A. Kapustin: D-Branes, Mirror Symmetry & all that 7/28/03

Motivation: Homological mirror symmetry (math)
Topological D-branes ... spacetime superpotential in CY compactifications .. closed vs open strings (physics)
A. Sen: can describe everything from open strings & condensing tachyons ... explicit in topological string theory.

(X,ω,B) Calabi-Yau manifold : complex manifold of Kähler type
Kähler form ω, $c_1(X)=0$
B-field: closed 2-form (in this situation)
Physics depends only on class $[B] \in H^2(X;\mathbb{R})/H^2(X,\mathbb{Z})$

$(X,\omega,B) \Longrightarrow N=2$ SCFT, construct only perturbatively in $\frac{1}{\text{volume}}$ [no proof it converges!]
... ∞-dim vector space of states, 2 copies of $N=2$ superVirasoro algebra.
gauge symmetry of string theory $N=4$ Super Vir. can talk e.g. about isoms of such as $N=1$ SCFT or of more refined $N=2$ isomorphisms.

When do two CYs (X,ω,B), (X',ω',B') give isomorphic $N=2$ SCFTs? often very distinct geometric data... e.g. for tori can determine explicitly - for ω, B constant - conditions for isomorphisms. e.g. T, $T^\vee$ with $B=0$ (dual tori) give isomorphic $N=2$ SCFTs. (T-duality)

$N=2$ superVir: even part J_n (R-current) $n\in\mathbb{Z}$
L_n (stress-energy)
odd part Q_{+n}, $\bar{Q}_{+n}$ $n\in\mathbb{Z}$
Symmetric wrt $J \longrightarrow -J$ $Q \longleftrightarrow \bar{Q}$:
mirror involution.

[not CY3s yet..]

$N=1$ isom of $N=2$ SCFTs : can require it to induce flip (mirror) on J, Qs.

If induces flip on <u>both</u> copies of SuperVir $\longleftrightarrow$ just reverse complex structure $X \longleftrightarrow \bar{X}$.

But if require <u>identity</u> on one $N=2$ supervir & mirror on other

$\Longrightarrow$ mirror symmetry!

$$(X, \omega, B) \rightsquigarrow N=2 \text{ SCFT} \xleftrightarrow[\text{mirror}]{?} N=2 \text{ SCFT} \rightsquigarrow (\hat{X}, \hat{\omega}, \hat{B})$$

For tori easy to argue...

[Mirror automorphism likely only out of $N=2$ preserving $N=1$ subalgebra]

Instead of SCFT let's look at its topologically twisted versions: B-model $\longleftarrow$ $N=2$ SCFT $\longrightarrow$ A-model

+ integrality condition

topological twist: First redefine spins of all fields (coupling to 2D gravity): one Q becomes spin 0, BRST operator

.... need redefined spins to be integral!
of all fields

A-model:

$$h' = h + \frac{q}{2}$$
$$\bar{h}' = \bar{h} - \frac{\bar{q}}{2}$$

B-model:

$$h' = h - \frac{q}{2}$$
$$\bar{h}' = \bar{h} - \frac{\bar{q}}{2}$$

For SUSY σ-model this integrality always holds, but eg for some Landau-Ginzburg theories only B-twist not A-twist makes sense!

Non CY: $N=2$ <u>nonconformal</u> σ-model, can do only A-twist..

[grading + $\mathbb{Z}/2$ grading have to be compatible!] (A

$$(X, \omega, B) \xleftrightarrow{\text{mirror}} (\hat{X}, \hat{\omega}, \hat{B})$$

B-model $\longleftrightarrow$ A-model

B-model

Mirror symmetry interchanges A, B models ...
Don't know what the $N=2$ SCFT is but can rigorously define A, B model.

TFT : graded supercomm. algebra with inner product $\langle u, v \cdot w\rangle = \langle uv, w\rangle$
Supercomm Frobenius algebra

A-model : $\bigoplus_{pq} H^{p,q}(X)$ with quantum cohomology product structure - deformation of cup product.

$tr\ u = \langle u, 1\rangle$

B-model $\bigoplus_{pq} H^{p}(\Lambda^{q} TX)$, usual product
trace = integration against holomorphic top-form

$$H^{p}(\Lambda^{q}TX) \xrightarrow[\text{top form}]{\sim} H^{n-q,p}(X)$$

Need richer structure to feel more of $N=2$ SCFT!
Replace algebras by categories, topological D-branes.

A-model : symplectic structure only
B-model : holomorphic structure only (forgetting B-field for now)

Kontsevich: A-model $\rightsquigarrow$ derived Fukaya category (A-type D-branes)

B-model $\rightsquigarrow$ $D^b(X)$ derived category.

All categories here will be additive, $\mathbb{C}$-linear
- morphisms are spaces of topological open strings between two D-branes.
Categories from physics: arrows are symmetric ... $\downarrow \Rightarrow \uparrow$??

$Coh(X)$ - (abelian) category of coherent sheaves on X

e.g. $\begin{array}{c} E, \bar\partial \\ \downarrow \\ X \end{array}$ holomorphic vector bundle

To make vector bundles into abelian category: need to add more objects, kernel & cokernel can jump..
Coherent sheaf := coker of hol map of vector bundles.

But torus & dual torus have same $N=2$ SCFT but different categories of coherent sheaves! $\Rightarrow$ replace by bounded derived categories, which will be equivalent.

$$D^b(X) : 0 \to \ldots \to A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \to \ldots \to 0$$

bounded complexes

Step 1: quotient morphisms by those homotopic to zero.
Think of complexes as $\mathbb{Z}$-graded sheaf, branes & anti-branes, differential $d : A^\bullet \to A^{\bullet+1}$, $d^2=0$ "BRST operator"
$[d,\varphi]=0$: morphisms are BRST closed.
Identify $\varphi \sim \varphi + [d,\alpha]$ α of deg -1.
ie quotient by BRST exact morphisms

Step 2 Identify complexes related by quasi isomorphism

Can think of $\mathbb{Z}/2$ graded objects: boundedness hypothesis means can add all even & all odd spaces, get something of finite rank

$$\underline{Ext^k(E,F)} = H^k(X, E^* \otimes F)$$

Fukaya category X symplectic manifold, $c_1(X)=0$
structure group is $Sp(n)$, universal ($\mathbb{Z}$) cover is infinite metaplectic group .. double cover appears in quantization .. Lift structure to $Mp_\infty(n)$.
obstruction lies in $H^2(X,\mathbb{Z})$ $\longleftrightarrow c_1(X)$.
If $c_1(X)=0$ Choose metaplectic structure
$Lag(X)$ bundle of Lagrangian Grassmannians of TX.

