Representation Theory — R. Kottwitz.
U. Chicago, Spring 2001
[Notes by D. Ben-Zvi, first two lectures missing]

4/3/01

$F(Fv \wedge v) = -(Fv \wedge v)$
$\Rightarrow (Fv\wedge v)^{q-1} = -1.$ $\xi^{q-1} = -1$ root
get $Fv \wedge v = \xi$.
$v = (x,y) \Rightarrow x^q y - y^q x = \xi$, or projectively
$x^q y - y^q x - \xi z^{q+1} = 0$

$GL_2(\mathbb{F}_q)$ acts on $\{v \mid (Fv\wedge v)^{q-1} = -1\}$ $= \coprod_\xi$ variety with fixed ξ
Pick $\xi \Rightarrow SL_2(\mathbb{F}_q)$ acts on $x^q y - y^q x = \xi$.
determinant permutes ξ's (in GL_2 action)

$\mathbb{F}_{q^2}^*$ also acts – comes from Weyl group cycle structure...
Fix $\xi \longrightarrow$ gets cut down to $SL_2(\mathbb{F}_q) \times A \circlearrowright \{x^q y - y^q x = \xi\}$
$A = \{\alpha \in \mathbb{F}_{q^2}^* \mid N\alpha = 1\}$ $N: \mathbb{F}_{q^2} \to \mathbb{F}_q$ norm.
$N: \alpha \longmapsto \alpha\cdot\bar\alpha = \alpha\cdot\alpha^q = \alpha^{q+1}$.

Note: $A = \{a \in \bar{\mathbb{F}}_q : a^{q+1} = 1\}$ $(q+1)$st roots of unity.
$\Rightarrow a^{q^2-1} = a$ since $q^2-1 = (q+1)(q-1)$..
Acts by rescaling $(x,y) \mapsto (ax, ay)$.

For SL_2 $\tilde{X}_w = \{(x,y) \mid x^q y - y^q x = \xi\}$
$\downarrow$ A-Galois $\Big\rangle$ both connected
$\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q) = X_w \subset \mathbb{P}^1$ open

Consider a character $\rho: A \longrightarrow \bar{\mathbb{Q}}_\ell^*$ (we're looking for a construction from ρ of a $(q-1)$-dim rep of $SL_2(\mathbb{F}_q)$).

$\rho \rightsquigarrow$ 1-dim local system $\mathcal{L}_\rho$ on X_w.
($\longleftrightarrow$ break up cohomology upstairs $\tilde{X}_w$ wrt A).
By analogy over $\mathbb{C}$: $Y \downarrow_G$ covering space (Galois) X, $\rho: G \to \mathrm{Aut}\, V$ $\mathbb{Q}$ vectorspace

$\Rightarrow H^*_c(X_w, \mathcal{L}_\rho)$ $SL_2(\mathbb{F}_q)$ rep. To get $GL_2(\mathbb{F}_q)$ rep need $\tilde\rho: \mathbb{F}_{q^2}^* \to \bar{\mathbb{Q}}_\ell^*$
lift to GL_2-equivariant local system.

<u>Claim</u> $h^i(X_w, \mathcal{L}_\rho) = \begin{cases} 0 & i=0 \\ q-1 & 1 \\ 0 & 2 \end{cases}$

Suppose X compact conn. R.S. genus g.

$S \subset X$ finite set. $H_c^\cdot(X \setminus S, \mathcal{L})$
$\mathcal{L}$ loc sys / $\mathbb{Q}$

What is the Euler characteristic? – independent of the local system (for fixed dim) cover $X \setminus S$ by fin. # of trivializing open sets

U V $H_c(U \cap V) \to H_c(U) \oplus H_c(V) \to H_c(X) \to \ldots$

χ is additive, so $H_c(X)$ agrees with that for trivial loc sys since pieces do.

$\Rightarrow \chi(H_c^\cdot(X \setminus S, \mathcal{L})) = \dim \mathcal{L} \cdot \chi(H_c(X \setminus S, \mathbb{Q})) = \dim \mathcal{L} \cdot (\chi(H_c^\cdot(X)) - \chi(H_c^\cdot(S)))$
$= \dim \mathcal{L}\,((2-2g) - |S|)$

In our case get $1 \cdot (2-(q+1)) = 1-q$.

