T. Bridgeland - Stability & Derived Categories 11/14/02

Stability condition on a triangulated category - motivation from physics. idea of M.R. Douglas on D-branes : Dirichlet branes, homological mirror symmetry & stability (ICM)

Space of stability conditions is roughly complex moduli of mirror variety...

Example: X smooth proj. curve $/\mathbb{C}$

$\mathrm{Coh}(X)$ - category of coherent sheaves on X

$E = T \oplus F$ T = torsion, F = locally free

(more canonically an extension $T \hookrightarrow E \twoheadrightarrow F$ splits)

T is finite length (extensions of skyscrapers)

$K(X) = K(\mathrm{Coh}(X)) \underset{\text{rank}}{\xrightarrow{\deg}} \mathbb{Z}$ rank $(T)=0$ $\deg(T)$ = length

Slope of $E \neq 0$ $\mu(E) = d(E)/rk(E)$, $\mu(T) = +\infty$.

$0 \neq E \in \mathrm{Coh}(X)$ is (semi)stable if for any $0 \neq A \subsetneq E$ one has $\mu(A) < \mu(E)$ $(\leq)$

- All torsion sheaves are semistable, stable iff skyscraper
- semistable $\rightarrow$ vector bundle or torsion

∴ if E not torsion slope $< \infty$ so can't have torsion subsheaf to be semistable

Any $0 \neq E \in \mathrm{Coh}\, X$ has a Harder - Narasimhan filtration

$0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = E$ with each $F_i = E_i/E_{i-1}$ semistable $\mu(F_1) > \mu(F_2) > \dots > \mu(F_n)$

- uniquely defined ... eg E semistable $\rightarrow E = E_1$.

$\sim$ any chain $\dots \subset E_2 \subset E_1 \subset E$ with $\mu(E_i) \geq \mu(E_{i-1})$ must stabilize ∴ the rank has to stabilize.

Construction: E_1 = subsheaf with largest slope...

- Category of semistable objects of fixed slope is finite length ....

- all generalizes to abelian category with slope function. [Char 0 not necessary]

Consider the triangulated category $D(X) = D^b(\mathrm{Coh}\, X)$ with truncation functor $\tau_{\leq i}$:

$\tau_{\le i}(\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \cdots)$
$= (\cdots \to M^{i-1} \xrightarrow{d^{i-1}} \ker d^i \to 0 \to \cdots)$

Then $H^j(\tau_{\le i} E) = \begin{cases} H^j(E) & j \le i \\ 0 & j > i \end{cases}$

Triangles
$$\begin{tikzcd} \tau_{\le i-1} E \arrow[r] & \tau_{\le i} E \arrow[d] \\ & H^i E[-i] \arrow[ul, dashed, "\deg 1"] \end{tikzcd}$$

In $D^b(X)$:
$N >> 0$
$$\begin{tikzcd} 0 = \tau_{\le -N} E \arrow[r] & \tau_{\le -(N-1)} E \arrow[r] \arrow[d] & \cdots \arrow[r] & \tau_{\le N-1} E \arrow[r] & \tau_{\le N} E = E \arrow[d] \\ & H^{-(N-1)} E[N-1] \arrow[ul, dashed] & & & H^N E[-N] \arrow[ul, dashed] \end{tikzcd}$$

Canonical filtration, way to construct object from extensions of sheaves. no morphisms between the steps.

Write Harder-Narasimhan in same way $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$
$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\ & F_1 \arrow[ul, dashed] & F_2 \arrow[ul, dashed] & & & F_n \arrow[ull, dashed] \end{tikzcd}$$

Combining these two ideas gives a filtration of any $0 \ne E \in D(X)$ by shifts of semistable sheaves.

Define a linear map $Z : K(X) \to \mathbb{C}$ or $\mathbb{R}^2$ ($\mathbb{R}$ appears as Dedekind cuts)

$$Z([E]) = -\deg E + \sqrt{-1} \operatorname{rank} E$$

$\longrightarrow$ this is where actual non-zero sheaves land: rank ≥ 0, if rank $= 0$ deg > 0.

Define $\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0,1]$
for any $0 \ne E \in \operatorname{Coh} X$ — phase of E (phase of torsion $= 1$)
Note $\phi(E_1) > \phi(E_2) \iff \mu(E_1) > \mu(E_2)$

Define $\phi(E[j]) = \phi(E) + j$ phase function
— <u>not</u> defined on all objects, eg $E \oplus E(1)$ has $Z = 0$ …

Now every $0 \ne E \in D(X)$ has a unique filtration

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with A_i shifts of semistable sheaves with $\phi(A_1) > \cdots > \phi(A_n)$

(more naturally ϕ's should be indices of filtered pieces).

