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M. Harris - Automorphic Galois representations & the Sato-Tate conjecture

(w/ Clozel, Taylor, Shepherd-Barron)

F totally real field. Dictionary:
$\Gamma_F = \mathrm{Gal}\ \bar{\mathbb{Q}}/F$

$$\left\{ \begin{array}{c} \text{Compatible irreducible systems} \\ \text{of } \ell\text{-adic reps of } \Gamma_F \\ n\text{-dim } (\rho_{\pi,\ell}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(certain) cuspidal} \\ \text{automorphic reps } \pi \\ \text{of } GL_n \mathbb{A}_F \end{array} \right\}$$

["certain" $\longleftrightarrow$ compatible]

Compatible system: all $\rho_{\pi,\ell}$ yield same L function $L(s, \rho_{\pi,\ell})$
Also demand $\exists S$ finite s.t. each $\rho_{\pi,\ell}$ unramified outside $S \cup \{\ell\}$

Demand: $L(s, \rho_{\pi,\ell}) = L(s,\pi)$ & RHS has analytic continuation & functional eqn. (usually entire, one exception).

Arrow usually goes RHS to LHS.

Clozel, Kottwitz, Harris-Taylor, Taylor-Yoshida:
$\pi = \pi_\infty \otimes \pi_f$ assume that
1. π_∞ is cohomological (so one of specific list, & infinitesimal character is regular)
2. $\pi \simeq \pi^\vee$
3. $\exists v_0$ s.t. π_{v_0} is discrete series (will be removed soon thanks to Laumon-Ngo)

$\Longrightarrow$ a. $\rho = \rho_{\pi,\ell}$ is geometric (de Rham in Fontaine sense ... in fact will come from algebraic geometry), Hodge-Tate regular

b. $\rho \otimes \rho \longrightarrow \mathbb{Q}_\ell(1-n)$ perfect pairing

c. Some local condition at v_0.

Fontaine - Mazur conjecture at this setting predicts can go back, given just one $\rho_{\pi,\ell}$ should get an automorphic rep.

Geometric = looks like an n-dim piece of M_ℓ of $H^d(Y, \mathbb{Q}_\ell)$ where Y is smooth projective (middle dimension! $d = \dim Y$)

ℓ-adic Hodge theory $\Rightarrow$ can recover Hodge numbers $h^{p,q}(M)$ from M_ℓ.

H-T regular : $h^{p,q}(M) = h^{p,q}(\rho_\ell) \leq 1 \quad \forall\, p+q=d$.

Reciprocity conjecture (Langlands, Fontaine-Mazur)
The above arrow is an equivalence preserving L-function.
(without conditions c / 3)

Example (Wiles, Taylor-Wiles, Breuil-Diamond-Conrad-Taylor)

$n=2 \quad F=\mathbb{Q} \qquad E/\mathbb{Q}$ elliptic curve,

$\rho_E = \rho_{E,\ell} = H^1(E, \mathbb{Q}_\ell)$, $d=1$, $h^{01} = h^{10} = 1$,

$\exists\ \pi_E = \pi(\rho_{E,\ell})$

Now consider $n > 2$. Assume E has no CM

$\rho_E^n = \mathrm{Sym}^{n-1} \rho_E \qquad (\rho_E = \rho_E^2)$. Y: sym. power of E

Supposed to be associated to cuspidal automorphic rep!

$d = n-1 \qquad h^{p,\, n-1-p} = 1 \quad$ others $= 0$

Reciprocity conjecture in this case: ρ_E^n cuspidal automorphic $\forall n$.

In particular $\quad L(s, \rho_E^n) = L(s, E, \mathrm{Sym}^{n-1})$

$$= \prod_{p \in S} (-) \prod_{p \notin S} \det(1 - \rho_E^n(\mathrm{Frob}_p) p^{-s})^{-1}$$

has analytic continuation & functional eqn. (... take unitary normalization of L, $L(s) \longleftrightarrow L(1-s) \cdot \varepsilon$ - factor

$\forall p \notin S$ α_p, β_p Satake parameters = normalized Frobenius eigenvalues,

& det terms above in L fn are $\prod_{i=0}^{n-1} \frac{1}{1-\alpha_p^i \beta_p^{n-1-i} p^{-s}}$

normalized: $|\alpha_p| = |\beta_p| = 1$ $\alpha_p \beta_p = 1$

write $\alpha_p = e^{i\theta_p}$ $0 \le \theta_p \le \pi$.

Sato-Tate conjecture The θ_p are equidistributed in $[0,\pi]$ wrt Sato-Tate measure $d\,ST(\theta) = \frac{2}{\pi} \sin^2\theta \, d\theta$ (pushforward of Haar measure on SU_2 under taking eigenvalues).