$\to$ this has ℓ-adic analog but statement more complicated – but wild ramification complicates things! (same as in Hurwitz formula).

$$\begin{array}{l} X \\ \downarrow f \\ Y \end{array} \qquad H_c^\cdot(X, \mathbb{Q}_\ell) = H_c^\cdot(Y, Rf_! \mathbb{Q}_\ell)$$
$X \to Y$ finite, fibers $(Rf_! \mathbb{Q}_\ell) = (\mathbb{Q}_\ell^{\dim \text{fibers}})$

– in our case ramification is tame!

look at monodromy around $\mathbb{P}^1(\mathbb{F}_q)$ points
$|A| = q+1$, rel prime to p so sits in tame part.

Other extreme: trivial local systems – i.e. study $H_c^\cdot(X_w, \mathbb{Q}_\ell)$
– virtual rep of $G(\mathbb{F}_q)$

Gl_2: $w=1$ $X_w = \mathbb{P}^1(\mathbb{F}_q)$ $w \neq 1$ $X_w = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$

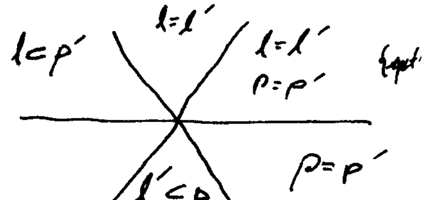
$G = Gl_3$: $W = S_3$, get six spaces. Virtual rep depends only on conj class $\to$ three of these.

$B = \begin{bmatrix} * & * & * \\ & * & * \\ 0 & & * \end{bmatrix}$ G/B = flags in V, $\dim V = 3$. $V = \overline{\mathbb{F}_q}^3$

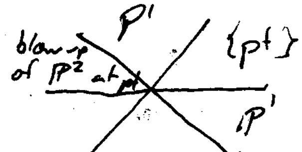
$F: V \to V$ $x, y, z \mapsto F_x, F_y, F_z = x^q\, y^q\, z^q$

$Y = G/B = \{\ell \subset P\}$ line in plane.

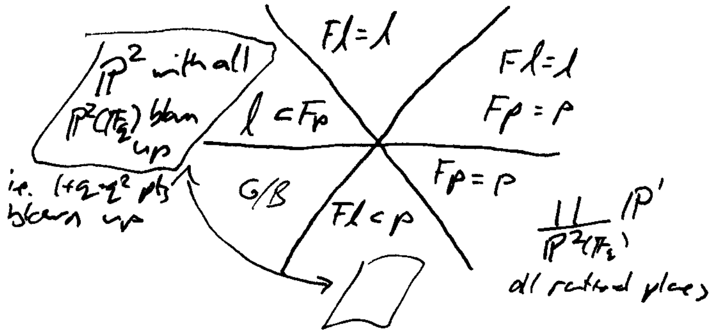
$\overline{G\text{-orbits on } Y \times Y}$ $(\ell \subset p) \times (\ell' \subset p')$



$\longleftrightarrow$ B orbits on X blow up of $\mathbb{P}^2$ at pt



$\overline{X}_w = \coprod_{\mathbb{P}(V)} \mathbb{P}^1$: ~~planes~~ containing fixed rational line



Corresponding Deligne-Lusztig varieties, $\overline{X}_w$

Schubert: blow up $\{\ell\} \in \mathbb{P}^2$: given ℓ, p, $\ell' \in \mathbb{P}^2$

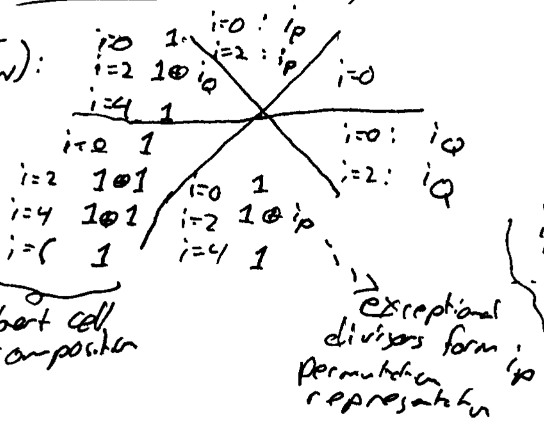
$\ell' \neq \ell \Rightarrow \ell, \ell'$ determine $p \ni \ell$, otherwise have to blow up.