$\downarrow$ π_1 (Lag Grass) is $\mathbb{Z}$, lift

X to this:

$c_1 = 0$ Can lift to universal cover

$$\begin{tikzcd} \widetilde{Lag}(X) \arrow[r, "\mathbb{Z}"] \arrow[d] & Lag(X) \arrow[dl] \\ X & \end{tikzcd}$$

Fukaya category: objects (Y, E, ∇) [B-field $= 0$ for now]

$Y \hookrightarrow X$ Lagrangian submanifold

E, ∇ Hermitian vector bundle with flat unitary connection

Metaplectic structure $\Rightarrow$ (TY) section of $Lag(X)|_Y \to Y$ over Y

Would like to lift it to $\widetilde{Lag}(X)|_Y \to Y$

Not always possible: e.g. 2D symplectic vector space, can't lift for

circle, twists around $\pi_1(Lag)$
- obstruction is Maslov class in $H^1(Y, \mathbb{Z})$

<u>On CY:</u> $\Omega \in H^{n,0}(X)$ holomorphic top-form

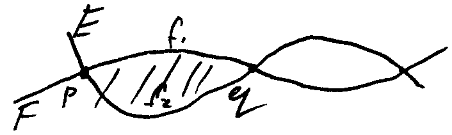
$$\Omega|_Y = f \cdot vol_Y$$ multiple of volume form

f is nonvanishing & Maslov class is $[\alpha = f^{-1} df]$

$\alpha = 0$ can define $\log f$ consistently; choice of log = grade

Obstruction vanishes $\Rightarrow$ "gradable Lagrangian submanifold"

<u>Morphisms</u>



generic intersections are points — assume transversal, if not deform via Hamiltonian flow

Hom space $V = \bigoplus_p Hom(E_p, F_p)$

Grade by integer part $\left[\frac{\log f_1 - \log f_2}{2\pi i}\right]$

Component of BRST operator between two points p, q whose grades differ by one:

$$d_{p,q} = \sum_{\substack{\text{hol maps} \\ D \to X}} \pm e^{-\int \omega} Hol_{qp}(E)^* \otimes Hol_{pq}(F)$$

holonomy from $p \to q$ using connection

- $Hol_{qp}(E)^{*} \otimes Hol_{pq}(F)$ defines map $E_p^{*} \otimes F_p \to E_q^{*} \otimes F_q$

Weight by ~~volume~~ symplectic area of disc.

Sum may not be finite! Define sign! Check $d^2 = 0$! (doesn't in general...)

Cohomology of d is Floer homology, make it into the hom space in our category. — Floer category

But really we want not cohomology but the complex itself as Hom space... No longer associative composition → A_∞ category. Kontsevich: take derived category of this A_∞ category, propose this as category of A-branes if ~~n=0~~:

$\overline{\mathcal{M}}_{g,n,h_1,\dots,h_m}$ dim $6g-6+2n+3m+\sum h_i$

h_i marked points on each of m boundary components

$3m$: to move ∂ component: move radius & center....

Something shifted to ∂: suspend, take symmetric power then suspend 3 times

Graded vector space at vertex: get shift of closed

Correl fns defined on $\mathrm{Sym}(s^{-2} C_{\text{closed}}) \otimes s^{-3}\,\mathrm{Sym}(s^{-1} C_{\text{open}})$

ex. $g=0=n$, $m=1$ $\Rightarrow$ dim h_1-3:

get $(h_1-1)!$ copies of Stasheff polyhedron K_{h_1-1} (depending on ordering of points)

Each component has nontrivial action of cyclic group C_{h_i}

$$\overline{\mathcal{M}}_{g,n,h_1,\dots,h_m} = \mathrm{Ind}_{C_{h_1}\times\dots\times C_{h_m}}^{S_{h_1}\times\dots\times S_{h_m}} \overline{\mathcal{M}}^*_{g,n,h_1,\dots,h_m}$$

fix cyclic order on marked pts in each ∂ component, cyclic gp acts

Stability: all $h_i>0$

So just look at $\overline{\mathcal{M}}^*$.

$g=0$ $n=1$ $m=1$: 1 pt in interior of disc, move to center $\Rightarrow$ dimension h_1-1

[Twisting in $N=2$: moving from one component of Picard to another

$N=2$ super RS: RS + 2 line bundles, their ratio $\cong$ square root of canonical

Normal bundle to sole: odd part splits into sum of 2 line bundles

(z,ξ) coords on $N=1$: $N=1$ SUSY $\overline{D} = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial z}$

Conn. kills change of coords $(x,\xi)(z,\xi)$, defines pseudogp of transformations

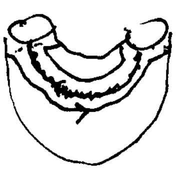
$N=2$: have two $\overline{D}$'s, their commutator is ∂]

$g=0$ $n=1$ $m=1$ W_{h_1} cyclohedron

$g=0$ $n=0$ $m=2$ Cylinder with points at each end $\Rightarrow$ Cardy condition : Z_{h_1, h_2} Cardyhedron

All spaces are homotopy equivalent to spaces with all $h_i = 1$, & homeomorphic to this $\times$ cube — in fact as stratified spaces.

Points collide $\Rightarrow$ disc bubbles off 



$Z_{1,1,i}$ Can do half-Dehn twist along curves relating 2 pts on boundary : Γ of Dehn twist along curve ∂ of tubular nbhd of curve

Strata $\longleftrightarrow$ simplices in curve complex! Curves that don't cross each other : collapse these curves to an

$Z_{1,1}$: 

maximal degenerations : 

+ one 1-dim stratum! forget one of these

Moduli of annulus = punctured disc

dim = 2

TFT : any two 0-dim strata give same correlation fns since they're cohomologous $\cdots \Longrightarrow$ WDVV eqs

 $\overline{\mathcal{M}}_{0,4} = S^2$ $\overline{\mathcal{M}}_{0,0,4}$ $\bullet\!\!-\!\!\!-\!\!\bullet$ (6 copies, though)

$\Rightarrow$ associativity in closed theory

3 ways to associate in commutative algebra

$\Rightarrow$ associativity in open theory

Stasheff associahedron

$\overline{\mathcal{M}}_{0,1,2}$ ~~$\bullet\!\!-\!\!\bullet$~~ pairing $2_*, 2^*$ between open/closed strings

$\overline{\mathcal{M}}_{0,1;2}$ ∘——∘ $2_*(ab) = 2_*(ba)$ $2: \mathcal{O} \to \mathcal{C}$

(~~[illegible]~~ 1 boundary condition, A associative

$2: A/[A,A] \longrightarrow$ bulk commutative algebra

Pair of ordin strata ⟹ equations

 ⟹ Cardy condition

Any 2 O-dim strata in any of these moduli can be joined by a sequence of moves made out of these.. --- need to know curve complex is connected...

A 

$2^* 2_a{}^a$ open ⟶ closed ⟶ open

 2 disks w/ 3 marked points, taking trace ...

$\sum a_{(2)} a_{(1)}$

Closed theory is deformation complex for open theory:

$\overline{\mathcal{M}}_{0,1;2}$ 1-dim: 2. ~~[illegible]~~
internal state ⟹ linear maps of degree one, deformation of differential

internal states deform A_∞ structure on ∂;
interior is BV algebra on Hochschild cohomology of the algebra...

WDVV Throw in pairs of states in interior ⟹ deformation of algebra

Same for boundary algebra: interior gives deformation of category

Frobenius algebra $A \cong A^*$ as A-bimodules,

A associative: A-bimodule $M \longleftrightarrow$ algebra (square-zero) on $A \oplus M$

Cyclohedron : link interior & boundary vertex, relate to deformation theory : internal theory ⟶
cyclic cochains on boundary

$$A(x)\,B(y) \sim \sum_n \frac{C_n^{\pm}}{(x-y)^n} \qquad C_n^{\pm} = [A\,B]_n^{\pm}$$

$x \gtrless y$ two different OPE's

V vector space $\qquad F(V)$ free Leibniz algebra on V

M = mapping cone of generators $V \to dV$
take free graded Lie algebra generated by this
inside take expressions of form $[dV, dV, dV, \ldots, V]$
brackets containing all dV's except one
– derived bracket $[dX, Y]$ on free Lie algebra.

