Khovanov homology

Note Title 10/17/2006

Goal: introduce some themes of 21st century representation theory $\longleftrightarrow$ 4d TFT.

Prehistoric rep theory:

G finite group $\rightsquigarrow$ number $|G/G|$ of conjugacy classes in G ($=|G|$ for G abelian)

1-dim TFT: $Z_1(Y^1) \in \mathbb{C}$

assign $\mathbb{C}$ to closed 1-manifold so that $\amalg \rightsquigarrow \cdot$ (monoidal ... $\emptyset \mapsto 1$).

not much data: just a single number, $Z_1(S^1)$. so $|G/G|$ defines a 1d TFT!

However: Note $|G/G| \in \mathbb{N}$!

19th Century rep theory: $|G/G|$ is the <u>dimension</u> of a vector space, the space of class functions $(\mathbb{C}G)^G$ ... in fact this is a commutative ring $\mathbb{C}G^G = Z(\mathbb{C}G)$, with trace function (evaluate at $1 \in G$) : commutative Frobenius algebra.

2d TFT: $Z_2(X^2) \in \mathbb{C}$ monoidal

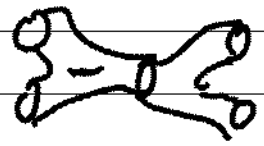
but also want to treat 1-manifolds:

natural rule $Z_2(Y^1) \in \mathrm{Vect}$ vector space

$\amalg \longleftrightarrow \otimes$ monoidal

orientation $\longleftrightarrow \times$

X w/boundary: $Z_2(X) \in Z_2(\partial X)$

& gluing laws 

<u>Theorem</u> 2d TFT $\longleftrightarrow$ comm Frobenius algebras

$Z \longmapsto Z(S^1), \mathbb{O}, \mathbb{O}$

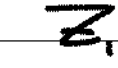
So $\mathbb{C}G^G$ defines a 1d TFT

[Labelling: $v \in Z_2(S^1) = V$ denote v $\in \mathbb{C}$

$= ($ $: V \to \mathbb{C})(v)$:

like having closed surface, but not vacuum state]

- Z_2 defines a Z_1 : $Z_1(Y) = Z_2(Y \times S^1)$

  ie $Z_1(S^1) = Z_2($  $) = \dim Z_2(S^1)$

Integrality:

Note $\mathbb{C}G^G$ has a natural basis χ_V (V irrep of G) of characters

with great positivity & integrality:

$$\chi_V \cdot \chi_W = \sum c_{VW}^U \chi_U \quad \text{with } c_{VW}^U \in \mathbb{N}$$

$\rightsquigarrow$ suggests thinking of c_{VW}^U as <u>dimensions</u>

<u>20th Century</u>: $\mathbb{C}G^G = \underline{\mathrm{Rep}\, G} = K(\mathrm{Rep}\, G)$:

representation ring is K-group of <u>rep. category</u>

$$\chi_V = [V], \quad \chi_V \cdot \chi_W = [V \otimes W]$$

$$= \left[\bigoplus \mathrm{Hom}(U, V \otimes W) \otimes U \right]$$

ie c_{VW}^U = multiplicities of occurrence

= dim of Hom spaces.

<u>Categorification</u>: numbers $\rightsquigarrow$ vector spaces
vect $\rightsquigarrow$ categories
$\overset{\in}{\text{vectors}} \rightsquigarrow \overset{\in}{\text{objects}}$
.... hinted at by integrality.

Rep G much richer object than $(\mathbb{C}G)^G$
.... eg multiplicity spaces have automorphisms!

<u>Recover G</u>: Forget: Rep $G \longrightarrow$ Vect
categorification of $\mathrm{eval}_1(\chi_V) = \dim V$
Faithful $\otimes$ functor

Can look at Aut (Forget) natural autos.

<u>Tannaka</u>: $G \cong$ Aut (Forget)

In fact (lin. alg.) groups $\Longleftrightarrow$ tensor categories $\mathcal{C}$,
$\underline{1}$, $\otimes$, Forget
$V \otimes W \xrightarrow{\sim} W \otimes V \xrightarrow{\sim} V \otimes W$
$\underbrace{\qquad\qquad}_{id}$
comm, assoc. constraints

$G \hookrightarrow$ Rep G (f.d. reps)

Symmetry: $V \in \mathcal{C} \Rightarrow V^{\otimes n} \circlearrowleft S_n$

i.e. for any $V \in \mathrm{Rep}\, G$,

$G \circlearrowright V^{\otimes n} \circlearrowleft S_n$ commuting,

or S_n acts on multiplicity spaces

<u>3d TFT</u> $Z_3(M^3) \in \mathcal{C}$ monoidal

$Z_3(X^2) \in \mathrm{Vect}$, $Z_3(M) \in Z_3(\partial M)$

functorial in diffeos

+ cobordism composition

$Z_3(Y^1)$: now need to have gluing

laws to decompose a vector space $\textcircled{\vdots}$

as a pairing $\textcircled{\cdot}\textcircled{\cdot}$

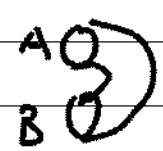
.... assign $Z_3(Y^1)$ a <u>category</u> $\left(Z_3(\emptyset^1) = \mathrm{Vect}\right)$

[$\mathbb{C}$-linear or dg or ...]