Fiber $\ell', p' \mapsto \ell'$ here is a $\mathbb{P}^1$.

Dimension same as Schubert variety & look similar....

In our case all $\overline{X}_w$'s nonsingular ... in general will need <u>intersection cohomology</u>

closed $H^i(\overline{X}_w)$: as rep



$G = GL_3 \supset P \supset B$, $\supset Q \supset B$ parabolics

$P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

$Q = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

G/P = lines in $V = \mathbb{P}^2$

G/Q = planes in $V = \mathbb{P}^2$

i_B: induced rep $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} 1$, same for P, Q.
- permutation rep of $\mathbb{F}_q$-rational points.

Action of Frobenius is by q^j on H^{2j} :

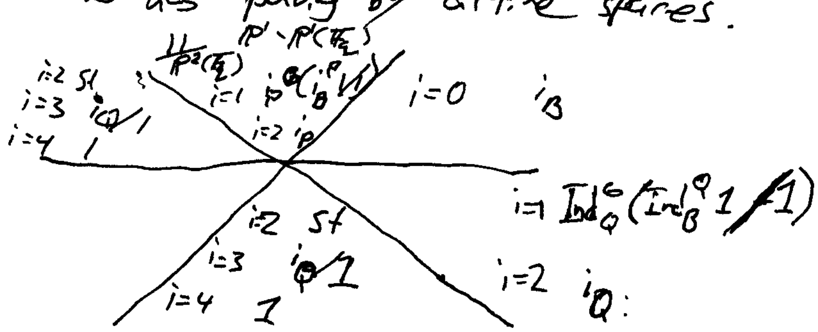
$H^i_c(\mathbb{A}^d) = \begin{cases} \mathbb{Q}_\ell & i = 2d \\ 0 & \text{otherwise} \end{cases}$ F is mult. by q^d.

... just q^d points in $\mathbb{A}^d(\mathbb{F}_q)$

Recall Lefschetz : $X/\mathbb{F}_q$, $\# X(\mathbb{F}_q) = \sum_{i=0}^{\dim X} (-1)^i \mathrm{Tr}(F; H^i_c(X))$

& G/B has paving by affine spaces.

open $H^i_c(X_w)$



$GL_3 \circlearrowright \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$

GL_3 permutes these around $\Longrightarrow$ <u>induced rep</u> for $\coprod_{\mathbb{P}^2(\mathbb{F}_q)} \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$

$H^2_c(\mathbb{P}^1 \setminus \text{pts}) = H^2_c(\mathbb{P}^1)$ ... induce trivial 1-d rep

GL_2: $\mathbb{P}^1 \supset \mathbb{P}^1(\mathbb{F}_q)$

deg 0 : 1 $\quad i_B^Q$

$\Rightarrow$ in complement get $i_B^Q - 1$

-- ie. $i_B^Q 1 / 1$

Steinberg for GL_2 is $i_B^P 1/1$

$H^*(\mathbb{P}^2$ w/ all $\mathbb{F}_2$-rat pts$)$ gave us $\begin{matrix} 1 \\ 1 \oplus i_Q \\ 1 \end{matrix}$

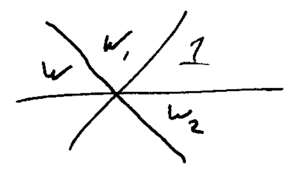
use this to get H_c^* of open piece...

~~Closed~~ all blown up part gets deleted:

so our open Deligne Lusztig is $\mathbb{P}^2 \setminus U$(rational ~~lines~~ hyperplanes)

-- some complicated configuration of lines in $\mathbb{P}^2$. . . .

$$X_w \xrightarrow{\text{open}} \overline{X_w} \xleftarrow{\text{complementary closed}} Y = \overline{X_{w_1}} \underset{GB(\mathbb{F}_2)}{\cup} \overline{X_{w_2}}$$



H_c^i

i	X_w	$\overline{X_w}$	closed Y
0	—	1	$\to 1$
1	—	—	~~$(i_P \oplus i_Q)/1$~~ i_B/i_P+i_Q
2	~~$i_P \oplus i_Q$~~ i_B	$1 \oplus i_Q$	$\to i_P \oplus i_Q$
3	$i_P/1$	—	—
4	1	1	—

$St = i_B/i_P+i_Q$

Y connected

~~sits diagonally in $i_P \oplus i_Q$~~

H^2 not affected by gluing

- cohomology of our X_w's is always sitting in middle dim & above!