$\boxed{\text{Clifford algebra of vector bundle } V \text{ on } M \text{ is Morita eq to } C^\infty(M) \text{ iff } V \text{ is roughly Spin}^c}$

– Morita context will give vector bundle of which Cliff is endomorphism, i.e. Spin^c structures,
Clifford modules will form a gerbe, in $\mathbb{Z}_2$
– the $\mathbb{Z}_2$ gerbe associated to V.

A. Kapustin II 7/29/03

Physical motivation for homological mirror conjecture

Kontsevich: A-model B-model

derived Fukaya category $DF(X)$ — $D^b(X)$

Consider SUSY σ-model on worksheets with boundary

∂ conditions — topological twist puts constraints on boundary conditions - must be preserved by BRST.

Σ

TFT $\leftrightarrow$ supercom Frobenius algebra

Objects of $D^b(X)$: complexes of coherent sheaves or of hol vector bundles (X smooth).

Neumann boundary condition in σ-model:

$\partial_{normal} \phi |_{\partial\Sigma} = 0$ preserves B-type BRST charge

$\phi: \Sigma \to X$ $\Rightarrow$ B type boundary condition

$\leftrightarrow$ D-brane wrapping whole manifold, doesn't constrain positions of boundary

Gauge field on brane must be zero $\to$ attach trivial bundle on X.

Witten: any holomorphic vector bundle $\Rightarrow$ B-type ∂ condition:

Line bundle (E, ∇) on X st. $\bar{\nabla}^2 = 0$ $\Longrightarrow$

boundary condition $\partial_n \phi = F \partial_t \phi$ correction to Neumann condition

$F = -i\nabla^2$ + fermion terms

This is a SUSY preserving (B type) boundary condition

$$g_{ij} \partial_n \phi^j = F_{ij} \partial_t \phi^j$$

Can generalize to E vector bundle, need extra degrees of freedom living on boundary

A-type boundary: Neumann bad, Dirichlet in all directions bad ... but let $Y \subset X$ be Lagrangian.

(E, ∇) vector bundle with $\nabla^2 = 0$ – can define A-type ∂ condition .... at least at classical level – in quantum case get anomalies in some A-branes. R-symmetry anomaly $\longleftrightarrow$ Maslov index

More general B-branes: (Y, E, ∇)
$Y \subset X$ complex submanifold, E holomorphic vector bundle on Y (∇ has $\bar{\partial}^2 = 0$)

$\mathbb{Z}/2$ graded complexes of vector bundles $\quad E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \quad FG = GF = 0$

E_0 branes E_1 antibranes, $F = G = 0$ just a brane & an antibrane, no relation. $F, G \neq 0$: tachyon mode connecting the brane & antibrane (from GSO projection)

$$Z = \int \underset{\varphi: \Sigma \to X}{\mathcal{D}\varphi} \underset{\text{fermion}}{\mathcal{D}\psi}\, \underset{\text{bulk action}}{e^{-S(x,\varphi)}}\, \underset{\text{holonomy operator}}{\mathrm{str}\, P\exp\Big(\int d\theta\, dz\,} \underbrace{A_\mu(\phi) D_\theta \phi^\mu}_{a}\Big)$$

$A_\mu(\phi)$ superconnection on superfield
$\phi^i = \varphi^i + \theta\psi^i$

So brane stack is really a superconnection ~~ϕ~~

a superconnection

$$a = \begin{pmatrix} A^1_\mu D_\theta \phi^\mu & T^\dagger(\phi) \\ T(\phi) & A^2_\mu D^\mu \phi \end{pmatrix}$$

T = tachyon field = $F + G^\dagger$
F holomorphic, G antihol for supersymmetry

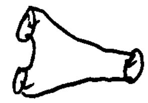
Why do boundary conditions form a category?
M, N branes $\Rightarrow$ quantize string on interval with boundary conditions M, N $\Rightarrow H_{MN}$ vector space of morphisms.

$M \longrightarrow N$

If M, N are vector bundles, $H_{MN} = \bigoplus_p H^p(X, N^* \otimes M)$ (Witten)
which are morphisms in derived category
Sharpe + Katz, Porter, Calderaru check for more complicated boundary conditions that do indeed get Ext groups.

In general haven't checked matching between morphism of branes & derived category!

TFT w/o boundary: quantize circle $\rightsquigarrow H$ vector space

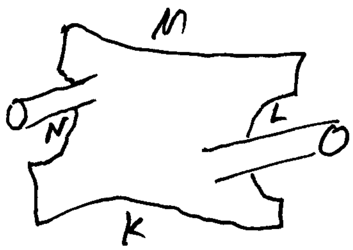


oriented surface $\Rightarrow$ path integral associates map $\underset{\text{incoming}}{\bigotimes H} \longrightarrow \underset{\text{outgoing}}{\bigotimes H}$

product $H \otimes H \to H$

counit/trace, identity, determine theory.

Boundary TFT:  $\rightsquigarrow H_{MN}$ $\quad O \rightsquigarrow H$



$H_{MN} \otimes H_{NK} \otimes H$
$\downarrow$
$H_{LM} \otimes H \otimes H_{KL}$

propagating boundaries labelled with ∂ condition (can't change on these)

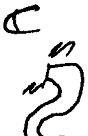
$H_{MN} \otimes H_{NK} \to H_{MK}$: category structure

Elements of H_{MN}: boundary changing operators, & composition law is that of boundary changing operators

Turn on tachyon: relevant perturbation, breaks boundary conformal invariance $\Rightarrow$ boundary RG flows, get two different descriptions in IR & UV: in one get (hopefully) resolution, in other get something supported on submanifold. Under RG flow, tachyon goes to ∞ wherever it's not an isomorphism, concentrates it to something with same cohomology

Have symmetry of B-branes: $H_{MN} \simeq H_{NM}^*$

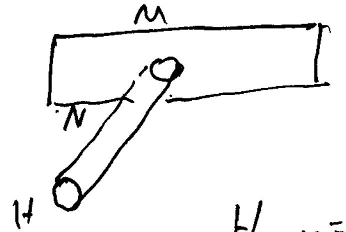
ie pairing $H_{MN} \otimes H_{NM} \to \mathbb{C}$ nondegenerate.



Not true for a general category!

By Serre duality Holds for Calabi-Yaus, not for general complex manifold (twist by canonical bdle)

H_{MN}  $\longrightarrow H_{NM}^*$



$H \otimes H_{MN} \longrightarrow H_{MN}$ module structure

CY : $H = \bigoplus_{p,q} H^p(\Lambda^q TX)$

$H_{MN} = \bigoplus_p \mathrm{Ext}^p(M,N)$

What is action of H on H_{MN}?

Kontsevich: integrated closed strings with open strings

H = Hochschild cohomology of $D^b(X)$.

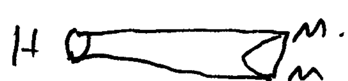
$\Delta \subset X \times X$ $\quad \mathrm{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \bigoplus_{p,q} H^p(\Lambda^q TX)$ as ~~alg~~ vector spaces

[PBW map, not algebra isomorphism]

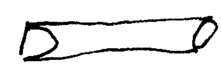
$\mathrm{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ acts on $\mathrm{Ext}^\bullet_X(M,N)$ every M, N

explicitly if $h : \mathcal{O} \longrightarrow \mathcal{O}[k]$ $\quad \varphi : M \to N[\ell] \Rightarrow M \otimes \mathcal{O} \xrightarrow{h\otimes\varphi} N \otimes \mathcal{O}[k+\ell]$

On level of Chern characters difference between $\bigoplus H^\bullet(\Lambda^\bullet TX)$ and $\mathrm{Ext}^\bullet(\mathcal{O},\mathcal{O})$ as rings is $\sqrt{\hat{A}}$ ... need to multiply by the Atiyah class .....