Note $Z(E[1]) = -Z(E)$ ..

<u>Def</u> A stability condition on a triangulated category $\mathcal{T}$ consists of a linear map $Z : K(\mathcal{T}) \longrightarrow \mathbb{C}$, the <u>central charge</u>, and full subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$ ("semistable objects of phase t") satisfying:

a. $\mathcal{P}(t)[1] = \mathcal{P}(t+1)$

b. $E \in \mathcal{P}(t) \Rightarrow Z(E) = m(E) e^{i\pi t}$ for some $m(E) > 0$
(ie $Z(E)$ has phase t)

c. If $t_1 > t_2$ and $E_i \in \mathcal{P}(t_i)$ then $\mathrm{Hom}_{\mathcal{T}}(E_1, E_2) = 0$
no homs to objects of smaller slope

d. $\forall\, 0 \neq E \in \mathcal{T}$ there are real numbers $\phi_1 > \phi_2 > \dots > \phi_n$ and a filtration

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$ $1 \leq i \leq n$

e. $Z\left(\bigcup_{t \in \mathbb{R}} \mathcal{P}(t)\right) \subset \mathbb{C}$ has no limit points. "finite mass gap"

... objects in $\mathcal{P}(t)$ are the branes.
Follows from axioms that filtration is unique in (d)
$|Z|$ = mass of the brane

Define $\phi^+(E) = \phi_1$, biggest, $\phi^-(E) = \phi_n$ smallest argument

$\phi^+(E) = \phi^-(E) \Longleftrightarrow E$ is semistable of phase $\phi^+(E)$.

Define "mass" $m(E) = \sum_i |Z(A_i)|$. $m(E) \geq |Z(E)|$, equality if E semistable.

Let $\Sigma(\mathcal{T})$ be the set of stability conditions on $\mathcal{T}$ up to equivalence (equivalent if semistable object in one is semistable of same phase & mass in other.)

Distance: $d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{T}} \left\{ |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, |\phi^-_{\sigma_1} - \phi^-_{\sigma_2}|, \log \left| \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)} \right| \right\} < \infty$

$d(\sigma_1, \sigma_2) = 0 \Rightarrow \sigma_1 = \sigma_2$

· action of autoequivalences of the category $\mathcal{T}$ by isometries & of $GL^+(2,\mathbb{R})_+$

$\Sigma(\mathcal{T}) \longrightarrow K(\mathcal{T}, \mathbb{C})^*$ $\quad (Z, \mathcal{P}) \longrightarrow Z$ is "covering map"

Def A slope function on an abelian category $\mathcal{A}$ is a linear map $Z : K(\mathcal{A}) \to \mathbb{C}$ such that $0 \neq E \in \mathcal{A} \Rightarrow$ $\operatorname{Im} Z(E) > 0$ or $\operatorname{Im} Z(E) = 0$ & $\operatorname{Re} Z(E) < 0$

Proposition A stability condition on a triangulated category $\mathcal{T}$ determines a bounded nondegenerate t-structure on $\mathcal{T}$ & a slope function on its heart. Conversely a bounded nondeg t-structure with a discrete slope function determines a stability condition
↳ image is discrete

t-structure on $\mathcal{T}$: full subcategory $\mathcal{T}^{\leq 0}$ with $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ s.t. inclusion $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ has right adjoint $\Rightarrow$ decomposition

$$\begin{tikzcd} A \arrow[r] & E \arrow[dl] \\ F, B & \end{tikzcd}$$

$, \operatorname{Hom}(\mathcal{T}^{\leq 0}, B) = 0$

Semiorthogonal decompositions give $\mathcal{T}$ structure.

Nondegeneracy: $\bigcap_{i \in \mathbb{Z}} \mathcal{T}^{\leq 0}[i] = 0$

$\Leftrightarrow$ object is nonzero iff nonzero object has a nonzero cohomology sh.

Given a stability condition & an interval $I \subset \mathbb{R}$ define $\mathcal{P}(I)$ to be the set of objects E with $\phi_i(E) \in I$ $1 \leq i \leq n$

$\Rightarrow$ $\mathcal{P}(>\phi)$ is a t-structure for any $\phi \in \mathbb{R}$

- family of t-structures parametrized by real numbers, compatible with translation

t-structure on $\mathcal{T}$ gives categories $\ldots \subset \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1} \subset \ldots$ "scale" on $\mathcal{T}$. Now we have categories $\mathcal{T}^{\leq a}$ for $a \in \mathbb{R}$ $a < b$ $\mathcal{T}^{\leq a} \subset \mathcal{T}^{\leq b}$, etc., compatible with shift by 1.