-- the X_w's are in fact affine varieties in our case (true always for q sufficiently large.)

& also nonsingular $\Rightarrow (H_c^i)^* = H^{4-i}$ - follows from vanishing ~~below~~ middle degree of ordinary cohomology ($\Leftarrow$ Morse theory/$\mathbb{C}$)

Steinberg: ~~$0 \to St \to i_B \to i_P \oplus i_Q \to 1 \to 0$~~

$i_P \hookrightarrow i_B \hookleftarrow i_Q$ overlap in trivial rep

$$0 \to 1 \to i_P \oplus i_Q \to i_B \to St \to 0$$

$$St = i_B - i_P - i_Q + 1$$

H_c^i(top X_w) complicated - but its Euler char is just given as virtual rep by

$$H_c^\bullet(\text{open } X_w) = 6 \cdot 1 - , \ldots$$

Decompose into irreps for i_B: $End_G(i_B)$ = Hecke algebra for $G = GL_3$

$\simeq \mathbb{C}[S_3]$ group algebra

regular rep of $S_3 = \underset{\text{triv}}{1} + \underset{\text{sign}}{\varepsilon} + \underset{\text{reflection}}{2r}$

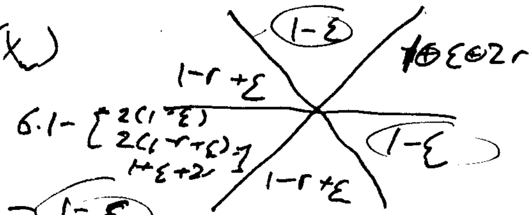
$r = S_3$ acting on $\mathbb{C}^3$ mod diagonal $\mathbb{C}$.

irreps of G contained in i_B are in 1-1 correspondence with $\hat{S}_3$'s: call them $\Pi(1)$, $\Pi(\varepsilon)$ & $\Pi(r)$

$P < G > Q$, $P > B < Q$ $\qquad$ $S_3 > S_2$, $S_3 > S_2$, $S_2 > 1 < S_2$ $\qquad$ $i_B = \Pi(1) \oplus \Pi(\varepsilon) \oplus 2 \cdot \Pi(r)$

Triv of S_2 induce to $S_3 \Rightarrow$ 3-dim rep
$\Rightarrow$ rep on $\mathbb{C}[S_3/S_2] = \mathbb{C}[\{1,2,3\}] = \mathbb{C}^3 = \mathbb{1} \oplus r$
So $i_P = \Pi(1) \oplus \Pi(r) = i_Q$
"names": $\Pi(1) = \mathbb{1}$ $\qquad$ $\Pi(\varepsilon) = St$

$H_c^{\bullet}(X_w)$



$= 1-\varepsilon$

Euler char of open $H_c^{\bullet}(X_w)$ is same as for (12), (23): only depends on conjugacy class!
— but Frob action depends on weight.
The (12) & (23) are conjugate via a single reflection & have same length...

$G \backslash (G/N \times G/N) = N \backslash G/N = N_G(T)$ $\qquad$ all defined $/\mathbb{F}_q$.

G-orbits $O(w) \subset G/B \times G/B$ $\qquad$ $\dot{w}$ lift of w to $N_G T$
$\uparrow \qquad \uparrow$ principal T-bundle
" $O(\dot{w}) \subset G/N \times G/N \circlearrowleft$ diagonal T action
Define action $(y_1, y_2) \cdot t = (y_1 t, y_2 w^{-1}(t))$
$(y_1,y_2) \in O(\dot{w})$, $t \in T$, $w^{-1}(t) = \dot{w}^{-1} t \dot{w}$

— this twisted action preserves $O(\dot{w})$ & is simply transitive on fibers $O(\dot{w}) \longrightarrow O(w)$.

$(y_1, y_2) \in O(\dot{w})$ $\qquad$ $y_1^{-1} y_2 \in N\dot{w}N \Rightarrow t^{-1} y_1^{-1} y_2 w^{-1}(t) \in N\dot{w}N$ also
(twisting)

Deligne-Lusztig $\quad X_w = \{x \in G/B : (x, Fx) \in O(w)\}$
$\uparrow$
$\tilde{X}_{\dot{w}} = \{y \in G/N : (y, Fy) \in O(\dot{w})\}$

~~fibers~~ This is a principal T^{F_w}-bundle:
w-twisted F action on T
$F_w : T \to T$ $\qquad$ $F_w = w \circ F$ $\quad$ (w as automorphism of T)
$F_w(t) = \dot{w} F(t) \dot{w}^{-1}$

F_w is new Frobenius for our split torus $T \simeq (\mathbb{G}_m)^l$
- ~~...~~ gives form of our torus T.

