M. Emerton   p-adic Langlands - Global Motivations   6/27/06

Fix a prime p.

Basic objects in number theory : $\rho : G_{\mathbb{Q}} \xrightarrow{\text{continuous}} GL_2(A)$

$G_{\mathbb{Q}} = \mathrm{Gal}\, \overline{\mathbb{Q}}/\mathbb{Q}$

A : p-adic ring / field

Assume ρ unramified away from p & ∞

Form the "representation variety" $\mathfrak{X}$ classifying such ρ. If $G_{\mathbb{Q}}$ were finitely presented, get a rep variety. it's not though but we can imagine it's the completion of a fin. presented group, it essentially behaves this way $\rightsquigarrow$ $\mathfrak{X}$ is a rigid analytic space --- due to inequalities on generators needed for continuous representations.

Instead it's traditional (following Mazur, '80s) to fix $\bar\rho : G_{\mathbb{Q}} \longrightarrow GL_2(k)$

(k a finite extension of $\mathbb{F}_p$) & look at

$\mathfrak{X}_{\bar\rho}$ = lifts of $\bar\rho$, get a formal scheme

$\mathfrak{X}_{\bar\rho} = \mathrm{Spf}\, R_{\bar\rho}$

this may over a ring $\mathcal{O}$ (p-adic integer ring) with residue field k.

<u>Examples of</u> ρ : come from cuspidal eigenforms on $GL_2\mathbb{Q} \backslash GL_2\mathbb{A}$ of level a power of p.

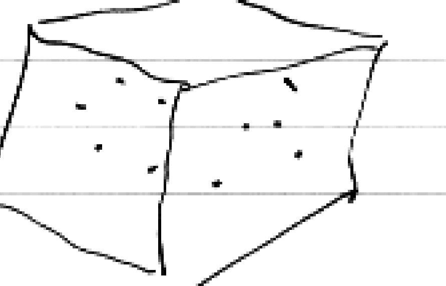
First classification: Maass ? (conjecturally), when $\det \rho(c) = 1$

or holomorphic ✓ $\det \rho(c) = -1$

... related to different structure at ∞.

[c = complex conjugation]

From now on assume $\det \bar\rho(c) = -1$.



f hol cusp. eigenform

$\longleftrightarrow \rho_f$ Galois reps. assume $\bar\rho_f = \bar\rho$ mod p

Q: Given a $\rho \in \mathfrak{X}_{\bar\rho}$ when does it occur as a ρ_f.

<u>Conjecture</u> (Fontaine-Mazur, Langlands)
$\rho \in \mathfrak{X}_{\bar\rho}$ is of the form ρ_f (up to Tate twist)
$\iff$ $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semistable in the sense of Fontaine.

(reps detected by Fontaine's ring B_{st})

Let $\mathrm{Spf}\, \mathbb{T} \hookrightarrow \mathrm{Spf}\, R$ be the Zariski closure of the ρ_f.
Aside on structure of $R, \mathbb{T}$: reps are determined via Čebotarev by traces of Frobenius

$$\mathcal{O}[\{T_\ell\}_{\ell \neq p}] \xrightarrow{\text{dense}} R \xrightarrow{\text{dense}} \mathbb{T}$$

$$T_\ell \longmapsto \text{trace of Frob}_p \longrightarrow T_\ell : \ell^{\text{th}} \text{ Hecke eigenvalue of } R$$

$\mathbb{T}$ = smallest quotient which acts on every cusp form
Points of $\mathrm{Spf}\, \mathbb{T}$ are p-adic eigenforms

<u>Conjecture</u> (Mazur) $\mathrm{Spf}\, \mathbb{T} = \mathrm{Spf}\, R$.
... i.e motivic points are dense.

Different than many $R = T$ claims: there we only consider weights which <u>are</u> potentially semistable in our reps varieties, get small R, T conjecture, close to Fontaine-Mazur. This conjecture is more about p-adic interpolation

Gouvea-Mazur: explain 3 ~~dimensionality~~ of these varieties
1 dim twisting by character
2 dims Hida / Coleman-Mazur families ($\sim$ spare-Ellis curve

Results: Bockle in some cases, Kisin-~~Emerton~~ in many cases ... in particular if $\bar\rho|_{\mathbb{Q}_p}$ $p \neq 2$ is irred

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One goal of p-adic Langlands: use rep theory of GL_2 to understand such questions.

At each ρ_f we have a $\Pi_p(f)$: smooth $GL_2\mathbb{Q}_p$-rep.

Can we interpolate these over $\mathfrak{X}_{\bar\rho}$?
No. but seems to be yes with slight modification.