— follows from stability condition, by forgetting mass info & linearity of Z function

Heart of $\mathcal{P}(>0)$ is $\mathcal{P}((0,1])$, precisely things mapped to upper half plane

Z of semistables has discrete image automatically but not necessarily for linear combinations!

$\widetilde{GL}^+(2,\mathbb{R}) = \{ (f,T) : f:\mathbb{R}\to\mathbb{R}$ increasing, $f(t+1)=f(t)+1$ & $T \in GL^+(2,\mathbb{R})$ s.t. f lifts T on space S^1 of rays in $\mathbb{R}^2$ to $\mathbb{R}\}$

universal cover of $GL^+(2,\mathbb{R})$ positive determinant

This acts by $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$, $Z' = T^{-1} \circ Z$

Translation part: $f(t)=t+a$: just moving phases, rotating central charges. 360° rotation ⟷ double shift $E \mapsto E[2]$.

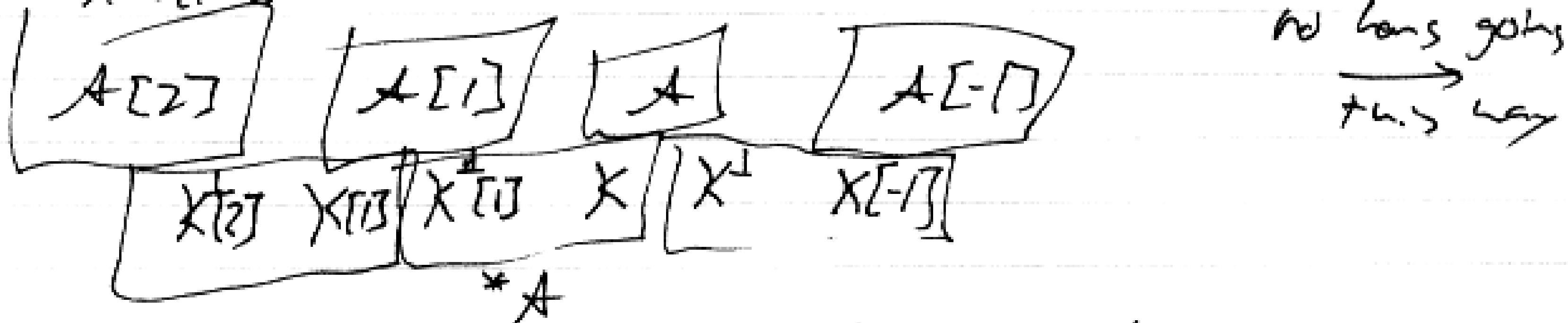
Action on heart is <u>tilting</u> $\mathcal{P}((0,1]) \to \mathcal{P}((a,a+1])$

<u>Def</u> A torsion theory in an abelian category $\mathcal{A}$ is a full subcategory $\mathcal{K} \subset \mathcal{A}$ such that if $\mathcal{K}^\perp = \{A \in \mathcal{A} : \mathrm{Hom}(K,A)=0 \ \forall K \in \mathcal{K}\}$ then every $E \in \mathcal{A}$ has a decomposition
$$0 \to K \to E \to L \to 0 \quad \text{with } K \in \mathcal{K},\ L \in \mathcal{K}^\perp.$$

"$\mathcal{K}$ = torsion sheaves, $\mathcal{K}^\perp$ = torsion free sheaves"

Suppose $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ is a t-structure & $\mathcal{K} \subset \mathcal{A}$ is a torsion theory on its heart $\Rightarrow$ new t-structure ${}^*\mathcal{T}^{\leq 0} \subset \mathcal{T}$ defined by $E \in {}^*\mathcal{T}^{\leq 0}$ iff $H^i(E)=0$ $i>0$ & $H^0(E) \in \mathcal{K}$.

Given $\mathcal{T}$ triangulated $\supset$ thick subcategory, t-structures which restrict to t-structures on sub ⟷ t-structure on sub & quotient, can shift each $\Rightarrow$ perverse t-structure



So as we translate stability structure get continuous family of tilting!

Fix μ, X a curve, ${}^*\mathcal{A}_\mu \subset D(X)$ defined by

$E \in {}^*\mathcal{A} \iff H^i(E)=0$ unless $i=0$ or -1,

H-N factors of $H^{-1}(E)$ have slope $\leq \mu$

" " " $H^0(E)$ have slope $> \mu$

In here stable objects of slope μ are simple -- equivalent to category of modules on NC scheme (Schofield)

NC moduli of sheaves of this slope.