$\alpha^*_M : H \longrightarrow H_{MM}$

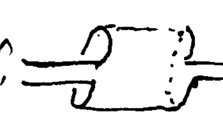


$\alpha_M : H_{MM} \longrightarrow H$

$V = \mathrm{Ext}^\bullet(M,N)$

$\mathrm{Ext}(N,N) \ni f_1$ $\quad$ $\mathrm{Ext}(M,M) \ni f_2$

Action $F(f_1, f_2) : V \longrightarrow V$ giving bi-module map

Cardy condition  $=$ f_1  f_2

$\mathrm{str}\, F = \langle \alpha_M(f_1), \alpha_N(f_2) \rangle$

Supertrace of F should be given by closed string sector

$f_1 = f_2 = \mathrm{Id}$: HRR theorem

Caldararu proves general case

eg.. Let M, N be vector bundles
$f_1, f_2 \in H^*(\mathrm{End}\, M) \_ H^*(\mathrm{End}\, N)$
$f_1 = f_2 = id$ get Chern character calculated via Hirzebruch RR.

Landau-Ginzburg Models $X = \mathbb{C}^n$, $W: X \to \mathbb{C}$
superpotentials, holomorphic to preserve $N=2$ supersymmetry, though not superconformal ...
$N=2$ LG $\underset{twist}{\rightsquigarrow}$ B-model only
$\mathcal{O} \rightsquigarrow H = \mathbb{C}[x_1 \ldots x_n] / (\partial_i W)$ Jacobi ring

1 Hori-Iqbal-Vafa: 2 conditions: need submanifolds where W is constant.
e.g $W = x^n$ stable branes live only at $x=0$.
Add fields $A^{(1)}, A^{(0)}, T$ as before $\xrightarrow{BRST}$ $\bar{\partial}_{1,1}^2 = 0$
$T = F + G^+$,

(F, G holomorphic) $E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$
$FG = W \cdot Id$ + const, same for $G \cdot F$:
"twisted complex": F, G components of BRST operator on boundary $D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$, $D^2 = W \cdot Id$
to cancel bulk contribution to D^2 ...
Rank E_0 has to be = rank E_1

A. Kapustin III

<u>D-branes in Landau-Ginzburg Models</u>

(B-type) - similar but simpler than CY case.
Closed string algebra = Hochschild cohomology of category of D-branes including Frobenius algebra structure, in LG & CY cases; recover closed string algebra from open strings!

<u>LG</u> $W: X \longrightarrow \mathbb{C}$ holomorphic, we'll restrict to case $X = \mathbb{C}^n$

Try Neumann boundary conditions:

$\phi: \Sigma \longrightarrow X$ $\quad \partial_n \phi^I|_{\partial\Sigma} = 0$

$N=1$ supersymmetry requires also $\psi_+^I - \psi_-^I|_{\partial\Sigma} = 0$

Variation of bulk action gives term $\qquad \epsilon$ Grassmann variables

$$\delta S_{bulk} = \int_{\partial\Sigma} \tfrac{i}{2}\,\bar{\epsilon}\,(\psi^{\bar{i}}\,\partial_{\bar{i}}\bar{W} + \psi^i \partial_i W) \quad + h.c.$$

- not invariant! Warner problem : breaks $N=2$ SUSY

To maintain SUSY, change boundary conditions;

one way: impose Dirichlet in some conditions, worldsheet boundary on level surfaces of W.

more general ∂ conditions: brane wraps all X but turn on gauge & tachyon field ... can set gauge $= 0$ (bundles trivial) but tachyon term is what will save us.

$$S_{bdry} = \tfrac{1}{2}\int d\tau\, d\theta \left(\bar{\Gamma} D_\theta \Gamma + \Gamma T + \bar{T}\bar{\Gamma}\right)$$

$Hom(E_0, E_1) \ni T = T(\Phi^I(\tau,\theta))$ $\qquad \Phi^I = \phi^I + \theta\psi^I$ boundary superfield

impose Neumann on Φ^I. $\qquad \Gamma(\tau,\theta)$ fermionic superfield on ∂

$E_0 \quad E_1$ brane & antibrane. $\quad \Gamma \in Hom(E_1, E_0)$

For simplicity take E_0, E_1 line bundles (unitary)

$\Gamma(\tau,\theta) = \eta(\tau) + i\theta P(\tau)$ $\qquad \eta$ Fermion P boson valued in $Hom\, E_0, E_1$

$$S_{bdry} = \frac{1}{2}\int_{\partial\Sigma} i\bar\eta\partial_0\eta - \frac{i}{2}\left(i\psi^{\bar I}\partial_{\bar I}T\right)\eta + h.c.) - \frac{1}{2}\bar T T$$

Kapustin-Li

special cases

Kachru-McGreevy-Hellerman

$N=1$ SUSY automatic. For $N=2$ study variation to cancel bulk variation:

$$\delta\eta = \varepsilon S_1 + \bar\varepsilon S_2$$

write... ~~...~~ $\bar T = F + G^+$

$F \in Hom(E_0, E_1)$ $\quad G \in Hom(E_1, E_0)$ both holomorphic

$S_1 = iF^+ \qquad S_2 = -iG$

Compute $\delta S_{bdry} = \frac{1}{2}\int \bar\varepsilon\left(\psi^{\bar i}\partial_{\bar i}(F^+G^+) - \psi^i\partial_i(FG)\right) + h.c.$

To ~~cancel δS_{bulk}~~ must take

$$\boxed{FG = GF = iW + \text{const}}$$

Spectrum of topological open strings / spaces of morphisms:

$$Q_{bdry} = -iF\eta + iG\bar\eta \qquad \{\eta, \bar\eta\} = 1$$

$$= \begin{pmatrix} 0 & iG \\ -iF & 0 \end{pmatrix} \qquad \text{Realize as } \eta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \bar\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$Q_{bdry}^2 \neq 0$ ... aspect of Warner problem:

$$(Q_{bulk} + Q_{bdry})^2 = 0$$

$$Q_{bulk} = \int d\sigma\left(g_{i\bar j}(\psi_+ + \psi_-)^{\bar j}\partial_0\phi^i + (\psi_+ - \psi_-)^{\bar j}\partial_1\phi^i - \frac{i}{2}(\psi_+ - \psi_-)^i\partial_i W\right)$$

$$Q_{bulk}^2 = -i\,\cancel{\oint_{\partial\Sigma}}\, W dx^0\Big|_{\partial\Sigma} = -Q_{bdry}^2$$

F, G holomorphic ensures $\{Q_{bulk}, Q_{bdry}\} = 0$

$\delta\phi^i = 0 \qquad \delta\phi^{\bar i} = -\bar\varepsilon\psi, \quad \partial\psi^i = -2i\bar\varepsilon\phi^i$

BRST invariant quantities: expressions of form $\mathcal{O}_f = \psi^{\bar i_1}\ldots\psi^{\bar i_k} f_{\bar i_1\ldots\bar i_k}(\phi)$

$$\{Q_{bulk}, \mathcal{O}_f\} = \mathcal{O}_{\bar\partial f}, \quad \text{~~...~~}\ldots$$

$f \in End(E_0 \oplus E_1)$

$$\{Q_{bdry}, \mathcal{O}_f\} = \mathcal{O}_{[Q_{bdry}, f]}$$

States are vectors $f \in \bigoplus_p \Omega^{0,p}(X) \otimes \mathrm{End}(E_0 \oplus E_1)$

$$Qf = \bar\partial f + [Q_{bdry}, f]. \quad : \quad \bar\partial^2 = [\bar\partial, Q_{bdry}] = 0,$$

Q_{bdry}^2 is central, so $[Q_{bdry}, [Q_{bdry}, f]] \sim [Q_{bdry}^2, f] = 0$

So $Q f = 0$.

