MK III

A_∞ CY 3-dim category $C/_k$

Assume $\sum_i Ext^i(E,F) < \infty$

$Ext^i(E,F)^* = Ext^{3-i}(F,E)$

in A_∞ case: cyclic A_∞-cat $\langle m_n(f_1,\dots,f_n), f_{n+1}\rangle$ is $\mathbb{Z}/_{n+1}$ sym. char $k=0$

also see MK & YS for char $k>0$

$\forall E$: minimal model: formal power series $(\alpha \in Ext^1(E,E))$

$\Psi_E(\alpha)$ "potential assoc to E" $= \sum_{n\geq 2} \frac{\langle m_n(\alpha,\dots,\alpha),\alpha\rangle}{n+1}$

$= O(\alpha^3) + \cdots$

weight $w(E) = (\mathbb{L}^{\frac{1}{2}})^{\sum_{i\leq 0}(-1)^i \text{rk}\, Ext^i(E,E)} \left(1 - \left[H^*(\text{Milnor fiber of } \Psi_E)\right]\right)$

$\mathbb{L} = H^*_c(\mathbb{A}^1)$

Milnor fiber: ~~the~~ basic discussion ... (notes omitted) ... H^* carries variation of MHS over $\mathbb{C}^\times$ quasi-unip monod.

Thom-Sebastiani Thm: external sum of fns $f_1 \boxplus f_2$ $\Longrightarrow$ Milnor fiber $= M_{f_1} \equiv M_{f_2}$

Hodge polynomial of associated graded of H^*
($\mathbb{Q}$-exponents)
$p+q \in \mathbb{Z}$

Denef-Loeser: abstract char $k=0$ story $X/_k$

formal Motivic milnor fiber $Y_\alpha \circlearrowleft \mu_N$ $N>>0$
(compatible with previous picture)

For any k, (char arbitrary) any formal family

SGA: Milnor fiber $\in K_0(\ell\text{-adic sheaf on } \mathrm{Gal}(k((t))))$

then extend to the torus at ∞, Fourier transform, get $\# \in \mathbb{Q}_\ell(\sqrt[?]{i})$

(Informal/claim: count objects with weight $w(\mathcal{E})$

$$(\text{char } k>0) \quad A_V = \sum_{\mathcal{E}\in\mathcal{C}_I} \frac{1}{\#\mathrm{Aut}(\mathcal{E})} \left(q^{\frac{1}{2}}\right)^{\sum_{i\leq 0}(-1)^i \mathrm{rk}\, \mathrm{Ext}^i(\mathcal{E},\mathcal{E})} \Big(1-\underbrace{\mathrm{Tr}_{\mathrm{Frob}}\ \underset{\text{inv.}}{\text{numer.}}\,(MF(\Psi_{\mathcal{E}}))\Big)}_{} \cdot e_{[\mathcal{E}]}$$

$[\mathcal{E}]\in\Lambda$

$I\subset\mathbb{R}$ interval $\quad V=\exp(iI)$

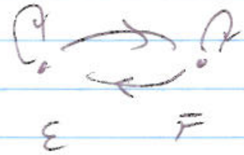
gives homomorphism from Hall alg to quantum torus

Basic identity $/\mathbb{F}_q$

$\forall \mathcal{E}, F$

$$\sum_{\alpha\in \mathrm{Ext}^1(F,\mathcal{E})} w(\mathcal{E}\oplus_\alpha F) = q^{?}\, w(\mathcal{E})\cdot w(F)$$

Picture:



consider $\mathrm{Ext}^1(\mathcal{E}\oplus F, \mathcal{E}\oplus F)$

$$= \mathrm{Ext}^1(F,\mathcal{E}) \quad V_1$$
$$\oplus\ \mathrm{Ext}^1(\mathcal{E},F) \quad V_2$$
$$\oplus\ \mathrm{Ext}^1(\mathcal{E},\mathcal{E})\oplus\mathrm{Ext}^1(F,F) \quad V_3$$

$\Psi=\Psi_{\mathcal{E}\oplus F}$: series at $0\in V_1\oplus V_2\oplus V_3$

$$\underbrace{+1 \quad -1 \quad 0}_{\text{homogeneity}}$$

After some miracles . . .
$$\sum_{\alpha\in V_1}\left(1-H^\bullet(\Psi|_{(\alpha,0,0)})\right) = q^{\mathrm{rk}\,V_1}\left(1-H^\bullet(MF \text{ of } \tilde{\Psi})\right)$$

$$\tilde{\Psi} = \Psi|_{(0,0,V_3)}$$

Can now reduce with Thom-Seb. theorem.

Proof (simple topology) think in $\mathbb{C}$

ψ on $V_1 \oplus V_2 \oplus V_3$

$$\text{LHS} = H^\cdot\left(\left\{ (z_1, z_2, z_3) \mid \psi(z_1,z_2,z_3) = t \quad |t| = \varepsilon \right.\right.$$
$$\left.\left. |z_1| \le \varepsilon^{-\frac{1}{10}}, \ |z_2|, |z_3| \le \varepsilon^{\frac{1}{100}} \right\}\right)$$

$$\text{RHS} = H^\cdot\left(\left\{ (0,0,z_3) \mid \psi(0,0,z_3) = t \quad |t| = \varepsilon \right.\right.$$
$$\left.\left. |z_3| \le \varepsilon^{\frac{1}{100}} \right\}\right) \cdot \underset{(z_1,0,0)}{H_c^\cdot(|z_1| < 1)}$$

$z_1 \to z_1 \lambda$

$z_2 \to z_2/\lambda$ $\qquad$ LHS $\supseteq$ piece $|z_1||z_2| > 0$

$\lambda \in \mathbb{C}^\times$ $\qquad$ $\cup$ $|z_1|, |z_2| = 0$ ... - proof too complicated to continue.

