MK . I

Motivic D-T invariants

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Plan 1) Intro + Lie alg aspects of stability
2) Ind constr. categories, motivic functions, motivic DT-invariants
3) Applications to cluster algebras

Intro : 3d CY X smooth, compact, Kahler, dim = 3, holonomy $\subset SU(3)$

Counting problems (equivalent):

1) GW invs: g genus, $d \in H_2(X, \mathbb{Z})$ $N_{g,d}^{GW} \in \mathbb{Q}$ (rational b/c of symmetries)

$$\rightsquigarrow F = \exp\left(\sum_{g,d} \lambda^{2g-2} q^d \cdot N_{g,d}^{GW} \right)$$

$$|\lambda| << 1, \quad q \in Hom(H_2, \mathbb{C}^*)$$

Mirror symmetry $\Rightarrow$ F hol fn on certain formal alg scheme

2) DT invs (in proper sense): hol CS theory's Casson invariant

Basic idea $\mathcal{E}$ hol v.b./X, deform of $\mathcal{E}$ $RHom(\mathcal{E}, \mathcal{E})$ as dg Lie alg

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ Ext^0 & Ext^1 & Ext^2 & Ext^3 \end{array} \rightsquigarrow T_{\mathcal{E}} \mathbb{Z} / \mathbb{C} \cdot id_{\mathcal{E}}$$

Ext^0: $\cong$ locally $\mathcal{O}_X$ if $\mathcal{E}$ 1 dim

$\underbrace{Ext^1 \; Ext^2}$ ideal situation only two terms

$Ext^3 \cong (Ext^0)^*$

Serre duality $\Rightarrow$ 0 dim moduli
$Ext^1 \leftrightarrow Ext^2$

Only one case when moduli is proper with $End(-) = \mathbb{C}$:

$\mathcal{E}$ coh sh = ideal sh of subscheme $Y \subset X$ dim $Y \leq 1$

$\mathcal{E}|_{X \setminus Y} = \mathcal{O}_X$ (so $Ext^0 = \mathbb{C}$ since Y codim 2)

$\rightsquigarrow$ Hilbert scheme (so compact)

DT numbers $\underset{\mathbb{Z}}{\overset{\uparrow}{N^{DT}_{n,d}}}$ depend on Chern class $ch(\mathcal{E}) = (1, 0, d, n)$

$\mathbb{Z}$ H^0 H^2 $H^4(X) = H_2(X)$ $\mathbb{Z} = H^6(X)$

<u>Hope</u>: extend DT invs to any class in K_0^{top} theory $\approx H^{ev}(X)$ up to torsion

Neither GW nor DT depend on $\boxed{\mathbb{C}\text{-str or Kahler class}}$

<u>topological</u>

<u>Conj</u> (MNOP) 1-1 corr GW $\leftrightarrow$ DT

There are better #'s Gopakumar-Vafa invs

$N^{GV}_{g,d} \in \mathbb{Z}$ $d \neq 0$

Pick an a.c. str (tamed by sympl. form)
Consider hol curves $\underset{\text{embedded}}{\subset} X$ will generically be isolated smooth ~~embedded~~

Count them! (with signs ± 1)

3) Fictional theory: count special Lags $L \subseteq X$ L = oriented Lag
special with arg $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$
if $\Omega^{3,0}|_{TL} = e^{i\varphi} \cdot$ pos real 3-form

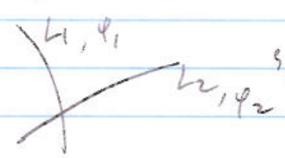
deformations $H^*(L, \mathbb{R})$
with $U(1)$-conn $H^*(L, \mathbb{C})$

count spec. Lags $\leftrightarrow$ DT invs
mirror symmetry

These are <u>not</u> invariants! (outside of analytic difficulties)

Should depend on $\mathbb{C}$-str

<u>why?</u>



spheres = S^3

$\int_{L_i} \Omega^{3,0} = e^{2\pi \varphi_i} \operatorname{vol}(L_i)$ (normalized)

$H^1(S^3) = H^2(S^3) = 0 \rightsquigarrow$ so stable

$\varphi_1 \neq \varphi_2$ vary $\underline{\underline{\mathbb{C}\text{-str}}}$ $\begin{matrix} L_1 \\ L_2 \end{matrix} \Big| \longrightarrow \begin{matrix} L_1 \\ L_2 \end{matrix}$ ~~st~~ $\arg L_1 = \arg L_2$

$\text{codim}_{\mathbb{R}} = 1$ $\quad L_1 \# L_2$ ← extra spec Lag!

$\# = 2 \rightsquigarrow \# = 3$

<u>naive formula</u>: $N^{DT}_{\gamma_1+\gamma_2}$ changes by $N^{DT}_{\gamma_1} \cdot N^{DT}_{\gamma_2} \cdot (\gamma_1 \cap \gamma_2)$ $(*)$

$\gamma \in H_3(X)$

<u>Conjecture</u> for CY X dim = 3

$\rightsquigarrow$ Numbers $N^{DT}_{\gamma} \in \mathbb{Q}$ $\quad \gamma \in H_3(X)$

1) do not depend on Kahler class
2) change if we vary $\mathbb{C}$-str according to $(*)$ below

$\mathcal{M}_{\mathbb{C}\text{-str}}$ = moduli of $\mathbb{C}$-str

$\pi_1 \neq 0$ + cusps

bundle of lattices $H_3(X)$
get function on $H_3(X)$
with connection

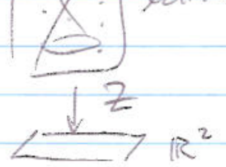
$\Lambda = H_3(X)/\text{torsion}$ $\quad \Lambda \otimes \Lambda \xrightarrow{\cap} \mathbb{Z}$ $\quad$ fix norm $\|\cdot\|$ on $\Lambda \otimes \mathbb{R}$