$X = \mathbb{C}^n$ affine, Dolbeault cohomology vanishes, so can restrict to 0-forms! write f in components $f = f_0 + f_1 + \dots + f_r$
$f_i \in \Omega^i \otimes \mathrm{End}(E_0 \oplus E_1)$

$$\bar\partial f + [Q_{bdry}, f] = 0 \implies f_r = \bar\partial \beta_{r-1} \quad \text{exact}$$

Redefine $f' = f - (\bar\partial + [Q_{bdry}, -]) \cdot \beta$

f' cohomologous to f but we decreased top degree occuring in f ... continue to get f Q-cohomologous to something in degree zero, $f = f_0$
f BRST closed $\implies$ $\boxed{\bar\partial f_0 = 0, \quad [Q_{bdry}, f_0] = 0}$

So space of physical states are holomorphic endomorphisms of $E_0 \oplus E_1$ commuting with Q_{bdry}.
r branes, r antibranes $\implies$ $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, entries $r \times r$ matrices

more precisely $f = \alpha_0 + \eta\alpha_1 + \bar\eta\alpha_2 + \eta\bar\eta\alpha_3$ $\quad \alpha_i$ combos of $ABCD$
$\implies B, C$ odd, A, D even.

$$D : f \longmapsto [Q_{bdry}, f] = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$$

<u>Example</u> A_{n-1} Minimal model $\quad W = z^n$
in IR flow to $N=2$ minimal models, $N=2$ R(S)CFT

ADE potentials $\longleftrightarrow$ finitely many indecomposables
$\rightsquigarrow$ finitely many indecomposable objects in category
ie Branes are quiver categories...

Objects : $E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$ (trivial v. bundles on $\mathbb{C}^n$)

$FG = W \cdot \mathbb{1}$ + const ; $G \cdot F = W \cdot \mathbb{1}$ + const

Morphisms : ~~endomorphisms~~ $E_0 \rightleftarrows E_1$ homs of $\mathbb{Z}/2$ graded spaces

$$\begin{tikzcd} E_0 \arrow[r, shift left] \arrow[d, "\varphi"'] \arrow[dr] & E_1 \arrow[l, shift left] \arrow[d] \arrow[dl] \\ E_0' \arrow[r, shift left] & E_1' \arrow[l, shift left] \end{tikzcd} \qquad \varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

On this space of endomorphisms have $D: \varphi \to [Q_{bdry}, \varphi]$

φ D-closed $\iff$ pass to complexes

D-cohomology $\iff$ derived category . = space of physical states
= BRST cohomology

This category of twisted complexes suggested for LG models by Kontsevich

DG category here has more info than homotopy category - eg can calculate 4/pt functions from here not from Q-cohomology.

topological field theory contained in homotopy category

topological string theory " " DG category ?

(couple to topological gravity ??)

Composition of morphisms / boundary OPE :

e.g. $W = z^n$, can take $F = z^{n-k}$ $G = z^k$

1 by 1 matrices , these generate...

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad D\varphi = \begin{pmatrix} Cz^k + Bz^{n-k} & (D-A)z^k \\ (A-D)z^{n-k} & Bz^{n-k} + Cz^k \end{pmatrix}$$

$[k \le \frac{n}{2}]$ Solve $D\varphi = 0$: $A = D$, $C = -Bz^{n-2k}$

D-exact : can shift $B \rightsquigarrow B + (\text{polynomial}) \cdot z^k$

$A \rightsquigarrow A + (\text{polynomial}) \cdot z^k$

Cohomology $H_D = \underset{\substack{\text{even} \\ \text{bosonic states}}}{\mathbb{C}[z]/z^k} \oplus \underset{\substack{\text{odd} \\ \text{fermionic states}}}{\mathbb{C}[z]/z^k}$

Algebra structure on H_D : in generators 1, $a = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$, $\theta = \begin{pmatrix} 0 & 1 \\ -z^{n-2k} & 0 \end{pmatrix}$

Relations $a^k = 0$, $\theta^2 = -a^{n-2k}$

e.g. $n=2$ $W = z^2$, $F = z$ $G = z$ only nontrivial object

Massive LG model ... endomorphisms $1, \theta, \theta^2 = -1$

form a Clifford algebra.

In fact for any quadratic superpotential $W = \sum_{ij} Q_{ij} z^i z^j$

Q_{ij} nondegen quadratic form

$Q_{bdry}^2 = W$ $Q_{bdry} = \Gamma_i z^i + \dots$ Γ linear

$\Longrightarrow$ Clifford algebra $\{\Gamma_i, \Gamma_j\} = Q_{ij}$.

semisimple case — So every brane gives $\mathbb{Z}/2$-graded Clifford module,

Only germ of W at critical point matters!

Clifford module corresp to D-brane captures all of its info

Clifford model for each critical point $\leadsto$ D-brane...

n nondeg critical pts $\Rightarrow$ full subcategory of

$\oplus$ categories of Clifford modules.

$\mathbb{Z}/2$-graded version of Segal-Moore classification of semisimple

Frob algebra

Knorrer periodicity! (Arlur)

$W \rightsquigarrow W' = W(x) + y_1^2 + y_2^2$ doesn't affect category of

D-branes: only cares about #(variables) mod 2

Change parity $\leftrightarrow$ orbifolding by $\mathbb{Z}/2$ symmetry

... shift complexes by 1

Orbifolded category: pairs $\left(E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \text{, isomorphism } E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \longrightarrow E_1 \underset{F}{\overset{G}{\rightleftarrows}} E_0 \right)$ shift

... brane + odd endomorphism β s.t. $\beta^2 = 1$.

$W = z^n$, basic brane is $F = z$ $G = z^{n-1}$ endomorphisms.

$1, \theta$ $\theta^2 = \begin{cases} 0 & n > 2 \\ -1 & n = 2 \end{cases}$: lose info once $n > 2$...

Kontsevich: must keep track of A_∞ products on cohomology

eg $n=3$ => nontrivial Massey product $m_3(\theta,\theta,\theta)=-1$.
More generally $m_n=(\theta,\theta,\dots,\theta)=-1$,
all other products vanish.

$n=2$ $m_2(\theta,\theta)=-1$ Clifford algebra
More generally get "n-Clifford algebra"
- category of branes is category of A_∞-modules for this Clifford algebra [Koszul duality]

A. Kapustin IV -

[Mirror symmetry for tori & generalized Kähler structures:

$T = V/\Gamma,\ g,\ \omega,\ I,\ B$

$T' = V'/\Gamma',\ g',\ \omega',\ I',\ B'$

Get isomorphic $N=2$ SCFT iff have isomorphism $T \oplus T^* \longrightarrow T' \oplus T'^*$ identifying generalized ~~Kähler~~ complex structures (B-field transforms of symplectic)]

Last time: factorizations $W = z^n$, $F = z$, $G = z^{n-1}$

$\Rightarrow$ algebra $\{1, \theta : \theta^2 = 0\}$ $\quad n > 2$

comes as cohomology of dga - but this carries extra structure of A_∞ algebra with zero differential.

