Q \subset \Delta$
~~& Y is smooth!~~
- non vanishing exts have to occur on reasonably large varieties!
Now $\Psi\Phi \mathcal{O}_y$ is supported at $y \in Y$ and has
hom dim $\leq n$. Need to check $H^0(\Psi\Phi\mathcal{O}_y) \cong \mathcal{O}_y$

(in principle Ext groups could range from 0 to n, at
endpoints get 0 outside Δ, on diagonal $\leq n$.
Adjoint functor used just to show Exts depend nicely on points!)

By adjunction get map $\Psi\Phi\mathcal{O}_y \to \mathcal{O}_y$, take cone
$C \to \Psi\Phi\mathcal{O}_y \to \mathcal{O}_y$ Just Need $H^0(C) = 0$ (by positive support of cohomology)

- apply $\text{Hom}_{D(Y)}(-, \mathcal{O}_y)$

and observe $\text{Hom}^1_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \text{Hom}^1_{D(Y)}(\Psi\Phi\mathcal{O}_y, \mathcal{O}_y)$
$=$ $\qquad\qquad\qquad\qquad ||$
this is Kodaira-Spencer map hence is injective! $\qquad \text{Hom}^1_{D^G(M)}(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_y})$

Not true in ~~h~~ higher dimensions! only expect the stack as resolution of singular variety, won't have crepant resolutions ... except in symplectic setting (Haiman) If don't have crepant resolution don't have candidate for smooth model of the stack..