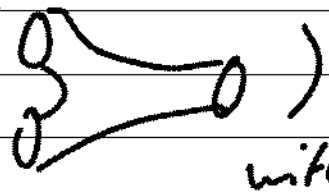
$Z_3(X \text{ w/ bdry}) \in Z_3(\partial X)$ object

$Z_3($ $): \mathcal{C} \times \mathcal{C}^{op} \longrightarrow Vect$

$$A \times B \longmapsto Hom(A,B)$$

ie labelled 

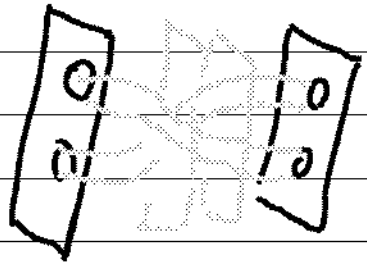
$= Hom(A,B)$.

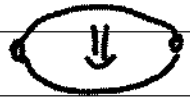
$Z_3($  $)$: Functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$

with composition

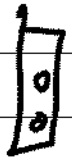
Looks like tensor category

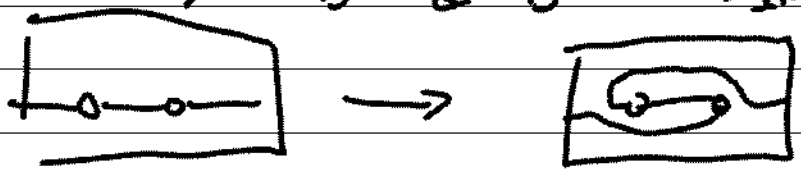
Given 3-manifold with corners make natural transformations between functors!

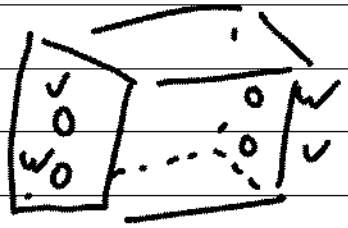


eg  $\times S^1$

$\rightsquigarrow$ natural transformation between

$A \otimes B$ & $B \otimes A$: self diffeo of 





$$V \otimes W \xrightarrow[\beta_{VW}]{\sim} W \otimes V$$

But nothing says $\beta_{vw}\beta_{wv} = id$:

get action of braid group on $V^{\otimes n}$

$PB_n \to B_n = \pi_1(\text{config of } n \text{ pts in } \mathbb{C} = \mathbb{C}^n - \Delta / S_n)$

$\downarrow$

S_n $\qquad PB_n \circlearrowright V_1 \otimes V_2 \otimes \dots \otimes V_n$

Satisfying braid relation $sts = tst$

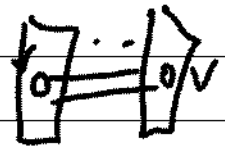
$\Rightarrow$ $\mathcal{C} = Z_3(S^1)$ is a <u>braided tensor category</u>

$=:$ <u>quantum group</u> ... more general than group, but not terribly so.

[Actually full 3d TFT $\leftrightarrow$ <u>modular</u> tensor category, ...]

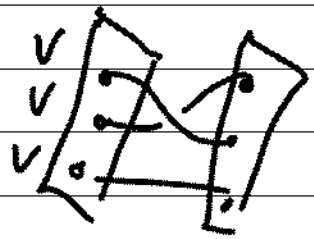
<u>Dimensional reduction</u> $\quad Z_2(Y) = Z_3(Y \times S^1)$

$Z_3(X^2 \times S^1) = \dim Z_3(X^2) \qquad = K(Z_3(Y))$



$id : V \to V$

Clear $\leadsto \Rightarrow$ $tr(id_V) = \dim V$

More generally 

Close up to a knot $\Rightarrow$ number, which is <u>true</u> of the operator $V^{\otimes 3}$

i.e. to $\beta \in B_n$ assign $tr_V \beta \in \mathbb{C}$

Under good conditions this is a <u>knot invariant</u>.

<u>Example</u> $Rep\, SL_n$ has natural deformation to a quantum group $Rep_q\, SL_n$ $q \in \mathbb{C}^\times$

.... generically, $Rep_q\, SL_n \simeq Rep\, SL_n$ as categories, $\otimes, 1,$ etc: only difference is in isomorphisms $V \otimes W \xrightarrow{\sim} W \otimes V$

2-dim rep of $SL_2 \Longrightarrow$ <u>Jones polynomial</u> $\in \mathbb{C}[q, q^{-1}]$ knot invariant.

