Kevin Costello - Renormalization & BV Formalism

Note Title 9/5/2007

M compact Riemannian manifold
Δ Laplacian (positive definite) & $m>0$.

Consider scalar field theory:
field $\phi \in C^\infty(M;\mathbb{R})$
action $S(\phi) = \left(\int_M \underbrace{\phi(\Delta+m^2)\phi}_{\text{mass } m}\right) + \underbrace{I(\phi)}_{\text{interacting part}}$

where $I_n(\phi)$ is a <u>local functional</u>:
a finite sum of expressions like
$\int_M \phi (D_1\phi)(D_2\phi) \cdots (D_n\phi)$
where D_i are differential operators
on $C^\infty(M)$. ..... integral of an
expression only depending on Taylor series
of ϕ at a point

$$I(\phi) = \sum_{n\geq 3} I_n(\phi)$$

Want to make sense of functional integrals of form:

$$Z(a) = \int_{\phi \in C^\infty(M)} e^{\langle \phi, (\Delta + m^2)\phi \rangle / \hbar + I(\phi + a)/\hbar}$$

with $a \in C^\infty(M)$ a background field

(can go from this to typical form with current $J \cdot \phi$ by change of variables but not conversely - this contains more information)

Heat kernel $K_t^0 \in C^\infty(M^2)$ for the Laplacian - ie kernel representing the operator $e^{-t\Delta}$

$K_t = e^{-tm^2} K_t^0$ representing $e^{-t(\Delta + m^2)}$

Propagator $P = \int_0^\infty K_t \, dt \in \mathrm{Dist}(M^2)$

... kernel representing the inverse operator $(\Delta + m^2)^{-1}$.

$t \longleftrightarrow$ proper time of a particle.

Regulated version:

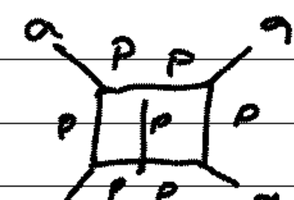
$$P(\varepsilon, T) = \int_{\varepsilon}^{T} K_t \, dt, \qquad \text{so } P = P(0,\infty).$$

Feynman graph expansion:

$$\hbar \log Z(a) = \sum_{\substack{\text{connected}\\ \text{graphs } \gamma\\ \text{with external}\\ \text{edges}}} \frac{\hbar^{b_1(\gamma)}}{|\mathrm{Aut}(\gamma)|} W_\gamma(a)$$

b_1 = Betti # of graph

Weight $W_\gamma(a)$:



put P on all internal edges & $\underline{a}$ on all external edges + interaction I at vertices

(no propagator on external edges)

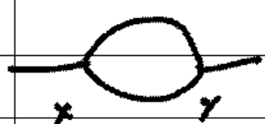
Simple example: if $I(\phi) = \int \phi^3$

$$\int_{x \in M} \int_0^\infty K_t(x,x)\, a(x)$$

This is ill defined:

for t small $K_t(x,x) \sim t^{-\dim M/2}$

$\Rightarrow$ integral diverges



$$\int_{x,y \in M} \int_{s,t=0}^{\infty} K_t(x,y)\, K_s(x,y)\, a(x)\, a(y)$$

$$= \int_{Met(\gamma)} \int_{Maps(\gamma, M)} \pi_1^* a \;\; \pi_2^* a$$

$\vdots$

maps where external maps get length zero... heat kernel comes from integrating Wiener measure on space of maps from γ to M.

General definitions

Let $\mathcal{O} = \mathcal{O}(C^\infty(M))$

$$= \prod_{n \geq 0} \mathrm{Dist}(M^n)^{S_n}$$ S_n-invariant distributions

"=" $\overline{\mathrm{Sym}^*}\left(C^\infty(M)^\vee\right)$

i.e. formal power series on the vector space $C^\infty(M)$.

$\mathcal{O}$ is an algebra under direct product of distributions.

If $A \in C^\infty(M^2)$ (think of as propagator)

$\Rightarrow$ second order diff op $\partial_A : \mathcal{O} \to \mathcal{O}$

by $\partial_A : \mathrm{Dist}(M^n) \to \mathrm{Dist}(M^{n-2})$

= contraction with A in all possible ways.

For any $I \in \mathcal{O}$ define

$$\Gamma(A, I)(a) = \hbar \log \left(e^{\hbar \partial_A} e^{I/\hbar} \right)(a) \in \mathcal{O}[[\hbar]]$$

$$= \sum_{\substack{\gamma \text{ connected} \\ \text{graph}}} \frac{\hbar^{b(\gamma)}}{|\mathrm{Aut}\,\gamma|} w_\gamma(a)$$

as before: put A on internal edges, I on vertices & $\underline{a}$ on external edges.

– always well defined for $A \in C^\infty(M^2)$ smooth...

What we had before is

$$\hbar \log Z(a) = \lim_{\varepsilon \to 0} \Gamma(P(\varepsilon, \infty), I)$$

So above formalism is shorthand for Feynman formalism.