-- simple objects are precisely stable vector bundles of slope μ! — instead of things supported at points...

$\mathcal{A}^* \iff$ sheaves on NC scheme! Can have Exts between objects supported at different points (space = moduli of stable bundles, with different points correlated... have Ext's..

- whole derived category here is derived of this abelian

$\iff$ modules over an NC algebra, endos of a projective generator.

X an elliptic curve, $K(X)$ is <u>not</u> finite rank group ($\mathbb{Z} + \mathrm{Pic}\, X$)

- degree & determinant.

A stability condition is <u>numerical</u> if

$Z(-) = \chi(\Theta, -)$ for some $\Theta \in K(X) \otimes \mathbb{C}$

$$\chi(E,F) = \sum (-1)^i \dim \mathrm{Ext}^i_X(E,F)$$

- ie factors via chern character

$$\begin{tikzcd} K(X) \arrow[r, "Z"] \arrow[dr, "ch"'] & \mathbb{C} \\ & H^*(X,\mathbb{Q}) \arrow[u] \end{tikzcd}$$

Stab(X) = set of numerical stability conditions

- closed subspace of all stability conditions

<u>Prop</u> The action of $\widetilde{GL}^+(2,\mathbb{R})$ on Stab X is free & transitive so Stab $X \cong \widetilde{GL}^+(2,\mathbb{R})$

(true for any curve genus >0)

String theorists! want $\dfrac{\text{Stab } X}{\text{Aut } D(X)} = \dfrac{\widetilde{GL}^+(2,\mathbb{R})}{SL(2,\mathbb{Z})} \longrightarrow \mathcal{H}/SL_2\mathbb{Z}$

$SL(2,\mathbb{Z}) \subset \mathrm{Aut}\, D(E)$

i.e. $\mathrm{Stab}\,X/\mathrm{Aut}\,\mathcal{O}(X)$ = "space of deformations of the mirror"
is in this case elliptic curves + 1-form.

Deformation

Lemma If $\sigma_1, \sigma_2 \in \Sigma(\mathcal{T})$ have the same central charge Z
then $d(\sigma_1, \sigma_2) \geq 1$

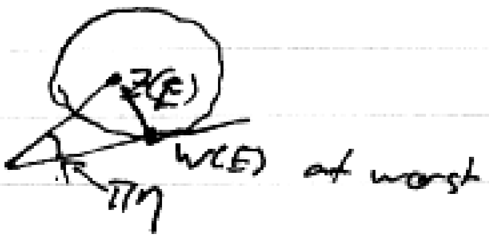
$$\Sigma(\mathcal{T}) \xrightarrow{Z} [K(\mathcal{T}) \otimes \mathbb{C}]^\vee$$ fibers are discrete

Def Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{T}$.
A perturbation of σ with parameter ε is a linear map $W : K(\mathcal{T}) \to \mathbb{C}$ s.t.

a. $|W(E) - Z(E)| < \varepsilon\, |Z(E)$ for all $E \in \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$

b. $W\left(\bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)\right) \subset \mathbb{C}$ has no limit points. -- apply to all family

Let $\eta = \frac{1}{\pi} \sin^{-1}(\varepsilon)$
- phase can change by at most η



An object $0 \neq E \in \mathcal{T}$ is <u>tight</u> if $\phi^+(E) - \phi^-(E) < 1 - 2\eta$.
For any such $\exists$ well defined branch of $\psi(E) = \frac{1}{\pi} \arg W(E) \in (\phi^-(E) - \eta, \phi^+(E) + \eta)$

all slopes lie in half plane, just follow them around too.

Def E tight is W-semistable if there is no triangle of tight objects
$$\begin{tikzcd} A \arrow[r] & E \arrow[dl] \\ B \arrow[u, dashed] & \end{tikzcd}$$
with $\psi(A) > \psi(E) > \psi(B)$

Theorem If $Q(\psi)$ denotes the subcategory of W-semistable objects of phase ψ then (W, Q) is a stability condition on $\mathcal{T}$.