Claim $t \mapsto F(t) \cdot w^{-1}(t)^{-1}$ is surjective, get

$$1 \to T^{F_w} \to T \to T \to 1$$

- $\boxed{\text{Lang's theorem}}$:

H connected alg group $/\mathbb{F}_q$. $H \to H$

$h \mapsto h^{-1}F(h)$ is a principal H^F-bundle (via left mult.)

eg $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, $\{1, F\}$. $\mathrm{Aut}\,\mathbb{G}_m = \pm 1$ nontrivial elt: $x \mapsto x^{-1}$ on $\overline{\mathbb{F}}_q^*$.

Twist by this, get new Frobenius $x \in \overline{\mathbb{F}}_q^*$

$F'(x) = F(x)^{-1} \to$ torus T

$$T^{F'} = T(\mathbb{F}_q) = \{x \in \overline{\mathbb{F}}_q^* : F(x) = x^{-1}\}$$
$$\Rightarrow F^2x = x$$

$\Rightarrow x \in \mathbb{F}_{q^2}$ with norm 1 $\quad xF(x) = 1$.

Lang's thm $\iff H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), H(\overline{\mathbb{F}}_q)) = \{1\}$.

e.g. H^1 with coeffs in orthogonal group classifies quadratic forms over your field - nothing new for fin. fields.

Proof (outline)

Consider twisted conjugation action of H on H

Form $H \rtimes \langle F \rangle \Rightarrow H \cdot F$ coset, invariant under H action by conjugation (write as right action)

Claim: all orbits are open:

fix $h' \in H$, orbit map $H \to H$

$$h \mapsto h^{-1}h'F(h)$$

- but differential of F is zero - so as if F factor is absent, so looks like differential of translation $h \mapsto h^{-1}h'$. $\to$ differential is isomorphism $\to$ map open in Zariski topology. $\blacksquare$

$\tilde{X}_w$ $\quad$ θ character of T^{F_w}

$$\begin{array}{l} \tilde{X}_w \\ \pi \downarrow \; T^{F_w} \xrightarrow{\theta} \overline{\mathbb{Q}}_\ell^* \\ X_w \end{array}$$

$\Rightarrow$ rank 1 local system $\mathcal{L}_\theta$ on X_w

Proper map $R\pi_! \overline{\mathbb{Q}}_\ell = R\pi_* \overline{\mathbb{Q}}_\ell = \pi_! \overline{\mathbb{Q}}_\ell \circlearrowleft T^{F_w}$

- decompose under T^{F_w} get $= \bigoplus_\theta \mathcal{L}_\theta$.

$T^{F_w} \circlearrowright H^i_c(\tilde{X}_{\dot w}, \overline{\mathbb{Q}}_\ell) = \bigoplus_\theta H^i_c(X_w, \mathcal{L}_\theta)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad H^i_c(\tilde{X}_{\dot w}, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}}$

& G^F acts on everything.

R^θ_w = virtual rep (character) of $\sum_i (-1)^i H^i_c(X_w, \mathcal{L}_\theta)$

... as fn on G^F

Given $f, f' : G^F \to \overline{\mathbb{Q}}_\ell$ virtual characters

($\mathbb{Z}$-linear combos of characters of irreps)

$\Rightarrow$ inner product $\langle f, f' \rangle = \frac{1}{|G^F|} \sum_{g \in G^F} f(g) f'(g^{-1})$

$f'(g^{-1})$ ... will be complex conjugate character (dual rep)

If Θ, Θ' characters of G^F $V' V$ reps

$\langle \Theta, \Theta' \rangle = \dim \mathrm{Hom}_{G^F}(V', V) = \dim (W^* \underset{\overline{\mathbb{Q}}_\ell}{\otimes} V)^{G^F}$

$\tilde{X}_{\dot w}$ $\qquad \tilde{X}_{\dot w} \xrightarrow{\cdot t^{\in T}} \tilde{X}_{\dot w \cdot \dot w'}$ $\qquad \dot w' = \dot w^{-1}(t^{-1}) F(t)$