Roughly $e, f, h \rightsquigarrow e, f, q^h = K$ group like

Towards 21st century math $\longleftrightarrow$ 4d TFT:

Lusztig: Canonical bases for irreps of $U_q\mathfrak{g}$ .. i.e for objects of $\mathrm{Rep}_q G$

... e.g for $\mathrm{Rep}\, G$ itself.

e, f, h operators act integrally positively on canonical bases: i.e can define reps of G over $\mathbb{N}$.

$\Rightarrow$ suggests each $V \in \mathrm{Rep}\, G \leftsquigarrow \mathcal{V}$ category with categorified G-action by functors...

$V^{\otimes n} \rightsquigarrow \mathcal{V}_n \circlearrowleft B_n$ category with braid action

$\circlearrowleft B_n$

$\Rightarrow$ can take categorified traces to get knot homologies.

4d TFT : haven't been rigorously constructed

.... Give category for surface,

2-category for circle ... $\mathbb{P}^1$ with an labels

$\Downarrow$

category

closed braid $\Rightarrow$ trace of operator on category $\Longrightarrow$ vector space.

In 4d: $N=2$ SYM $\Rightarrow$ Donaldson

$N=4$ SYM $\Rightarrow$ Geometric Langlands

Gukov talk ...

Crane-Frenkel program: try to construct by categorifying reps of quantum group

BFK / Khovanov / Stroppel, ..

ex. $\binom{n}{k} = \dim \Lambda^k \mathbb{C}^n$

q-analogue: $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

= graded dim of $[n]$ of $\mathfrak{sl}_2$

$[n]_q! = [1]_q \cdot \ldots [n]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!}{[k]!\,[n-k]!}$

$= \dim_q \Lambda^k \mathbb{C}^n_{\mathfrak{sl}_2}$ n-dim rep of $\mathfrak{sl}_2$

Geometric source: realize
n-dim rep of $\mathfrak{sl}_2$ as $H^*(\mathbb{P}^{n-1})$
(Lefschetz)

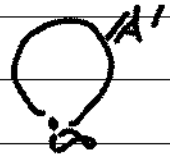
[BLM]
[G]

- grading gives q-analog.

How to categorify? Natural basis for homology given by B orbits on $G/P = \mathbb{P}^n$.

Can look at a category of <u>sheaves</u> on $\mathbb{P}^n$ locally constant on each orbit / B-equivariant.
[or derived category...]

This is a version of one of the most basic categories in rep theory
- category $\mathcal{O}$ = highest weight modules.

$\mathbb{P}^1$ case: 

three basic objects:
$\mathbb{C}$, δ, $j_* \mathbb{C}$ extension.

$\Leftrightarrow$ highest weight reps of $\mathfrak{sl}_2 = \{e, h, f\}$
with given Casimir $ef + fe + \frac{1}{2}c^2$:

$\mathbb{C} \longleftrightarrow$ (fin) $= \Gamma_{\mathbb{P}^1}(\mathcal{L})$

$j_! \mathbb{C} \longleftrightarrow \Gamma_{\mathbb{C}}(\mathcal{L}) \simeq$ polynomials in $f \leftrightarrow z$
e acts as $\frac{\partial}{\partial z}$

..... ( )

Quotient is δ-functions = poles parts at ∞.

[BFK] K-group 2-dimensional realize 2-dim rep of SL_2.

$\mathbb{C} \subset j_* \twoheadrightarrow \delta$

n-dim rep: use singular category $\mathcal{O}$ for SL_n:
$B \backslash G / P$ versus..

Braid group actions: comes from Kazhdan-Lusztig theory & Springer theory.

$S_n \longleftrightarrow B \backslash GL_n / B$ Schubert varieties on Fl_n

$\mathbb{Z} S_n = K(\mathcal{P})$ $\mathcal{P} = B$ perverse sheaves on G/B [$\xleftrightarrow[BB]{}$ category $\mathcal{O}$]

Algebra structure comes from convolution

$$\begin{tikzcd} & GL_n \times GL_n \arrow[dl] \arrow[d] \arrow[dr] & \\ GL_n & GL_n & GL_n \end{tikzcd}$$

$\mathbb{Z}[q, q^{-1}] S_n = K(\mathcal{P}_{gr})$ graded version: remember shifts in pushforward

$\| $

$\mathcal{H}$ Hecke algebra! $T_i^2 = 1$ deforms to $(T+1)(T-q) = 0$.

+ braid relations still hold.

$\mathcal{P}$ action: richer than S_n or $\mathcal{H}$ action, only gives B_n action in general!

· acts on $Shv(P\backslash G/B)$ from right!

K-L: constructed interesting basis for $\mathbb{Z}S_n$ (or really for $\mathfrak{gl}$) using combinatorics, found transition matrix to standard was with $\mathbb{N}[q]$ coefficients .... - interpret as local IH groups of Schubert varieties

K-L conjecture: $P_{ww'} = [M_w : L_{w'}]$
multiplicity spaces in category $\mathcal{O}$
= Ext groups between objects. .