T. Bridgeland II

Sheaves on K3

K3 : projective smooth surface / k with $\omega_X = \mathcal{O}_X$ trivial canonical bundle, & $H^1(X, \mathcal{O}_X) = 0$

Properties! $H^2(X, \mathbb{Z}) = \mathbb{Z}^{\oplus 22}$ with intersection pairing signature $(3,19)$

Mukai - extend to $H^{2*}(X, \mathbb{Z}) = \underset{\overset{\shortparallel}{\mathbb{Z}}}{H^0(X, \mathbb{Z})} \oplus H^2(X, \mathbb{Z}) \oplus \underset{\overset{\shortparallel}{\mathbb{Z}}}{H^4(X, \mathbb{Z})}$

$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1$: add hyperbolic plane, get Mukai lattice in 24-dim space

Torelli : Two K3s are isomorphic iff there's a Hodge isometry $H^2(X_1, \mathbb{Z}) \xrightarrow{\sim} H^2(X_2, \mathbb{Z})$ - preserves $(,)$ & takes $H^{2,0}(X_1) \longrightarrow H^{2,0}(X_2)$

Orlov Two K3s have equivalent derived categories iff extended lattices $H^{2*}(X_i, \mathbb{Z})$ are Hodge isometric. (put $H^0(X, \mathbb{Z})$ & $H^4(X, \mathbb{Z})$ inside $H^{1,1}$)

- automorphisms of a K3 are same as automorphisms of Hodge structure ⊕ positive cone ... ie index two in Hodge automorphisms.

Equivalences of derived categories given by complex on product, to which one can then assign an equivalence of ~~the~~ cohomology groups (Chern character of universal object) commuting with Chern character maps. Can mix degrees of cohomology !

$\Phi : D(X_1) \xrightarrow{\sim} D(X_2) \Rightarrow$ Orlov:
$$\begin{tikzcd} & \mathcal{P} \arrow[d, phantom, "\downarrow"] & \\ & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

given by $Rp_{2*}(\mathcal{P} \overset{L}{\otimes} p_1^*(-))$

Gives correspondence on cohomology $p_{2*}(ch(\mathcal{P}) \cdot p_1^*(-))$ up to Todd class ... gives map

$$\phi : H^*(X_1, \mathbb{Q}) \longrightarrow H^*(X_2, \mathbb{Q})$$

on K3 also preserves integral cohomology
- preserves difference $p-q$ in Hodge structure.

Riemann-Roch $E, F \in D(X) \Rightarrow$ relative Euler char.

$$\chi(E,F) = \sum_i (-1)^i \dim_k \operatorname{Hom}^i_{D(X)}(E,F)$$ sum of dims of ext groups

$$= \int_X ch(E^\vee)\, ch(F)\, td(X)$$

$$= -\langle v(E), v(F) \rangle$$ Mukai pairing

where $v(E) = (r(E),\ c_1(E),\ \underbrace{ch_2(E) + r(E)}_{\text{chern character}})$
Mukai vector (rank) $\quad ch_2(E)+r(E) =: s(E)$

$v : K(X) \to H^*(X, \mathbb{Z})$.

special to K3's: no denominators here, only $\frac{1}{2}c_1^2$ & intersection form on K3 is even.

Serre duality $\operatorname{Hom}^i_{D(X)}(E,F) = \operatorname{Hom}^{2-i}_{D(X)}(F,E)^\vee$ $\quad (\omega_X = \mathcal{O}_X)$

$\Rightarrow \langle\ ,\ \rangle$ is symmetric

Stability: L ample line bundle on X.

$$\mu(E) = \frac{c_1(E)\cdot L}{r(E)} \qquad \nu(E) = \frac{s(E)}{r(E)} = \frac{ch_2 + r}{r}$$

A torsion-free sheaf E is L-stable if $\forall\, 0 \neq A \subsetneq E$
$(\mu(A) < \mu(E))$ or $(\mu(A) = \mu(E),\ \nu(A) < \nu(E))$

Faltings: a vector bundle on a curve is semistable iff $\exists$ another v.b. with no Homs or Exts between them
- no such condition on surfaces ... $\Rightarrow$ more interesting space of stability conditions.

Theorem (Mukai) Let $N(X) = K(X)/$numerical eq - numerical lattice of X
$(N(X) \overset{v}{\hookrightarrow} H^*(X,\mathbb{Z}))$ $\quad E_1 \sim E_2 \Leftrightarrow \chi(E_1, F) = \chi(E_2, F)\ \forall F \in D(X)$

If $v \in N(X)$ is primitive then for general ample line bundle L the moduli of stable sheaves $M_{(X,L)}(v)$ is a smooth projective complex symplectic variety of dim $\langle v,v \rangle + 2$

Symplectic form: $T_E M_{(X,L)}(v) = \operatorname{Ext}^1(E,E)$.

pair $\operatorname{Ext}^1(E,F) \times \operatorname{Ext}^1(E,F) \to \operatorname{Ext}^2(E,F) = \operatorname{Hom}(E,E)^\vee = \mathbb{C}$
(E simple)