$\downarrow T^{F_w}$ $\qquad G^F \times T^{F_w}$-equivariant $\qquad$ -- varying t get all lifts of w this way

X_w

map $\_ \cdot t$ defined over $\mathbb{F}_q \Leftrightarrow F(t) = t \Leftrightarrow t \in T(\mathbb{F}_q)$

(supposing $\tilde{X}_{\dot w}$ defined over $\mathbb{F}_q$: $\mathcal{O}(\dot w)$ defined $/\mathbb{F}_q$ when $\dot w$ is, $F\dot w = \dot w$ -- but given $w \in W$ have a lift $\dot w$ defined $/\mathbb{F}_q$ by Lang's theorem, applied to T)

Split group: $w^F = w$ all w, W defined over $\mathbb{F}_q$ -- not true for non-split.

$\to$ So except for Galois action, choice of lifting $\dot w$ irrelevant, (say G^F reps).

To calculate $(W^* \otimes V)^{G^F}$ geometrically:

take spaces where W & V live, take product & quotient out by diagonal action.

$\Rightarrow$ study $G^F \backslash X_w \times X_{w'}$, $X_w \times X_{w'} \subset X \times X$

& $X \times X = \coprod_{w_1 \in W} \mathcal{O}(w_1)$ G-orbits on $X \times X$.

$gF(xy)g^{-1}$
$= g\ F$

$gFg^{-1}(xy)$ $\quad gFg^{-1}(x)\ gFg^{-1}(y)$

$gFxg^{-1}\ gFg^{-1}y = gx^Fg^{-1}y$

~~Define~~ $Y_{w,w',w_1} := G^F \backslash [(X_w \times X_{w'}) \cap O(w_1)]$

$\Rightarrow$ pairs

$$\begin{tikzcd} B_1 \arrow[r, "w_1"] \arrow[d, "w"'] & B_2 \arrow[d, "w'"] \\ FB_1 \arrow[r, "Fw_1"'] & FB_2 \end{tikzcd}$$

$Fw_1 = w_1$ (split groups)

$- \xrightarrow{w} -$ denotes relative posn w

$$G \backslash \left\{ \begin{tikzcd} B_1 \arrow[r, "w_1"] \arrow[d, "w"'] & B_2 \arrow[d, "w'"] \\ B_1' \arrow[r, "w_1"'] & B_2' \end{tikzcd} \right\}$$

$\parallel$

normalize B_1, B_2 using G action

$$T(N \cap N') = B \cap B' = \mathcal{C} \backslash \left\{ \begin{tikzcd} B \arrow[r, "w_1"] \arrow[d, "w"'] & B' = w_1 B w_1^{-1} \arrow[d, "w'"] \\ \tilde{B} \arrow[r, "w_1"'] & \tilde{B}' \end{tikzcd} \right\}$$

B standard Borel

(to be precise should phrase for stacks: must account for stabilizer.)

$$Z_{w,w',w_1} := \left\{ \begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ \tilde{B} \arrow[r] & \tilde{B}' \end{tikzcd} \right\}$$

principal $\mathcal{C}$-bundle

So can form fiber product

$$\begin{tikzcd} D \arrow[r] \arrow[d] & Y_{w,w',w_1} \arrow[d] \\ Z_{w,w',w_1} \arrow[r] & \mathcal{C} \backslash Z_{w,w',w_1} \end{tikzcd}$$

Claim: $Y_{w,w',w_1} = G \backslash \{ (\Phi, B_1, B_2) : \Phi \in G \cdot F \subset G \rtimes \langle F \rangle$

i.e. $\Phi : G \to G \quad x \mapsto gFC$

$$\begin{tikzcd} B_1 \arrow[r, "w_1"] \arrow[d, "w"'] & B_2 \arrow[d, "w'"] \\ \Phi B_1 \arrow[r, "w_1"'] & \Phi B_2 \end{tikzcd} \Big\}$$

By Lang G acts transitively on the Φ's & stabilizer of standard $\Phi = F$ is $G^F \Rightarrow$ get just Y again.

$$\left\{\begin{array}{ccc} B_1 & \xrightarrow{w_1} & B_2 \\ \downarrow w & & \downarrow w' \\ \Phi B_1 & \xrightarrow{w_1} & \Phi B_2 \end{array}\right\} \longrightarrow \left\{\begin{array}{ccc} B_1 & \xrightarrow{w_1} & B_2 \\ w \downarrow & & \downarrow w' \\ B_1' & \xrightarrow{w_1} & B_2' \end{array}\right\}$$

is a principal G-bundle!