. n^{th} Massey product: defined canonically if all lower ones vanish, otherwise just in general.

$Q_{bdry} = \begin{pmatrix} 0 & z^{n-1} \\ z & 0 \end{pmatrix}$

DGA of endomorphism of brane: $\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $D : \varphi \longmapsto [Q_{bdry}, \varphi]$ $\quad$ (Q_{bdry}^2 central)

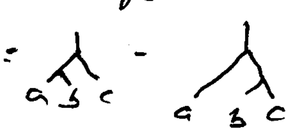
Cohomology inherits A_∞ structure ... Massey products:

~~e.g. n=3 ...~~

$[a],[b],[c]$ cohomology classes of dga $\quad m_3([a],[b],[c])$: $\quad [a]\cdot[b] = [b]\cdot[c] = 0$

choose representatives $a \cdot b = \partial\alpha$, $bc = \partial\beta$

$m_3([a],[b],[c]) = [\alpha c - (-1)^a a\beta]$:



well defined indep. of choices

Lo, $n=3$ example: $\quad \theta = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $\theta^2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = D\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

$$m_3(\theta,\theta,\theta) = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -1.$$

higher products vanish.

General n: all $m_i = 0$ $i < n$ $\quad m_n = -1$.

superpotential is degree n for open string moduli, potential for deformation of objects in dg category

$\{u_i\}$ a degree 1 cohomology of a dg algebra
W superpotential defined on virtual tangent space to deformation moduli space (space of solutions to Maurer-Cartan equation

$$W(u) = \sum_{k=0}^{\infty} \langle u_{k+1}, m_k(u, \ldots, u_k)\rangle$$

$u = \sum t_i u_i$

Lazaroiu: Cubic string field theory: $(\mathcal{H}, d, \cdot)$ dGa
$\langle\cdot,\cdot\rangle$ bilinear nondeg form $\mathcal{H}\otimes\mathcal{H} \to \mathbb{C}$

$$\langle du, v\rangle + (-1)^{|u|}\langle u, dv\rangle = 0$$

$$\langle u, vw\rangle = \langle uv, w\rangle$$

Take minimal model of $(\mathcal{H}, d, \circ)$ = minimal A_∞ algebra homotopy equivalent to $(\mathcal{H}, d, \cdot)$
Kontsevich - Soibelman: can realize minimal model on BRST cohomology $(H_d(\mathcal{H}), m_*)$
$\underline{\underline{m_1 = 0}}$

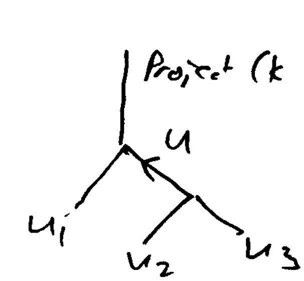
Cohomological splitting of $(\mathcal{H}, d)$

$$\mathcal{H} = \underbrace{K \oplus S}_{\ker d} \oplus L \qquad U: S \to L$$

$\operatorname{Im} d$

$$U = \frac{1}{d}\operatorname{Project}(S)$$

$m_k(u_1, \ldots u_k)$:



$+ \cdots$

$$e^{-W_{eff}} = \int \mathcal{D}[\varphi]\, e^{-S(\varphi)} \qquad S(\varphi) = \frac{1}{2}\langle\varphi, d\varphi\rangle + \frac{1}{3}\langle\varphi, \varphi\varphi\rangle$$

W_{eff} function on $K = H_d(\mathcal{C})$
= virtual tangent space to moduli of solns of Maurer Cartan $d\varphi + \varphi\cdot\varphi = 0 \iff \frac{\delta S}{\delta\varphi} = 0$

$$W(\varphi) = \sum_{n}^{\infty} \frac{(-1)^{?}}{n}\langle\varphi, m_n(\varphi^{\otimes n})\rangle$$

Spacetime Superpotential
W expresses structure of algebra with invariant inner product: relation of cyclic & Hochschild cohomology.

$\boxed{LG}$ $FG = W - W_0 = GF$ W_0 constant

NO SUSY string states stretch between different values of W_0: categories decompose according to value W_0.
If W_0 not critical value of W won't get any cohomology - no $\neq 0$ endomorphisms of object...
so categories live only at critical pts of W.
Structure of category depends only on structure of W near critical points.

Orlov: consider equation $W = W_0$: $\mathbb{C}[x_1 \ldots x_n] / W(x) - W_0$
scheme with nilpotents X_{W_0}
$\Rightarrow$ associated derived category of coherent sheaves on here,

inside subcategory of perfect complexes, take quotient: measure of singularity
$D_{W_0} = D^b(X_{W_0}) / \mathrm{Perf}(X_{W_0})$
category of singularities.
W_0 noncritical $\Rightarrow$ get zero.

Orlov theorem: category of D-branes $FG = W - W_0 = G \cdot F$ is equivalent to D_{W_0}

Gives definition of LG branes when X is not affine!

$W = Z^n$ X_{W_0} 0-dimensional $\mathbb{C}[x]/x^n$, all interesting modules are $\mathbb{C}[x]/x^i$ $i \leq n$ (Jordan form)
Only perfect complex $\leftrightarrow$ $i = n$ so are left with only these indecomposables $\mathbb{C}[x]/x^i$ $i < n$ $\longleftrightarrow \{z^k, z^{n-k}\}$ $k < n$

D-type minimal model $W = X^n + XY^2$, E type
still only finitely many branes up to isom

How to construct (some of) these branes?

Write $W = W_1 + W_2$ $\qquad D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$

Try $D_1^2 = W_1 \quad D_2^2 = W_2 \qquad D = D_1 \otimes 1 + 1 \otimes D_2$

doesn't work since D_1, D_2 commute..

Put in Clifford algebra $\quad D = \eta_1 D_1 \otimes 1 + 1 \otimes D_2 \eta_2$

$\eta_1^2 = 1 \quad \eta_2^2 = 1 \quad \{\eta_1, \eta_2\} = 0 \quad \Longrightarrow \quad \underline{D^2 = W}.$

So can ~~factorize~~ break apart superpotential into pieces, and construct some (maybe all) branes this way.

Computing correlators  Prod & Ann $\overset{M}{\underset{M}{[\,]}}$ trace

give all disc correlators with ∂ insertions

to give bulk insertions also need bulk $\leftrightarrows$ boundary

$H_{mm'}$  H

Vafa: compute correlators of B-type

Claim path integral localizes on constant maps.

Assume superpotential has only nondegen critical points

saddle pt approx exact, calculate at these points.

Degenerate critical pts: calculate as limit of nondeg ones.

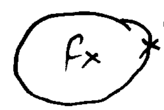
Sphere correlator 1pt function $\quad f \times$  $\longleftrightarrow$ $f \in H$  trace map

$$\langle f \rangle = \mathrm{res} \frac{f \, dx_1 \ldots dx_n}{\partial_1 W \ldots \partial_n W}$$

$f \in \mathbb{C}[\vec{x}]/\partial_i W \qquad \left[\begin{array}{l} = \text{integral over Lagrangian } n\text{-cycle} \\ \quad |\partial_i W| = \varepsilon_i \text{ small} \end{array} \right]$

$\Rightarrow$ trace on Jacobi ring

Gives nondegenerate trace. $\quad \langle fg \rangle = \langle f, g \rangle$.