$Z = \int_{\Omega^{3,0}} 3\text{-form} : \Lambda \to \mathbb{C} = \mathbb{R}^2$

fix $p \in M_{\mathbb{C}\text{-str}}$ get N^{DT}_{γ}

if $N^{DT}_{\gamma} \neq 0 \Rightarrow c|Z(\gamma)| \geq \|\gamma\|$

$\exists$ L spec Lag $|Z(\gamma)| = vol(L)$ so $\int_{\Omega^{3,0}} 3\text{-form} \leq \text{const} \cdot vol(L)$



can choose good fn on H_3
so that it's positive on γ with $N^{DT}_{\gamma} \neq 0$

$\{ Z(\gamma) \in \mathbb{C} \mid N_\gamma^{DT} \neq 0 \}$ discrete with poly density at ∞

<u>Change of $\mathbb{C}$-str</u> : <u>Wall-crossing formula</u>

For given $Z : \Lambda \to \mathbb{C}$ (choice of $\mathbb{C}$-str)

given Q quad form on $\Lambda \otimes \mathbb{R}$ s.t. $Q|_{(\ker Z)\otimes\mathbb{R}} < 0$

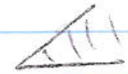
Say $N^{DT} : \Lambda \to \mathbb{Q}$ <u>controlled</u> by Q if

$$N_\gamma^{DT} \neq 0 \Rightarrow Q(\gamma) > 0$$

<u>Vary.</u> Z_t $t \in [0,1]$ Assume $Q|_{(\ker Z_t)\otimes\mathbb{R}} < 0$

assume $N_\gamma^{DT,t}$ controlled by same Q, nearby t.

Given Z, N_γ^{DT}, pick any strict angle sector $V \subset \mathbb{R}^2 = \mathbb{C}$



Lie alg $\mathfrak{g}_\Lambda = \bigoplus_{\gamma \in \Lambda} \mathbb{Q} \cdot e_\gamma$ $\quad [e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\gamma_1 \wedge \gamma_2} \gamma_1 \wedge \gamma_2 \, e_{\gamma_1 + \gamma_2}$

pro-nilp. (without sign $\rightsquigarrow$ Lie alg of fns on $T = \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$)

Lie alg $\mathfrak{g}_V = \prod_{\gamma \in \Lambda \cap \text{convex hull of } (Z^{-1}(V) \cap Q^{-1}(\mathbb{R}_{>0}))} \mathbb{Q} \cdot e_\gamma$

$G_V = \exp(\mathfrak{g}_V)$ formal symplectomorphisms of T

Element $A_V \in G_V$ $\quad A_V = \prod_{\substack{\text{rays } l \subset V \\ \text{clockwise order}}} \exp\left(\sum_{\substack{\gamma \\ Z(\gamma) \in l}} N_\gamma^{DT} \left(\sum_{k \geq 1} \frac{e_{k\gamma}}{k^2} \right) \right)$

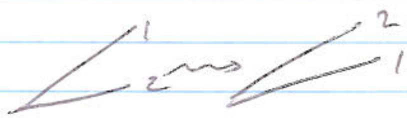
Satisfy $A_V = A_{V_1}\cdot A_{V_2}$ if $V = \underset{\text{left}}{V_1} \sqcup \underset{\text{right}}{V_2}$

$G_{V_1}, G_{V_2} \subset G_V$

Parallel transport axiom : A_V stays constant as long as for γ with $Q(\gamma)>0 \Rightarrow Z(\gamma)\notin \partial V$

Then whole story well-defined

Consider 2-dim ~~case~~ $\Lambda_2 \simeq \mathbb{Z}^2 \subset \Lambda$ with "flop"



~~$\Lambda_2 \to \mathbb{R}\oplus i\mathbb{R}$~~

γ_1,γ_2 basis of Λ_2
$k = \langle\gamma_1,\gamma_2\rangle$

(A) if $\langle\, ,\rangle|_{\Lambda_2} = 0 \Rightarrow$ numbers do not change

(B) if $\langle\, ,\rangle|_{\Lambda_2} \neq 0 \Rightarrow$ ~~suppose~~ let x,y variables s.t. $\{x,y\} = kxy$ $\quad k\in\mathbb{Z}\neq 0$

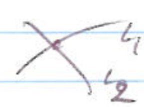
$T_{a,b}\in \mathrm{Aut}\,\mathbb{Q}[[x,y]]$ $\quad x\mapsto (1\pm x^a y^b)^{kb}x$
$\qquad\qquad\qquad\qquad\quad y\mapsto (1\pm x^a y^b)^{ka}y$

Formula (*) $\displaystyle\prod_{a/b\uparrow} T_{a,b}^{N^{DT,\,left}_{a\gamma_1+b\gamma_2}} = \prod_{a/b\downarrow} T_{a,b}^{N^{DT,\,right}_{a\gamma_1+b\gamma_2}}$

$\underline{k=1}$: $T_{1,0}\cdot T_{0,1} = T_{0,1}\cdot T_{1,1}\cdot T_{1,0}$

L_1 | L_1
L_2 | L_2
| $L_1 \# L_2$

$\underline{k=2}$ $\quad T_{1,0}\cdot T_{0,1} = T_{0,1}\cdot T_{1,2}\cdot T_{2,3}$
$\qquad\qquad \cdots\; T_{1,1}^{-2}\cdot T_{2,2}^{+1} \;\cdots\; T_{3,2}\,T_{2,1}\cdot T_{1,0}$



(Moore: appears in S-W)

$\underline{k=3}$ super complicated... but understandable
algebraic, hypergeometric