Given $(B_1', B_2') \in O(w_1)$ so is $F(B_1, B_2) \in O(w_1)$

$\Rightarrow \exists g$ s.t. $(B_1', B_2') = \Phi(B_1, B_2)$ for $\Phi = gF$ by Lang

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$$Y_{w,w',w_1} = G^F \setminus [(X_w \times X_{w'}) \cap O(w_1)]$$

$$= G \setminus \left\{ (\Phi, B_1, B_2) \in G\cdot F \subset G \rtimes \langle F \rangle \times X \times X : \begin{array}{ccc} B_1 & \xrightarrow{w_1} & B_2 \\ w \downarrow & & \downarrow w' \\ \Phi B_1 & \rightarrow & \Phi B_2 \end{array} \right\}$$

$B = TN/\!\!\!\!\,$ std

$B' = w_1 B w_1^{-1}$

$C = B \cap B' = TN_{w_1}$

So B_1 is in X_w for new Frobenius Φ etc

$$= C \setminus \left(\left\{ \Phi \in G\cdot F : \begin{array}{ccc} B & \xrightarrow{w_1} & B' \\ w \downarrow & & \downarrow w' \\ \Phi B & \xrightarrow{w_1} & \Phi B' \end{array} \right\} =: S \right)$$

$$= C \setminus S$$

$$\begin{array}{ccc} Y_{w,w',w_1} = C \setminus S & \xleftarrow{\psi_1} & S \\ & & \downarrow \psi_2 \\ & & Z_{w,w',w_1} = \left\{ (\tilde{B}, \tilde{B}') \in X \times X : \begin{array}{ccc} B & \xrightarrow{w_1} & B' \\ w \downarrow & & \downarrow w' \\ \tilde{B} & \xrightarrow{w_1} & \tilde{B}' \end{array} \right\} \end{array}$$

ψ_1, ψ_2 both principal C bundles -- two actions of C on S which commute -- multiply Φ by left or right

... $\Phi \in G\cdot F$ right $\Phi c B = \Phi B \quad \Phi c B' = \Phi B'$

left $c B = B, \quad c B' = B'$

ψ_1 = quotient of S by $\Phi \longmapsto c \Phi c^{-1} \quad c \in C$

ψ_2 = quotient of S by $\Phi \longmapsto \Phi c^{-1} \quad c \in C$

-- these two actions don't commute though!

Factorize further:

- divide just by T

... a_1, a_2 T-bundles

$$\begin{array}{ccccc} C \setminus S & \xleftarrow{b_1} & S_1 & \xleftarrow{a_1} & S \\ & & & & \downarrow a_2 \\ & & & & S_2 \\ & & & & \downarrow b_2 \\ & & & & Z \end{array}$$

T not normal so b_1, b_2 not principal bundles but are affine bundles, fibers $\cong N_{w_1} = N \cap w_1 N w_1^{-1}$

$\longrightarrow$ doesn't affect cohomology with compact support, except for shift.

$\Longrightarrow$ want to compare cohomologies of S_1, S_2.

– need more info about the T-bundles.

Claim a_2 naturally trivial -- in fact two natural sections $S_2 \underset{a_2}{\overset{s}{\rightleftarrows}} S$

Aside: where does this come from! closely related

T-bundle has a section ...

$G/B \supset BwB/B$ Schubert cell

$T \circlearrowright \uparrow \pi$

G/N T-bundle trivializes on any Schubert cell

Section for π over BwB : choose $\dot w$ lifting w to $N_G(T)$

get section using N action

Describe image of section $s : S_2 \longrightarrow S$!

