Jochen Heinloth - Geometric Satake, continued

Note Title 1/5/2007

$$\mathrm{Perv}_{G[[t]]}(\mathcal{G}r_G) \simeq \mathrm{Rep}\ {}^LG \quad (/\mathbb{C})$$

Proof by Tannaka duality:

1. Show LHS is a $\otimes$ category (r.s.c)
2. Construct an exact $\otimes$ functor LHS $\longrightarrow$ Vect

   $\Rightarrow$ LHS $\cong$ Rep (some group)
3. Identify the Tannakian group with ${}^LG$.

How to construct $\otimes$?

First idea:
$$\begin{tikzcd} G((t)) \times G((t))/G[[t]] \arrow[r] \arrow[d, "p"] & G((t)) \times^{G[[t]]} G((t))/G[[t]] \arrow[d, "m"] \\ Gr_G \times Gr_G & Gr_G \end{tikzcd}$$

Given $P_1, P_2 \in \mathrm{Perv}_{G(\mathcal{O})}\mathcal{G}r$,

$p^*(P_1 \boxtimes P_2)$ descends to $P_1 \tilde{\boxtimes} P_2$ in middle,

$P_1 * P_2 := R\pi_! \, P_1 \tilde{\boxtimes} P_2.$

why should $P_1 * P_2$ be perverse & why symmetric??

<u>Nearby cycles definition</u>:

C curve, eg $\mathbb{A}^1$

Recall: $G((t))/G[[t]]$ = moduli of G-bundles on C + trivialization outside a point

Gr_C : choice of $x \in C$ + G-bundle trivialized outside x

$$\begin{array}{c} Gr_C \\ \downarrow \\ C \end{array} \quad \text{fibers} \cong Gr_G.$$

$Gr_{C \times C} = \{ (V, x, x', \varphi) \quad V$ G bundle on $C \ni x, x'$, φ triv of V on $C \setminus x, x'$ $\}$

Fiber over (x,x) is Gr, over $(x \neq x')$ is $Gr \times Gr$

$$\begin{tikzcd}
\widetilde{Gr}_{C\times C} \arrow[d, "p_1"] \arrow[dd, bend right, "\tilde{f}"'] \\
Gr_{C\times C} \arrow[d, "f"] \\
\mathbb{A}
\end{tikzcd}$$

$\widetilde{Gr}_{C\times C} = \{ (V, V', x, x', \varphi, \varphi') : \varphi : V|_{C\setminus x} \xrightarrow{\sim} \mathrm{triv}|_{C\setminus x}$

$\varphi' : V'|_{C\setminus x} \xrightarrow{\sim} V|_{C\setminus x} \}$

fibers $\cong Gr_0 \times Gr_{Gr}$

smooth locally trivial fibration

Define $P_1 * P_2 = \Psi_f (P_1 \boxtimes P_2)$

$f : Gr_{C\times C}|_{\text{antidiagonal}} \longrightarrow \mathbb{A}^1$

for antidiagonal $\mathbb{A}^1 \xrightarrow{(t,-t)} \mathbb{A}^1 \times \mathbb{A}^1$.

Thm 1. $P_1 * P_2$ is perverse!

2. $P_1 * P_2 \cong P_2 * P_1$ functorially

3. This gives the same convolution product

$\Psi_{\tilde{f}}\ P_1 * P_2 = P_1 \tilde{\boxtimes} P_2 \qquad \widetilde{Gr}_{C\times C} \xrightarrow{\tilde{f}} C\times C$

& $P_0 = m$ in previous definition

& Ψ commutes w/ proper pushforward

4. Ψ_t commutes w/ proper pushforward

$\Rightarrow \quad H^*(Gr, P_1 * P_2) = H^*(Gr, P_1) \otimes H^*(Gr, P_2)$

Problem Calculate $H^*(Gr, IC(Gr_\mu))$

Recall: Satake looked at $N((t))$ orbits in $G((t))/G[[t]]$

Let $S_\nu = N((t))\, \nu(t)\, G[[t]] \,/\, G[[t]]$

$T_\nu = N^-((t))\, \nu(t)\, G[[t]] \,/\, G[[t]]$

Theorem (Mirkovic-Vilonen)

a. $S_\nu \cap Gr^\mu = \begin{cases} \emptyset & \nu(t) \in \overline{Gr^\mu} \\ \text{equidimensional of dim} \\ \rho(\nu+\mu) & \mu \text{ dominant} \end{cases}$

b. $P \in Perv_{G[[t]]} Gr \Rightarrow$

$H^x_c(S_\nu, P|_{S_\nu}) = 0$ unless $x = \rho(\nu+\mu)$

& $\quad \cong H^*_{T_\nu}(Gr, P)$ in general

<u>Corollary</u> $H^*(Gr_G, P) = \bigoplus_{\nu} H^*(S_\nu, P|_{S_\nu})$

is an exact faithful functor.

in particular get fiber functor into

$\mathrm{Rep}\ {}^LT$, get weight decomposition!