Jochen Heinloth - Geometric Hecke Operators

Note Title 1/5/2007

G reductive group (eg GL_n)

$\cup$

B Borel, torus

$N^0 \cup$

T

Langlands: study functions on $G(k(C)) \backslash \prod_{x \in C}' G(K_x)$

& decompose as a representation of $G(\mathbb{A}) = \prod_{x \in C}' G(K_x)$

Simplify: look at smaller function spaces

$$\mathrm{Fun}\left(G(k(C)) \backslash \prod_{x \in C}' G(K_x) / \prod G(\mathcal{O}_x) \right)$$

What is left here from action of $G(\mathbb{A})$?

still have action of $\mathrm{Fun}_{cpt}(G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x))$

by convolution (for all $x \in X$)

Note: $G(k(C)) \backslash \prod' G(K_x) / \prod G(\mathcal{O}_x) = \mathrm{Bun}_G(\ )(k)$

k-points of moduli of G-bundles on C:

use $\prod' G(K_x)$ as local gluing data for a G-bundle

Identify $K_x = k((t))$ $\mathcal{O}_x = k[[t]]$

$$G(K_x) = \coprod_{\substack{\mu \in \\ \mathrm{Hom}(\mathbb{G}_m, T)^+}} G(\mathcal{O}_x)\, \mu(t)\, G(\mathcal{O}_x)$$

dominant cocharacters

double coset decomposition

GL_n case: $G(k((t))) = \coprod_{d_n \geq \dots \geq d_1 \in \mathbb{Z}} G(k[[t]]) \begin{pmatrix} t_1^{d_1} & & \\ & \ddots & \\ & & t_n^{d_n} \end{pmatrix} G(k[[t]])$

In particular $\mathcal{H} = \mathrm{Fun}_{cpt}(G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x))$

has a basis given by characteristic fns of these double cosets,

Theorem (Satake) $\mathcal{H} \simeq \mathbb{C}[\mathrm{cochar}(T)]^W$

($= R({}^LG)$ representation ring of the dual group)

... in particular $\mathcal{H}$ is commutative.

Satake's trick: act $\mathcal{H}$ on $\mathrm{Fun}\left(T(\mathcal{O}_x) N(K_x) \backslash G(k((t))) / G(k[[t]])\right)$

.... this is a $\mathbb{C}[\text{cochar } T] \times \mathcal{H}$ bimodule & is free of rank one over LHS

$\Rightarrow$ get map $\mathcal{H} \longrightarrow \mathbb{C}[\text{cochar } T]$

as $h \longmapsto \mu(h)$ where $1 * h = \mu \cdot 1$.

(Note $G(K_x) = \coprod_{\substack{\mu \in \\ \text{cochar } T}} N(K_x)\, \mu(t)\, G(\mathcal{O}_x)$)

<u>Geometrization</u> $G[[t]]$ is a group scheme $/k$:

$G[[t]](R) := G(R[[t]])$ (note $\mathbb{A}^1[[t]] \simeq \mathbb{A}^\infty$ is a scheme)

OTOH $G((t))$ is a group <u>ind-scheme</u>:

eg $\mathbb{A}^1((t))$ is not a scheme: must require all but fin many negative coeff $=0$, not algebraic condition

$G((t))/G[[t]]$ is an ind-scheme$/k$, the <u>affine Grassmannian</u> Gr_G

Moreover $Gr_\mu = G[[t]]\,\mu(t)\,G[[t]] / G[[t]] \subset Gr_G$
is finite dimensional for any $\mu(t)$ cocharacter.

let $\overline{Gr_\mu}$ = its closure in Gr_G.

<u>Theorem</u> (Geometric Satake Isomorphism)
(Ginzburg, Mirkovic - Vilonen, Lusztig, Drinfeld ...)

$$\mathrm{Perv}_{G[[t]]}\, Gr_G \xrightarrow{\sim} \mathrm{Rep}({}^LG) \text{ as tensor categories}$$

Aside: Perverse sheaves. X any variety $\Rightarrow$ abelian category $\mathrm{Perv}(X) \subset D^b_{et}(X)$ with marvelous properties

1. For $Z \subset \overline{Z} \subset X$ with Z smooth & L a local system on X $\Rightarrow$ canonical object $IC(L) = j_{!*} L \in \mathrm{Perv}(X)$, &
$\mathrm{Perv}(X)$ is generated by such objects.

$H^*(X, IC(L))$ is dual to $H^*(X, IC(L^*))$

7. $X \xrightarrow{f} Y$ proper $\Rightarrow$

$$Rf_* IC(X) \in \bigoplus_i \underbrace{Perv}_{\text{semisimple}}[i] \in D^b(X)$$