- note $\langle v,v\rangle = -\chi(E,E) = \dim \overbrace{\mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^0(E,E)}^{=1}$
$\quad$ so $\dim M_E = \langle v,v\rangle + 2$ $\qquad - \dim \underbrace{\mathrm{Ext}^2(E,E)}_{=1}$

(Stability: look at $\chi(E\otimes L^n)/r(E)$)

E stable $\Rightarrow$ $\langle v(E), v(E)\rangle \geq -2$
If $v(E)^2 = -2$ $\quad$ E is called *spherical*,
moduli space is just a point! $\quad$ satisfy
$\mathrm{Hom}(E,E) = \mathbb{C}$, $\quad \mathrm{Ext}^2(E,E) = \mathbb{C}$ $\quad \mathrm{Ext}^1(E,E) = 0$ like sphere!

Stability conditions

$\mathcal{T} = D(X)$ $\quad$ *Numerical* stab condition: $Z : N(X) \to \mathbb{C}$ linear
+ subcategories $P(\phi) \subset D(X)$ $\quad \phi \in \mathbb{R}$
$\quad Z(P(\phi)) \subset \mathbb{R}_{>0}\, e^{\pi i \phi}$ $\quad$ etc.
$\mathrm{Stab}(X) = $ set of numerical stability conditions on $D(X)$, + topology/metric.
$\quad$ There is a continuous map
$\quad Z : \mathrm{Stab}(X) \longrightarrow [N(X)\otimes\mathbb{C}]^\vee$

Let $\Delta = \{\delta \in N(X) : \delta^2 = -2\}$ $\qquad$ Mukai vectors of spherical objects

Fibres in Minkowski space! proj from $\mathbb{R}^{2,2}$ to $\mathbb{R}^{3,2}$)

$\mathcal{P}^+(X) = \{ Z : N(X)\otimes\mathbb{R} \to \mathbb{C} :$ Ker Z negative definite
$\qquad\qquad$ & Ker $Z) \cap \Delta = \emptyset$ empty $\}$
- open subset of $[N(X)\otimes\mathbb{C}]^\vee$ with 2 components

Theorem There is a connected component of $\mathrm{Stab}(X)$ whose image under Z is $\mathcal{P}^+(X)$ & the induced map
$\quad Z : \mathrm{Stab}^0(X) \longrightarrow \mathcal{P}^+(X)$ is a covering space

Conjecture $\mathrm{Stab}(X) \xrightarrow{Z} \mathcal{P}^+(X)$ is the universal cover

... and automorphisms of $D(X)$ acting trivially on H^* are deck transformations of this covering!
$\quad \pi_1 = \mathrm{Ker}\,(\mathrm{Aut}\, D(X) \longrightarrow \mathrm{Aut}\, H^*(X,\mathbb{Z}))$

... aut K3 is a complicated group, this is saying that it is built up from Aut (lattice)
+ $\pi_1(\mathcal{P}_+)$
$\qquad$ hy probe complaint.

$\mathcal{P}_+$ is space where Borcherds' automorphic forms live...

Why is Z a covering space?
Suppose $(Z, \mathcal{P}) \in \operatorname{Stab} X$, $Z \in \mathcal{P}^+(X)$
Need to show that if $W = Z + \delta Z$ small deformation,
then $|(W-Z)(E)| \leq \varepsilon |Z(E)|$ for all
stable objects E, then apply theorem from last time.

Take an orthogonal basis of $N(X) \otimes \mathbb{R}$ $e_1, \dots e_n$ with
$e_1^2 = e_2^2 = 1$, $e_i^2 = -1$ $i \geq 3$ & $Z(e_i) = 0$ $i \geq 3$.
Any stable E satisfies $\langle v(E), v(E) \rangle \geq -2$

$$\Rightarrow \sum_{i \geq 3} v_i^2 \leq 2 + v_1^2 + v_2^2 \leq K |Z(v)| \quad \text{some } K$$

Write $W(E) = Z(E) + \langle u, v(E) \rangle$ $u \in N(X) \otimes \mathbb{C}$
Then $|W(E) - Z(E)| = |\langle u, v(E) \rangle| \leq \|u\| \|v(E)\| \leq \varepsilon |Z(v)|$
$\|u\| < \frac{\varepsilon}{K}$

— Very special to K3 surfaces, probably not true in general higher dim.

$\pi_1(\mathcal{P}^+)$ is generated by loops around the spherical vectors
($\mathcal{P}^+$ is hyperplane complement)
So conj gives description of Aut K3.