(future)
<u>Pro-theorem</u> (Colmez, following a strategy of Kisin)
If $\bar\rho|_{G_{\mathbb{Q}_p}} \neq \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \Longrightarrow$

ω = mod p cyclotomic

$\exists$ a pro-free Banach module $\hat\Pi / R_{\bar\rho}$ w/ action of $GL_2\mathbb{Q}_p$
s.t. the fiber of $\hat\Pi$ at a classical point ρ_f (where f has wt ≥ 2) is a completion of $\Pi_p(f) \otimes$ algebraic

... proof will be purely local, on deformation functors.
Bad case above: on Galois side has singular deformations,
on automorphic forms trouble finding good candidate

Another structure involving same ingredients: cohomology of modular curves

A rings $\quad H^1_A := \varinjlim_r H^1_{et}(X(p^r), A) \circlearrowleft GL_2\mathbb{Q}_p$, $G_{\mathbb{Q}}$

<u>Theorem</u> (Eichler-Shimura, Igusa, Langlands, Deligne, Carayol)
$$H^1_{\bar{\mathbb{Q}}_p} := \bigoplus_{f \text{ of wt } 2} \rho_f \otimes \Pi_p(f)$$

For $H^1_{\mathcal{O}}$ this is not at all true: different cusp forms have congruences between them: can take differences, divide by p, get forms that can't be decomposed as sums.

$\hat{H}^1_\mathcal{O}$: p-adic completion of $H^1_\mathcal{O}$

... could a priori have been zero,

$G_\mathbb{Q} \times GL_2\,\mathbb{Q}_p$ but in fact $H^1_\mathcal{O} \hookrightarrow \hat{H}^1_\mathcal{O}$

In fact $\mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} H^1_\mathcal{O} = (\mathbb{Q}_p \otimes \hat{H}^1_\mathcal{O})_{\text{smooth}}$:

ie smooth vectors are just the obvious ones,
rest of rep isn't smooth.

Let $\hat{H}^1_{\mathcal{O},\bar\rho} = \bar\rho$-part of $\hat{H}^1_\mathcal{O}$: direct summand

defined by considering $H^1_\mathcal{O} \cap \bigoplus \rho_f \otimes \pi_p(f)$

$\rho_f \equiv \bar\rho \bmod p$

Away from p all Hecke operators T_ℓ act at every level, hence at limit, hence on $\hat{H}^1$.

In fact $\mathbb{T}$ acts on $\hat{H}^1_\mathcal{O}$, faithfully.

..... thus Zariski closure of all f's of weight 2 gives same as for all weights! secretly know about other weights.

We can think of $\hat{H}^1_\mathcal{O}$ "over" $\operatorname{Spf} \mathbb{T} \hookrightarrow \operatorname{Spf} R$.

What is the structure of $\hat{H}^1_\mathcal{O}$? guess some kind of direct integral of deforming Galois reps $\otimes$ interpolating $G(\mathbb{Q}_p)$-reps.

<u>Conjecture</u> Assume $\bar\rho$ abs. irred, $\bar\rho|_{G_{\mathbb{Q}_p}} \not\cong \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$
so that $\hat\pi$ is defined.

Then $\hat{H}^1_\mathcal{O} \cong \rho_{R_p} \underset{R_p}{\otimes} \hat\pi \otimes \operatorname{Hom}(R, \mathbb{Z}_p)$

put in all needed torsion
since rest of ~~the~~ factors are free.

$\Rightarrow$ any ρ that looks it should be classical is.
Use Jacquet modules, smooth vectors etc.
Once have this conjecture since LHS is something you control.

<u>Theorem</u> ($\bar\rho$ as in the conjecture)
Let $X = \mathrm{Hom}_{R_{\bar\rho}[G_{\mathbb{Q}} \times GL_2 \mathbb{Q}_p]} (\rho_{R_{\bar\rho}} \otimes_{R_{\bar\rho}} \hat\Pi, \hat H'_{0,\bar\rho})$.

Then: 1. X is supported on $\mathrm{Spf}\, \Pi$.
2. The evaluation map $\rho_R \otimes \hat\Pi \otimes X \xrightarrow{ev} \hat H'_0$ is a surjection after $\otimes \mathbb{Q}_p$
3. If $\mathrm{End}(\bar\rho|_{G_{\mathbb{Q}_p}}) = k$ then ev. is an $\cong$.

What about the funny case $\bar\rho|_{G_{\mathbb{Q}_p}} = \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$

$\exists !$ nontrivial extension
$$0 \to \mathrm{Ind}_{\bar B}^{GL_2} \omega^{-1} \otimes \omega \to E \to St \to 0$$

but E is not the right rep to put in: need also a 1-dim trivial rep .... get a $\mathbb{P}^1$ worth of nontrivial extensions ..... resolve singularity in deformation space to get all