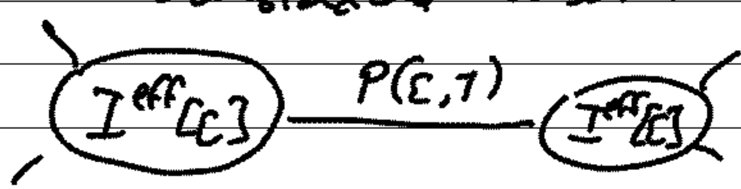
Def An effective interaction $I^{eff}[\varepsilon]$ at scale ε tells us what happens at scale $< \varepsilon$

Def A system of effective actions is

$$I^{eff}[\varepsilon] \in \mathcal{O}[[\hbar]] \quad \forall \varepsilon > 0 \quad \text{such that}$$

1. $\Gamma(P(\varepsilon,T), I^{eff}[\varepsilon]) = I^{eff}[T]$

(renormalization group equation, nonstandard version) ($\varepsilon, T \leadsto$ distances)

i.e 

gives $I^{eff}[T]$

2. $\exists$ some $J(\varepsilon)$ which is local, ε-dependent s.t.

$$\lim_{\varepsilon \to 0} I^{eff}[\varepsilon] - J(\varepsilon) = 0$$

Wilson: to write QFT should have effective action at every scale... original interaction corresponds to scale 0 ... ie nonsensical!

Theorem There is a bijection between local functionals $I = I^{(0)} + I^{(1)} + \cdots\cdots$ ($I^{(i)}$ local) & systems of effective actions

... ie any "nonsensical" interaction I corresponds to a system of effective actions.

Idea! Given I, we construct a canonical* series of counter terms,

$I^{CT}(\varepsilon)$, ε-dependent local functionals, s.t.

$$\lim_{\varepsilon \to 0} \Gamma(P(\varepsilon, T), I - I^{CT}(\varepsilon)) \text{ exists } \underline{\forall T}.$$

This limit is $I^{eff}[T]$.

Similar process conversely: subtract singular terms in limit using counterterms.

*: Canonical: there's a universal choice, which applies to all theories, which we make one:

Choice of a way to define "singular part" of certain functions of ε.

(things like dilogarithms ...)

Philosophy: $I^{eff}[\varepsilon]$ are fundamental, but I & I^{CT} are less natural (depend on this universal choice).

How to discuss gauge theory in this language — low energy gauge symmetries don't form a group...

$\rightsquigarrow$ pass to BV formalism, have a quantum master equation at each scale.

Yang-Mills in 4d, 1st order formulation:

$\mathfrak{g}$ Lie algebra with a pairing

Fields $\quad \Omega^1(M) \otimes \mathfrak{g} \quad \Omega^2_+(M) \otimes \mathfrak{g}$

(+ = self-dual) $\quad \overset{\shortparallel}{A} \quad \overset{\shortparallel}{B}$

Action: $\underbrace{\int B\wedge F(A)_+}\ + \int B^2$

at critical points $B = F(A)_+$
& second integral enforces YM equation.

Gauge fields: $\Omega^0(M)\otimes\mathfrak{g}$.

BV: add antifields $\Omega^3(M)\otimes\mathfrak{g}$ $\Omega^2_+(M)\otimes\mathfrak{g}$
(dual to fields)

& antighosts $\Omega^4(M)\otimes\mathfrak{g}$

$$
\begin{tikzcd}
-1 & \Omega^0\otimes\mathfrak{g} \arrow[d, "d"] & \\
0 & \Omega^1\otimes\mathfrak{g} \arrow[dr, "d_+"'] & \Omega^2_+\otimes\mathfrak{g} \arrow[dl, "d"'] \arrow[d, "\mathrm{Id}"] \\
1 & \Omega^3\otimes\mathfrak{g} \arrow[d, "d"] & \Omega^2_+\otimes\mathfrak{g} \\
2 & \Omega^4\otimes\mathfrak{g} &
\end{tikzcd}
\quad \mathcal{E}
$$

$\mathcal{E}$ has a differential Q discussed above

Pick a gauge-fixing operator

$$Q^\dagger : \mathcal{E} \longrightarrow \mathcal{E}, \quad (Q^\dagger)^2 = 0$$

& $H = [Q, Q^\dagger]$ is 2nd order & elliptic

(eg comes from metric on M).

$K_t \in \mathcal{E} \otimes \mathcal{E}$ heat kernel for H

$$P(\varepsilon, T) = \int_\varepsilon^T Q^\dagger K_t$$

(Hodge theoretic version of Q^{-1})

Proceed as before. If $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$

can define

$$\Gamma(P(\varepsilon,T), I)$$

As before have a bijection between effective actions & local functionals I.

Gauge invariance! for each scale t have a BV operator $\Delta_t : \mathcal{O}(\mathcal{E}) \circlearrowleft$

$\Delta_t^2 = 0$, Δ odd order 2.

... version of usual BV operator diffused by the heat kernel.

$I^{eff}[t]$ satisfies scale t quantum master equation

if $(Q + \hbar \Delta_t) e^{I^{eff}[t]/\hbar} = 0$

RG flow takes solutions of one QME to another:

<u>Lemma</u> If $I^{eff}[\varepsilon]$ satisfies QME at scale ε

$\Rightarrow \Gamma(P(\varepsilon, T), I^{eff}[\varepsilon])$ satisfies scale T QME,

$\Longrightarrow$ def of QME which is good as Δ_0 is ill-defined.

To find solns of QME, term by term,
solve obstruction problem in homological algebra.