Open string case: calculate  Ψ one bulk one boundary

- localize to constant maps. assume superpotential nondegenerate

$D = z_i \Gamma_i + \ldots \qquad z_i$ coords, $\quad \{\Gamma_i, \Gamma_j\} = Q_{ij} \qquad$ Clifford alg

$\varphi \in \mathrm{End}(M)$

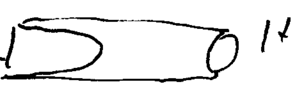
Kapustin-Li

Calculate $\langle f\varphi\rangle = \mathrm{res}\ \dfrac{f\cdot \mathrm{str}((\partial D)^n\varphi)}{\partial_1 W\cdots\partial_n W}$

$n = \dim \mathbb{C}^n$

$D = \begin{pmatrix} 0 & G \\ F & 0\end{pmatrix}$ matrix valued function. ∂D is matrix hol 1-form, $(\partial D)^n =$ n^{th} exterior power = matrix holomorphic n-form $\leftrightarrow$ ~~seeb~~ matrix valued function, multiply by φ & take supertrace.

[$D^2 = W \qquad dD\cdot D + D\cdot dD = dW$
so dD & D anticommute up to exact terms. ...... make various checks of answer]

Bulk-boundary map H_{MM}  H

$\varphi \longmapsto \mathrm{str}((\partial D)^n\varphi)$ generalized Chern character ... $\partial D \rightsquigarrow$ curvature of a superconnection..

[$\varphi = 1 \longmapsto$ boundary state $[M] \in H$.
$\Rightarrow$ lattice of boundary states in H, integral structure.]

$K(\text{category}) \to$ bulk Frob alg

$\langle \varphi_1, \varphi_2\rangle = \mathrm{res}\ \dfrac{\mathrm{str}((\partial D)^n\varphi_1\varphi_2)}{\partial_1 W\cdots\partial_n W}$ nondegenerate pairing (hopefully)

analogs of Serre duality.

Cardy condition: $\alpha \in H_{MM} \circlearrowright H_{MN} \circlearrowleft H_{NN} \ni \beta$ $\qquad f_{\alpha\beta} : H_{MN} \to H_{MN}$

Cardy: supertrace of $f_{\alpha\beta}$ can be calculated via bulk-bdry maps for M, N.

Case $\alpha = \beta = 1$ ("Hirzebruch-RR")

$$\chi(H_{MN}) \stackrel{?}{=} \mathrm{res}\ \frac{\mathrm{str}((\partial D_M)^n)\ \mathrm{str}((\partial D_N)^n)}{\partial_1 W\cdots\partial_n W}$$

Closed strings = Hochschild cohomology of category of branes

Write our category as modules over an algebra:

$E_0 \rightleftarrows E_1 \qquad F \cdot G = G \cdot F = W$

$\Rightarrow$ <u>curved dg algebra</u> - special case of A_∞ algebra

DGA: $\quad A$ algebra $\quad d: A \to A$ derivation (super Leibniz), $d^2 = 0$

Module over dga: graded A-module M, $D_M : M \circlearrowleft \ldots \; D_M^2 = 0$

eg $A = \Omega^\bullet$, $M =$ v. bundle with flat connection

CDG algebra: $\quad A,\ d: A \to A,\ B \in A$. $\ d^2 = [B, \cdot]$

$B =$ curvature

CDG module: $\quad M, D_M$ derivation, $\quad D_M^2 = B \cdot \_$

A CDG algebra is not a module over itself:

$[B, -]$ vs $B \cdot \_ \ \ldots$

Our case: $\quad A = \mathbb{C}[x_1 \ldots x_n]$ (all even). $d = 0$, $B = W$ central

CDG Module over this: $D_M^2 = W$, like we want.

Have derived category of CDG modules.

Hochschild cohomology: $\quad A$ is $A \otimes A^{op}$ - module, $HH^*_A = \mathrm{Ext}^*_{A \otimes A^{op}}(A, A)$

A is not a CDG A-module but it <u>is</u>.

a CDG $A \otimes A^{op}$ - module: $d^2 a = Wa - aW = 0$,

$B = W \otimes 1 - 1 \otimes W$

Expect $HH^*([A, 0, B])$ to be closed string algebra & its trace

Calculate that indeed the cohomology is Jacobi ring of singularity

$n=1$ $A = \mathbb{C}[x]$

$$D = \begin{pmatrix} 0 & \frac{W(x)-W(y)}{x-y} \\ x-y & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} f(x,y) & 0 \\ 0 & f(x,y) \end{pmatrix} \Big/ f \sim f + (x-y) + \frac{W(x)-W(y)}{x-y}$$

set $X = Y$, get $W'(x)$ set to zero.

Algebra/~~Cyclic~~ operad : elt of

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}(P(n) \otimes_{S_n} V^{\otimes n}, V) = \sum_{n=0}^{\infty} m_n$$

Suppose P is a cyclic operad - eg A_∞, L_∞, C_∞ (not Gerstenhaber or G_∞)

& V has a nondeg inner product

$\Rightarrow$ get subspace $\bigoplus_{n=0}^{\infty} P(n) \otimes_{S_{n+1}} V^{\otimes n+1}$

$$\cap$$

$\bigoplus \mathrm{Hom}(P(n) \ldots)$ above

<u>Cyclic</u> algebra if product lies in this subspace

Restrict to subspace of generators : An subspace spanned by generators

Free operad on generators $V(n)$

$\bigoplus_{n=0}^{\infty} V(n) \otimes_{S_{n+1}} V^{\otimes n+1}$ $\ni$ elts of this are same as cyclic alg structure.

A. Kapustin V 8/1/03

Derive closed strings from open strings, LG case:
$H = HH^*(A,A)$ Jacobi ring as Hochschild cohomology of curved DG algebra ... $A = (\mathbb{C}[x], 0, W)$
D-branes = CDG modules over this CDG algebra

Frobenius structure on H: Regard $A \otimes A^{op} = (\mathbb{C}[x,y], 0, W(x)-W(y))$ CDG algebra.
A is CDG $A \otimes A^{op}$-module.
Write free CDG resolution of A: $\mathbb{C}[x,y] \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}[x,y]$
(deformation of Koszul, $W=0$) $D = \begin{pmatrix} 0 & \frac{W(x)-W(y)}{x-y} \\ x-y & 0 \end{pmatrix}$
... L. Rozansky suggestion
exterior algebra $\rightsquigarrow$ Clifford algebra.

