Jochen Heinloth - Class Field Theory

Note Title 1/4/2007

~~#~~ fields $\rightsquigarrow$ curves / $\mathbb{F}_q$ $\rightsquigarrow$ curves / $\mathbb{C}$

$C/\mathbb{F}_q$ smooth projective curve, geometrically connected
$k(C)$ = meromorphic functions on C

$x \in C$ point $\mathcal{O}_x$ = complete local ring at x
$\cong \mathbb{F}_q[[t]] \subset \mathbb{F}_q((t)) = K_x$

Artin's reciprocity law (unramified version)

$$\left\{ \begin{array}{c} \text{finite quotients of} \\ \text{idele class group} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Galois groups of} \\ \text{abelian unramified} \\ \text{extensions of } k(C) \end{array} \right\}$$

$$k(C)^* \backslash {\prod}' K_x^* / \prod \mathcal{O}_x^* \qquad \left\{ \begin{array}{c} \text{autom. groups of abelian} \\ \text{Galois covers } \tilde{C} \to C \end{array} \right\}$$

$$= \qquad\qquad\qquad\qquad =$$

$$\text{principal divisors} \backslash \bigoplus_{x \in X} \mathbb{Z} \cdot x \qquad \left\{ \begin{array}{c} \text{finite abelian quotients} \\ \text{of } \pi_1(C) \end{array} \right\}$$

divisors on C

$$=$$

$$\mathrm{Pic}_C(\mathbb{F}_q)$$

Construction: use Frobenius elements at points of C. to get reciprocity law (independence of principal divisors) use geometry.

Geometrize: • replace groups by their representations

RHS: representations of $\pi_1(C) \longleftrightarrow$ local systems on C

LHS: Characters of $\mathrm{Pic}(\mathbb{F}_q) \subseteq$ functions on Pic

Grothendieck: interesting functions should arise from sheaves!

Theorem (Deligne) There is a natural correspondence

$$\left\{ \begin{array}{c} \text{1-dim local} \\ \text{systems on } C \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{finite characters} \\ \mathrm{Pic}\, \mathbb{F}_q \longrightarrow \bar{\mathbb{Q}}_\ell^* \end{array} \right\}$$

Deligne

function-sheaf

$+ : C \times \mathrm{Pic} \longrightarrow \mathrm{Pic}$ (over any field)

$x, \mathcal{L} \longmapsto \mathcal{L}(x)$

$$\left\{ \begin{array}{c} \text{local systems } \mathcal{A}_L \text{ on Pic} \\ \text{with } \mathcal{A}_L|_0 \text{ trivialized} \\ \& \ (+)^* \mathcal{A}_L \simeq L \boxtimes \mathcal{A}_L \\ \text{for } L \text{ local system on } C \end{array} \right\}$$

Functions-Sheaves Dictionary

$f:\tilde{C} \to C$ is étale if smooth with 0-dim fibers
($f^* \Omega_C = \Omega_{\tilde{C}}$)

Example: points $\tilde{x} \to x$: $\tilde{x} = \operatorname{Spec} k[x]/p(x)$
Smoothness $\iff$ $p(x)$ is separable

May assume $\tilde{x} = \operatorname{Spec} \mathbb{F}_{q^n} \to \operatorname{Spec} \mathbb{F}_q$
$\operatorname{Aut} \tilde{x}/x = \langle \mathrm{Frob} = (\,)^q \rangle$

• $\pi : \tilde{C} \to C$
$\coprod x' = \pi^{-1}(x) \mapsto x \in C$

$$\begin{tikzcd}
\operatorname{Aut} \tilde{C}/C \arrow[r, phantom, "\supset"] & \operatorname{Stab} x' \arrow[d, "\wr"] \\
\langle \mathrm{Frob}_x \rangle \arrow[u, phantom, "\cup"] & \operatorname{Aut} x'/x \\
 & \langle \mathrm{Frob} \rangle \arrow[u, phantom, "\in"] \arrow[uul, bend left]
\end{tikzcd}$$

Frob_x well defined
up to conjugation
(depends on preimage
x' of x).

$\pi_1(C) = \varprojlim_{\tilde{C}/C \text{ Galois}} \operatorname{Aut} \tilde{C}/C \supset \{\mathrm{Frob}_x\}_{x \in C}$ conjugacy classes

Local system := representation of $\pi_1(C)$ $\rightsquigarrow$ $D^b_{et}(-)$

Get a map $D^b_{ét}(C) \longrightarrow \mathrm{Fun}\left(\coprod_N C(\mathbb{F}_{q^N})\right)$

$$K^\bullet \longmapsto x \longmapsto \mathrm{tr}(\mathrm{Frob}_x, K^\bullet|_x)$$

$$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\sum (-1)^i \mathrm{tr}(\mathrm{Frob}_x, H^i(K^\bullet|_x))}$$

Facts: $\mathrm{tr}_{K^\bullet \oplus L^\bullet} = \mathrm{tr}_{K^\bullet} + \mathrm{tr}_{L^\bullet}$

$\mathrm{tr}_{K^\bullet \otimes L^\bullet} = \mathrm{tr}_{K^\bullet} \cdot \mathrm{tr}_{L^\bullet}$

$f: X \longrightarrow Y$ $\quad \mathrm{tr}_{f^* K^\bullet} = f^* \mathrm{tr}_{K^\bullet}$

hard! $\quad \mathrm{tr}_{Rf_! K^\bullet} = f_! \mathrm{tr}_{K^\bullet}$ = sum over fibers

(Grothendieck-Lefschetz trace formula)

Čebotarev: $(\mathrm{Frob}_x)_{x \in X}$ dense in $\pi_1(X)$

$\Longrightarrow$ $\rho: \pi_1(X) \to GL_n \overline{\mathbb{Q}}_\ell$ irred

is determined by the corresponding function!

$\Longrightarrow$ injection on K-groups.

Proof of Deligne's Theorem

let $\mathcal{L}$ be a 1-dim local system on C

$C^d/S_d = C^{(d)} \xrightarrow{\pi_d} \mathrm{Pic}^d$ effective divisors of deg d

$\mathcal{L}$ local system on $C \rightsquigarrow \mathcal{L}^{\boxtimes d}$ loc sys on C^d

$$\mathcal{L}^{(d)} = \left(\mathrm{sym}_* \mathcal{L}^{\boxtimes d}\right)^{S_d}$$

So stalk of $\mathcal{L}^{(d)}$ at $\sum n_i x_i$

is $\bigotimes_i \left(\mathcal{L}|_{x_i}\right)^{n_i}$ .... in fact $\mathrm{sym}^* \mathcal{L}^{(d)} = \mathcal{L}^{\boxtimes d}$.

Fiber of π_d over $\mathcal{L}$ is $\mathbb{P}(H^0(C, \mathcal{L}))$
& this is a fibration for $d > 2g-2$. ..

BUT $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$ are simply connected!

so $\pi_1(C^{(d)}) = \pi_1(\mathrm{Pic}^d)$ $\quad d > 2g-2$

$\Rightarrow \mathcal{L}^{(d)}$ descends to a local system
$A_{\mathcal{L}}$ on Pic^d !

Moreover

$$\begin{tikzcd} C \times C^{(d)} \arrow[r, "\tilde{+}"] \arrow[d] & C^{(d+1)} \arrow[d] \\ C \times \mathrm{Pic}^{d} \arrow[r, "+"] & \mathrm{Pic}^{d+1} \end{tikzcd}$$

$\Rightarrow$ $(+)^* \mathcal{A}_L = L \boxtimes \mathcal{A}_L$.

Now extend to all d by additivity. $\blacksquare$