Theorem Suppose $E/\mathbb{Q}$ has non-integral j-invariant (version of condition 3 above which should be removed). Then ST conjecture holds for E.

Serre: Reciprocity conjecture for ρ_E^n $\forall n \Rightarrow$ Sato-Tate conjecture. In fact enough to know that the $L(s, \rho_E^n)$ all meromorphic, & hol. nonvanishing for $\mathrm{Re}\, s \ge 1$ ... like prime number theorem.

In fact enough (by Brauer's theorem on induced characters, solvable base change, & Shahidi nonvanishing) to ~~assume~~ know $\forall m \exists$ totally real Galois extension $F = F_m/\mathbb{Q}$ s.t. for even $n \le m$ $\rho_E^n|_{\Gamma_F}$ is automorphic.

Example Let $Z = \mathbb{P}^1 \setminus \{\mu^{n+1}, \infty\}$ $n+1$ st roots of unity.

$t \in Z(F)$ consider $X_t \subset \mathbb{P}^n$

$$P_{t,n}(x_0, \dots, x_n) = \sum_{i=0}^{n} x_i^{n+1} - (n+1)\, t\, x_0 \cdots x_n$$

Calabi-Yau (Fermat at $t=0$, union of hyperplanes at $t=\infty$).
smooth for $t \in Z$, with understood monodromy.
$n=2$: family of elliptic curves with marked 3-division points.

$$H = \{(\zeta_0, \dots \zeta_n) \in \mu_{n+1}^{n+1} \mid \prod \zeta_i = 1\} / \Delta(\mu_{n+1})$$ diagonal

acts on family $/Z$.

Let $V_{\ell,t} = H^{n-1}_{\acute{e}t}(X_{t,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^H$

$\dim V_{\ell,t} = n$, $\quad h^{i,n-1-i} = 1$, others zero.

(Fontaine-Mazur)
<u>Reciprocity conjecture</u>: The $V_{\ell,t}$ are automorphic as $GL(n, \mathbb{A}_F)$ $\quad t \in Z(F)$

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3 kinds of deformation:

- vertical: lifting a mod ℓ representation to ℓ-adic
- horizontal: moving between ρ_ℓ & $\rho_{\ell'}$ in compatible system
- geometric: moving between different t's in Z.

Wiles' strategy: take $\bar{\rho} = \rho \pmod{\ell} : \Gamma_{\mathbb{Q}} \to GL_2 \bar{\mathbb{F}}_\ell$.

Vertical deformation: first order classification (liftings to square zero) given by Galois cohomology

Key definition: ρ is <u>residually automorphic</u> if $\bar{\rho}$ admits (at least) one reasonable lifting to char 0; $\tilde{\rho}$ (with fixed HT numbers) st $\tilde{\rho} = \rho_\pi$ for some automorphic π

$$\tilde{\rho} : \Gamma_{\mathbb{Q}} \longrightarrow GL_2 \mathcal{O}, \quad \mathcal{O}/m_{\mathcal{O}} = \mathbb{F}_\ell \quad (\text{or } \bar{\mathbb{F}}_\ell$$

Rough modular lifting theorem (Wiles, Taylor-Wiles): (dim 2)
If ρ is residually automorphic & Im $(\bar\rho)$ is "big"
then any reasonable lifting of $\bar\rho$ to char. 0 is
automorphic ... in particular ρ is automorphic.

- Uses Mazur's deformation theory + class field theory + Čebotarev density + comm algebra + modular forms

Why imagine something residually automorphic? Since
when $l=3$ there is always an automorphic lifting
$\tilde\rho$ (by luck).

Wiles strategy in 3 steps:

1. (Taylor-Wiles) modular lifting theorem for "minimal" lifting
2. (Level-raising) same for all liftings
3. Take $l=3$ to get started ... by theorems of Langlands - Tunnell.

Kisin: avoid separating steps 1, 2

$n>2$:

1. (Clozel Harris-Taylor) OK for (residually) auto reps satisfying condition C
2. Replaced by Taylor article (expanding Kisin)
3. must substitute: Harris-Taylor-Shepherd-Barron. (parity of n appears only in step 3 where need even n).

Idea: One class of n-dim Galois reps always known
to be modular, for n even:
Let $[K:\mathbb{Q}]=n$ be a CM field, cyclic & irreducible (imaginary quadratic)
$\chi : K^*_{\mathbb{A}}/K^* \longrightarrow \mathbb{C}^*$ algebraic Hecke character
(ie χ_∞ is cohomological)
Since Weil have known $\chi \rightsquigarrow \chi_l \;(\rho_{\chi_l})$ 1-dim reps
of Γ.