<u>Construction of stability conditions on K3</u>

$Z \in \mathcal{P}^+(X)$, $Z : N(X) \otimes \mathbb{R} \to \mathbb{C}$ $\operatorname{Ker} Z < 0$
$Z(E) = \langle \Pi, v(E) \rangle$ $\overset{\Pi}{\underset{\pi_1 + i\pi_2}{\shortparallel}} \in N(X, \mathbb{C})$ by nondegeneracy

π_1, π_2 span pos def plane in $N(X)$...
Acting by $GL_2 \mathbb{R}$ normalize so that $\pi_1^2 = \pi_2^2 > 0$,
$\langle \pi_1, \pi_2 \rangle = 0$.
Still have a $\mathbb{C}^*$ factor, normalize so that $Z(\mathcal{O}_x) = -1$.
— gotten rid of GL_2 action.

Let $\mathcal{P}^+_{red}(X) = \{ Z = \langle \pi_1 + i\pi_2, - \rangle \in \mathcal{P}^+(X) \}$
$\pi_1, \pi_2, Z(\mathcal{O}_x)$ as above

$\Rightarrow \Pi = e^{B+iJ} = \left(1, B+iJ, \frac{B^2-J^2}{2} + \langle B,J\rangle i\right)$

Then $Z(E) = \frac{1}{2r}\left(v(E)^2 - (\Delta - rB)^2 + r^2J^2\right) + iJ\cdot(\Delta - rB)$

if $v(E) = (r, \Delta, s)$

- need abelian category so that this lies in 

$\longrightarrow$ use tilting to define t-structure compatible with Z.

- torsion free

Define subcategories $\subset$ Coh X, $K = \{E \in \text{Coh } X : \text{All H-N factors of } E/\text{tors } E \text{ have slope} > BJ\}$

$K^\perp = \{E \in \text{Coh}(X) : E \text{ torsion free, all HN factors of slope} \leq BJ\}$

Slope/stability defined wrt J vector, $\mu(E) = \frac{c_1(E)\cdot J}{r(E)}$

~ASSUME J ample : we're working on an open subset of $\mathcal{P}^+$....

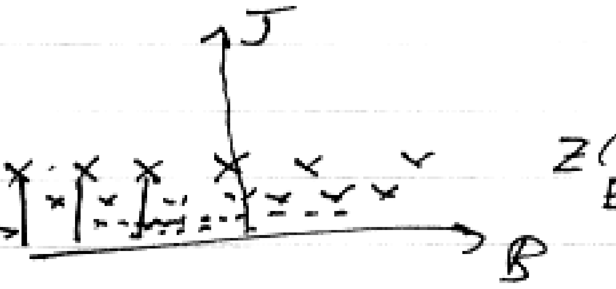
Tilting gives an abelian category $\mathcal{A} \subset D(X)$.
$\mathcal{A} = \{E \in D(X) : H^0(E) \in K, H^{-1}(E) \in K^\perp, H^i(E) = 0 \text{ if } i \notin \{0,-1\}\}$

$\operatorname{Im} Z(E) = r\cdot(\mu(E) - BJ)$. So $\operatorname{Im} Z(E) \geq 0$ for all $E \in \mathcal{A}$ - so this does the trick to land us in upper half plane!

What if $\mu(E) = B\cdot J$? Need such E to satisfy $Z(E) \in \mathbb{R}_{>0}$ (so that when shifted by 1 land in $\mathbb{R}_{<0}$)

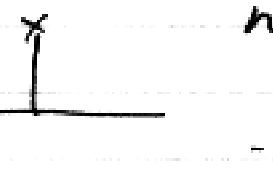
- problem when $v(E) = -2$ ...

Assume for simplicity $\text{pic}(X) = \mathbb{Z}$



$Z(E)=0$
E spher

- so have to remove  half lines starting from these points ! get large open subset



monodromy around these spherical bundle sheaf correspond to "twist functors"
... only branch cuts.

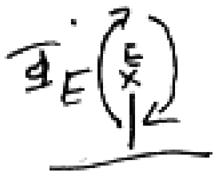
Have autoequivalence taking things out of this subset into our subset...

If E is spherical $\Rightarrow$ twist functor $\Phi_E : D(X) \longrightarrow D(X)$
defined by triangle

$$\begin{tikzcd}
\mathrm{Hom}(E,F)^\bullet \otimes E \arrow[r] & F \arrow[dl] \\
\Phi_E F \arrow[u, dotted] &
\end{tikzcd}$$

Claim Φ_E is <u>equivalence</u>
if E is spherical
--- reflection on K-theory ... need $\mathrm{Hom}_{D(X)}(E,E)^i = \left\{ \begin{array}{c} \mathbb{Z} \\ 0 \\ \vdots \\ 0 \\ \mathbb{Z} \end{array} \right.$
(use Serre duality functor to prove autoequivalence)
 — Seidel & Thomas

- Split category into E & its right orthogonal ..
$D(X)$ is like Hilbert space can do this on q-ch derived category $D(X) = \langle E \rangle + E^\perp$, functor is identity on $E^\perp$ and $E \longmapsto E[n-1]$ on $\langle E \rangle$
 Acts as reflection on K theory .. its square is "monodromy" around E.