Corollary We obtain a map

$$\mathrm{Stab}(\mathcal{C}) \longrightarrow \mathrm{Stab}\left(\begin{matrix}\text{quantum} \\ \text{torus}\end{matrix} \quad \underset{\text{for curve } \mathcal{C}/\mathbb{C}}{\overset{\text{coeffs}}{Q(z_1, z_2)}} \quad e_{\gamma_1} \cdot e_{\gamma_2} = L^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} e_{\gamma_1+\gamma_2}\right)$$

Candidates for $\mathcal{C}$: (1) X noncompact CY3
$\cup$
Z proper $\quad \mathcal{C} = \mathrm{Perf}_{\mathrm{supp}\, Z}(X)$

Stab condition for Z curve can be constructed (not yet for X itself)

(2) A A_∞-alg finite dim $\qquad \mathcal{C} = \mathrm{Perf}(A\text{-mod})$

CY 3 $\quad A^i \otimes A^{3-i} \to \mathbb{R}$

(3) M closed oriented 3d C^∞-mfd

Assume $M = K(\pi, 1) \quad \rightsquigarrow \quad \mathcal{C} = D_{\substack{\text{finite} \\ \text{dim cohom}}}(\mathbb{C}[\pi]\text{-modules})$

$\rightsquigarrow$ invariant of M 3d mfd.

(4) Quivers with potentials (huge class of examples)

Calculation examples:

(1) $\mathcal{C} =$ generated by one object $(\mathcal{E})$, spherical & $Ext^\bullet(\mathcal{E},\mathcal{E}) = H^\bullet(S^3)$ $\Rightarrow$ crucial condition

so

$\mathcal{C}^{ss} = \mathcal{E},\ \mathcal{E}\oplus\mathcal{E},\ \ldots$

$K_0(\mathcal{C}) = \mathbb{Z} \longrightarrow \mathbb{H}$

$\operatorname{Im} Z(\mathcal{E}) > 0$

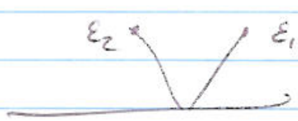
$\forall V$ $L(x)$: $A_V = \sum_{n\geq 0} \frac{q^{n^2/2}}{\# GL_n(\mathbb{F}_q)} x^n$ $\qquad x = e_{[\mathcal{E}]}$

quantum dilogarithm

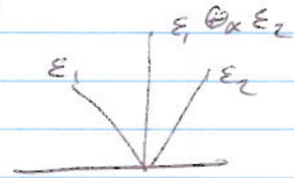
(2) Now two 3-spherical objs $\mathcal{E}_1, \mathcal{E}_2$

and $Ext^1(\mathcal{E}_2, \mathcal{E}_1) =$ 1-dim α $\quad$ others $= 0$

Indecomp ss. objs



$\mathbb{H}$ $\rightsquigarrow$



consider $xy = q\,yx$

then $L(x)\,L(y) = L(y)\,L(xy)\,L(x)$

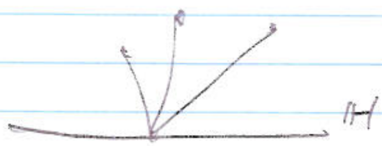
(3) $(\mathcal{E}_i)$ 3-spherical objs $\quad C = \langle (\mathcal{E}_i) \rangle$

rk $Ext^\alpha(\mathcal{E}_i, \mathcal{E}_j) = 0 \quad \alpha \neq 1, 2$, $i\neq j$

rk $Ext^1(\mathcal{E}_i, \mathcal{E}_j) = 0 \rightsquigarrow$ rk $Ext^2(\mathcal{E}_i, \mathcal{E}_j) = 0$ $\qquad$ (directed quiver)

A_∞ CY str $\Longleftrightarrow$ potential $\quad \ldots \quad$ mutations $\rightsquigarrow$ identities one explicit identity  + generic potential

mutation $\rightsquigarrow$ same quiver



quantum torus $\supset A_{\mathbb{H}} = \sum_{i,j,k \geq 0} c_{ijk} \cdot$ basis $x_1^i x_2^j x_3^k$

$c_{ijk} \in \mathbb{Q}(q^{1/2})$

$c_{000} = 1$

then $\exists\ b_{m_1, m_2, n} \in \mathbb{Q}(q^{1/2})$ $\quad m_1 \geq 0 \quad |m_2| \leq m_1$, $n \geq 0$

s.t. $\sum_{l \geq 0} q^{l(n_2 - n_1)} \frac{}{[l]!} \ldots$

incomprehensible formula

some quivers $\rightsquigarrow$ cluster transformations

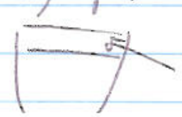
<u>Final words</u>: $q \to 1$ no limits : divergent

consider $Ad(A_V)$ aut of quantum alg

has limit as $q \to 1$, $q^{1/2} \to -1$ $\rightsquigarrow$ symplectomorphisms in limit

are DT-invs.

<u>meaning of symplectomorphisms</u>; agrees with Behrend story

glue from pieces a complex sympl. mfd $\dim = \operatorname{rk} \Lambda$  cone near ∞ "hypermultiplet moduli space"