$I(\chi)_\ell := \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\chi_\ell)$, of dim n,
can be made to satisfy 1, 2, 3.

$I(\chi)_\ell$ is automorphic, comes from an automorphic induction $\pi(\chi)$ ... Kazhdan, generalized by Arthur-Clozel, on $GL(n, \mathbb{A}_{\mathbb{Q}})$.

Suppose you know $\exists \ell'$ such that $\overline{\rho^n_{E,\ell'}} = \overline{I(\chi)_{\ell'}}$
.... then can apply modular lifting & get that $\rho^n_{E,\ell'}$ is modular. By compatibility of L-functions, so is $\rho^n_{E,\ell}$.

Idea: look for totally real field for which there's an ℓ' for which this happens.
.... find a compatible family $\{\sigma_\lambda\}$ for Γ_F,
Some totally real Galois $F/\mathbb{Q}$ & ℓ, ℓ'
with $\overline{\sigma}_\lambda \simeq \overline{\rho}^n_{E,\ell}|_{\Gamma_F}$, $\overline{\sigma}_{\ell'} \simeq \overline{I(\chi)_{\ell'}}|_{\Gamma_F}$.
Modular lifting over F ok if ℓ, ℓ' unramified in F (ℓ, ℓ' large)
$\Rightarrow \sigma_{\ell'}$ automorphic $\Rightarrow \sigma_\lambda$ residually automorphic
$\Rightarrow \rho^n_{E\lambda}$ automorphic!

Only known way to get compatible family: from motives ... cohomology of variety $/F$.

$\mathcal{M}_{\ell,\ell'}$ = moduli of CY hypersurfaces in the family X_t + twisted ℓ level structure
$V[\ell](X_t) := H^{n-1}(X_t, \mathbb{Z}/\ell\mathbb{Z})^H \xrightarrow{\sim} \overline{\rho}^n_{E,\ell}$
$V[\ell'](X_t) \simeq \overline{I}(\chi)_{\ell'}$

<u>Theorem</u> (Beukers-Heckman, Matthews-Vaserstein-Weisfeiler)
$\exists$ integer N_0 s.t. if $\ell, \ell' > N_0$ then
$\mathcal{M}_{\ell,\ell'}$ is geometrically irreducible.
...... explicit determination of monodromy of the cohomology
... & monodromy mod ℓ full for almost all ℓ

Local-Global principle (Moret-Bailly).

Let $S = S_1 \amalg S_2$ be a finite set of places of $\mathbb{Q}$, $\infty \in S_1$, & $\mathcal{M}/\mathbb{Q}$ geometrically irreducible & assume (*): For all $v \in S_1$ $\mathcal{M}(\mathbb{Q}_v)$, $v \in S_2$ $\mathcal{M}(\mathbb{Q}_v^{unram})$ } is nonempty

Then $\exists\, F_1/\mathbb{Q}$ Galois in which all { $v \in S_1$ split completely, $v \in S_2$ unramified } s.t. $\mathcal{M}(F_1) \neq \emptyset$.

(F_1 totally real)

... strong geometric form of weak approximation.

Now it suffices to find ℓ, ℓ' with good properties for S_2

$\Rightarrow$ $\mathcal{M}_{\ell,\ell'}(\mathbb{Q}_\ell^{unram}) \neq \emptyset$, same for $\mathbb{Q}_{\ell'}^{unram}$.

Also must worry about condition 3.

... This can be done! ∎

Theorem Π cuspidal automorphic rep of $GL_n K$, F totally real. $[K:F]=2$ totally imaginary.

Suppose 1. Π_∞ cohomological

2. $\Pi^\vee \cong \Pi \circ c$ (complex conjugation)

3. $\exists$ v_0/F split in K/F s.t. Π_{v_0} is discrete series.

then

a. (Clozel, Kottwitz, Harris-Taylor, Taylor-Yoshida) $\exists$ compatible system $\rho_\ell = (\rho_{\ell,\Pi})$ s.t. $\forall \ell$ $\forall$ primes v of $\mathbb{Z}$ not dividing ℓ, $\rho_{\ell,\Pi}|_{\Gamma_v} = \mathcal{L}(\Pi_v)$ local Langlands rep + HT numbers & $\rho^\vee = \rho \otimes \chi_{cyc}^{1-n}$

--- construct in cohomology of unitary Shimura varieties.

b. (Harris-Labesse) The ρ_ℓ can be realized in $H^{n-1}(Sh(V(v)), L(\Pi_\infty))$

Cohomology of Shimura varieties in ℓ-adic local system