<u>Lemma</u> An object $E \in D(X)$ with $r(E) > 0$, $\mu(E) > B \cdot J$ is stable in σ_{B+inJ} for all $n >> 0$ (large volume limit)
$\Longleftrightarrow$ E is a B-twisted stable sheaf (up to shift) (in Gieseker sense)
— Stability condition is never Gieseker stability, only in this asymptotic direction.
 In large volume limit branes are Gieseker stable sheaves.

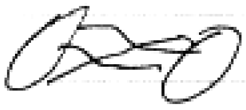
<u>Physics</u> X a Calabi-Yau $\Rightarrow$ SCFT moving in nice moduli space, "moduli of CFTs" - supposed to be smooth .. no global geometric picture of this moduli space.

$\boxed{\begin{array}{l} X \text{ Calabi-Yau} \\ + \text{extra data} \end{array}} \Rightarrow \boxed{\text{SCFT}}$, Roughly extra data is a complex structure on X
and a complexified Kähler class $B + iJ \in H^2(X, \mathbb{C}) / H^2(X, \mathbb{Z})$
 J Kähler, B field defined up to integer shift.
 - up to some discrete equivalence.
This data defines an open subset of moduli of SCFTs.

different open subsets from CYs
 ? - open piece not coming from CYs

- get open nbhd of cusp ($J \longrightarrow \infty$) from CYs
but have other things inside, eg orbifolds
Slice of things with fixed complex structure
should correspond to space of stability conditions
on $D(X)$. Mirror symmetry is auto. of this
space (?) not preserving slice...

To each SCFT string theorists associate a category
of branes - boundary conditions in this theory. B_1, B_2 branes
$\mathrm{Hom}(B_1, B_2) = Q$ cohomology strings from B_1 to B_2
of Hilbert space of



where Q is BRST conformal algebra.

There is also a topological twisting of $N=2$ SCFT (pass
to Q-cohomology, consider as CFT)
$\mathcal{P}$, cat of Branes in topologically twisted CFT are just $D(X)$
pe
// --- independent of other data $B + iJ$

$\bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$

In CY3 case all deformations of derived category come from
those of X, not true in K3 case.

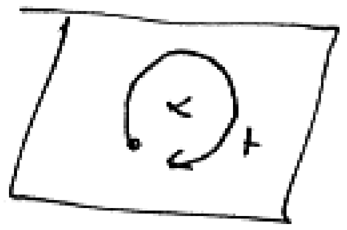
Gerbe case: if take nontorsion gerbes won't have
any coherent sheaves or perfect complexes
- will look like quantum torus,
q root of unity get Azumaya over huge algebra
but generically have no f.d. reps $uv = q\, vu$
$|q| = 1 \Rightarrow C^*$ algebras ..

K3 : $H^2(X, \mathcal{O}) \neq 0$ have gerbe deformations.
For formal deformations no problems...

In string theory, category $\mathcal{P}$ is well defined as abstract category
but inclusion $\mathcal{P} \subset D(X)$ is <u>not</u>, get monodromy!
- have a NC geometry i.e. A_∞ categories
but not commutative geometry - description of $D(X)$
- can have different ones underlying

- String theory gives an algebra category A s.t.
  $D(A) \xrightarrow{\sim} D(X)$

- have monodromy in this identification



$\mathcal{P}_t$ family with embeddings

$$\begin{tikzcd}
\mathcal{P}_t \arrow[r, hook, "\phi_t"] \arrow[d, equal] & D(X) \arrow[d, "\Phi"] \\
\mathcal{P}_1 \arrow[r, hook] & D(X)
\end{tikzcd}$$

Φ nontrivial autoequivalence – embedding changes by monodromy

Universal cover of moduli space should be space of deformations of $\mathcal{P}$ & embedding in $D(X)$.

$$\begin{tikzcd}
K(X) = \bigoplus H^{p,p}(X) \arrow[r, equal] \arrow[d, "Z"] & H^3(X^\vee) \arrow[d, "\int \Omega"] \\
\mathbb{C} & \mathbb{C}
\end{tikzcd}$$

mirror symm
period of hol 3 form