Disc correlator φ – extract closed string 1-pt function from open string ...
$$\mathrm{Res}\ \frac{\mathrm{str}\,((\partial D)^2\, \varphi)}{\partial_x W(x)\ \partial_y W(y)}$$
$\varphi = \begin{pmatrix} A(x) & 0 \\ 0 & A(x) \end{pmatrix}$ (any φ is this up to exact)
$$\overset{?}{=}\ \mathrm{res}\ \frac{A(x)}{\partial W(x)} \quad \text{closed string 1-pt function}$$
(calc) $\mathrm{res}\ \frac{A(x)}{W'(x)\,W'(y)}\ \ W'\frac{(x)-W'(y)}{x-y}$

[assume only nondeg critical points: $\mathrm{res}\ \frac{f}{\partial_1 W \cdots \partial_n W} = \sum_{x_i\ \mathrm{crit}} \frac{f(x_i)}{\det \frac{\partial^2 W}{\partial x_i \partial x_j}}$]
$$= \sum_i \frac{A(x_i)}{W''(x_i)\,W''(x_i)} \cdot W''(x_i) = \sum \frac{A(x_i)}{W''(x_i)} = \mathrm{res}\ \frac{A(x)}{W'(x)} \quad \blacksquare$$
[Crit points: $(x,y) = (x^i, x^j)$, only remaining terms are those $x^i = x^j$]

Algebraic construction of open-closed theories
A algebra (dg, cdg, A_∞, ...)

$\Rightarrow D^b(\mathrm{Mod}\text{-}A)$ derived category
$H = HH^*(A,A)$ closed string, acts on morphism spaces, get open-closed maps —

get all data except inner products, $H_{MN} \otimes H_{NM} \to \mathbb{C}$
- need to construct these & then check Cardy condition. (closed $\to$ open [illegible] in center is automatic)

Expect this to work (maybe) for A-model also.
Serre duality, HRR hold more generally...
but we get them for open-closed TFT in CY case - where do they come from? What is nontopological version of Cardy condition.....
Twisted version of axioms, with nontrivial canonical class,....

$\dot{S}\dot{S}$ Two spaces of closed strings -- HH^* & cyclic homology ?!

<u>LG orbifolds</u> Relate to CYs via Gepner models -- eg quintic $x_1^5 + \dots + x_5^5 = 0$ in $\mathbb{P}^4$
$\iff$ $W = x_1^5 + \dots + x_5^5$, $X = \mathbb{C}[x_1 \dots x_5] / \mathbb{Z}/5$ LG orbifold

deforming Kähler class of quintic relates quintic to this LG orbifold...
(equivariant CDG modules over CDG alg with a group action..)

$\to$ Q. Show derived category of quintic is equivalent to category of LG branes - ~~derived~~ category of equivariant CDG modules } Orlov

UV: Fano hypersurface in toric variety σ-model $\xrightarrow[\text{RG flow}]{\text{IR}}$ collection of vacua, one of which eg is massless - LG orbifold IR
Category of topological branes unaffected by RG flow: get full subcategory of derived category this way.....

A-branes There are more branes than objects in Fukaya category! Kapustin-Orlov: construct susy A-branes from coisotropic (generalized complex) submanifolds – K-theory of category gets bigger! Can check from mirror symmetry predictions, coherent sheaf category is bigger than Fukaya eg for some tori.
$\rightsquigarrow$ Fukaya category is the wrong one...

Primitive middle cohomology $H^{\frac{n}{2}}(X)^{prim}$ is sometimes smaller than B-brane K-theory...

(Y, E, ∇) Y submanifold, E, ∇ Hermitian line bundle + unitary connection, $\nabla^2 = \frac{1}{2\pi i} F$

Can formulate classical boundary conditions preserving $N=1$ invariance
To go to $N=2$: enough to show $N=1$ + R-charge invariance

R-charges: left moving $J = \omega(\psi_+, \psi_+)$ $\bar{J} = \omega(\psi_-, \psi_-)$ right moving
ω 2-form, ψ sections of pullback of tangent bundle.
Require $J + \bar{J}|_{\partial\Sigma} = 0$ conservation of R-charge.

$\leftrightarrow$ Check $N=1$ conditions preserved by R-symmetry:
pass to basis $\psi^i = \frac{1}{\sqrt{2}}(\psi_+^i + \psi_-^i)$ $\in$ tangent bundle
$\rho_i = \frac{1}{\sqrt{2}} g_{ij}(\psi_+^j - \psi_-^j)$ $\in$ cotangent bundle of X

Under R-symmetry, variation is
$\delta\rho = i\omega\psi$ $\delta\psi = -i\omega^{-1}\rho$

R-charge current $\begin{pmatrix} \delta\psi \\ \delta\rho \end{pmatrix} = i\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \rho \end{pmatrix}$ see symplectic generalized $\mathbb{C}$-structure appear.

B-model R-charge – get other generalized $\mathbb{C}$-structure

Usual boundary conditions: $\psi \in TY \subset TX$

$\rho - F\psi \in N^*Y$ conormal

Require for this subspace of $TX \oplus T^*X$ to be preserved by R-charge $\longrightarrow$ generalized $\mathbb{C}$-submanifold (in special case $F=0$)

$\Longleftrightarrow$ 1. Y coisotropic 2. $F|_{\ker \omega|_Y} = 0$

~~kernel~~. ∇ flat on nullfoliation of ω

3. $\underline{\omega_\perp = -F_\perp \omega_\perp F_\perp}$ descended forms on quotient by foliation

Can replace $F \rightsquigarrow F - B$ everywhere, get right answer in presence of B-field.

Leave B but set $F=0$ $\Rightarrow$ generalized $\mathbb{C}$-submanifold

<u>Def</u> generalized complex brane:

(Y, E, ∇) $F = \frac{1}{2\pi i}\nabla^2$ (E line bundle)

& require subbundle $\mathcal{R} \subset TX \oplus T^*X|_Y$:

defined by (ψ, ρ) $\psi \in TX|_Y$ $\rho \in T^*X|_Y$, $\psi \in TY$, $\rho - F\psi \in N^*Y$

to be preserved by the generalized $\mathbb{C}$-structure $\mathcal{J} \in \mathrm{End}\, TX \oplus T^*X$

– Gives right A & B-branes!

B-branes $\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, Y is complex submanifold

$(F - B|_Y)^{(0,2)} = 0$

Condition $F\omega^{-1}F = -\omega$

(just requirement of worldsheet supersymmetry)

Get integrable $\mathbb{C}$-structure in transverse direction to foliation, in case ω^{-1} & F^{-1} are <u>compatible</u> Poisson brackets (using a result of Gelfand-Dorfman)

$\Rightarrow F_\perp + i\omega_\perp$ holomorphic symplectic form.

Higher rank

Chern character for B-branes

J complex structure on X, $(1,1)$ tensor

$i_J : \Omega^p \longrightarrow \Omega^p$. contraction with J

Chern character of B-brane $\mathrm{tr}\, e^F$

pushed forward to X : $i_J \cdot \mathrm{tr}\, e^F = 0$

comes from $J - \bar{J}|_{\partial\Sigma} = 0$

, A-branes $(\omega - i_{\omega^{-1}})(\mathrm{ch}\, A) = 0$

Morphisms

1. Formal view of Floer homology!

path space

$$\begin{tikzcd} & PX \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_f"] & \\ X & & X \end{tikzcd}$$

X symplectic $\Rightarrow$ 1-form α on path space : $\alpha_\gamma(\delta\gamma) = \int_0^1 \omega(\dot\gamma, \delta\gamma)\, dt$

transgression of ω on X.

$$d\alpha = \pi_f^* \omega - \pi_i^* \omega$$

Restrict to paths between Lagrangians

$$\begin{tikzcd} & PX(L_1,L_2) \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_f"] & \\ L_1 & & L_2 \end{tikzcd}$$

$d\alpha|_{PX(L_1,L_2)} = 0 \Rightarrow$ what is cohomology of $\Omega^\cdot(PX(L_1,L_2))$ wrt differential $d + 2\pi\alpha$?

(like Floer middle dim cohomology...) ...