Choose $\dot w$. $\Phi = g.F \in S$

$$\begin{tikzcd} S & S_1 \arrow[l, "b_1"'] & S \arrow[l, "a_1"'] \\ & S_2 \arrow[u] \arrow[ur, "s"'] \arrow[d, "b_2"] & \\ & Z & \end{tikzcd}$$

$g \in BwB \Rightarrow \Phi$ can be written uniquely as hFt

for $h \in N\dot w N$, $t \in T$

$\Rightarrow$ get section of form $\{\Phi = hF = hF\cdot 1 \ (t=1)\}$

Now consider $a_1 \circ s : S_2 \longrightarrow S_1$:

Fiber through $\underset{n_1 \dot w n_2 \cdot F}{\underset{\shortparallel}{\Phi}} \in s(S_2)$:

$$= \{ t \in T : \underset{\substack{\shortparallel \\ t n_1 \dot w n_2 F t^{-1} \\ \shortparallel \\ n_1' \dot w n_2' t^w F(t)^{-1} F}}{t\Phi t^{-1}} \in s(S_2) \}$$

– can use to show fiber nonempty

& in fact $= \{ t \in T : F(t) = t^w \} = T^{Fw}$

so $a_1 \circ s$ is principal T^{Fw}-bundle

So $Y = C \setminus S$ not too far from Z !

In fact
Action of finite group T^{Fw} on S_2 extends to an action of the torus $\{(t_1, t_2) \in T \times T : F(t_2) = t_1^w\}$
(T^{Fw} is diagonal part)

This acts on S_2 not preserving fibers,
$H^*(S_1)$ is T^{Fw} invariants in $H^*(S_2)$.

But this extends to action of connected torus
$\longrightarrow$ trivial on cohomology, so T^{Fw} acts trivially on cohomology

$$\therefore \quad \boxed{H^i_c(Y_{w,w';w_1})} = H_c^{i+2d}(S_1)(d) \dashrightarrow \text{Tate twist}$$

$d = \dim$ of N_{w_1} fibers

$$= H_c^{i+2d}(S_2)(d)^{TF_w} = H_c^{i+2d}(S_2)(d) = \boxed{H^i_c(Z_{w,w';w_1})}$$

(F acts on $H^{top}(\mathbb{A}^d)$ by mult. by q^d – Tate twist $\mathbb{Q}_l(d)$)

- Note. we obtained S from $\begin{matrix} NwN/N \\ \downarrow \\ BwB/B \end{matrix}$

 ... can also get S from $B' = TN'$, $\begin{matrix} N'wN'/N' \\ \downarrow \\ B'wB'/B' \end{matrix} \longrightarrow w_1 w' w_1^{-1}$ sets correct

$\Rightarrow \{\Phi = gF \in S \text{ st } g \in N' w_1 w' w_1^{-1} N'\}$

Put $w'' = w_1 w' w_1^{-1}$.

$S_2 \xrightarrow{a_1 \circ s'} S_1$ principal $T^{Fw''}$-bundle, action extends ...

[$\Rightarrow$ new isom $H^i_c(Y_{w,w';w_1}) = H^i_c(Z_{w,w';w_1})$]

$$\Rightarrow \begin{tikzcd} & S_3 \arrow[dl] \arrow[d] \arrow[dr] & \\ S_2 \arrow[dr, "a_1 \circ s"'] & {} \arrow[d] & S_2 \arrow[dl, "a_1 \circ s'"] \\ & S_1 & \end{tikzcd}$$

fiber product, $S_3 \to S_1$ principal $T^{Fw} \times T^{Fw''}$-bundle, & this action extends to action of

$$H = \left\{ (t_1, t_2, t_1'', t_2'') \in T^4 : \begin{matrix} F(t_2) = t_1^w \\ F(t_2'') = t_1''^{w''} \\ t_1/t_1'' = t_2/t_2'' \end{matrix} \right\}$$

$t_1 \overset{w}{—} t_2$

$t_1'' \underset{w''}{—} t_2''$

$$= \left\{ (t_2, t_2'') : \frac{F_w(t_2)}{F_{w''}(t_2'')} = \frac{t_2}{t_2''} \right\}$$

$$\Leftrightarrow \{ t_2^{-1} F_w t_2 = (t_2'')^{-1} F_{w''} t_2'' \}$$

H closed subgroup of torus — but might not be connected

$$1 \longrightarrow T^{Fw} \times T^{Fw''} \longrightarrow H \longrightarrow T \longrightarrow 1$$

The <u>identity component</u> of H acts trivially on cohomology $H^i_c(S_3)$.

But is $T^{Fw} \times T^{Fw'}$ in identity component, or which part of it is?

Let H^0 = identity component of H, $1 \to H^0 \to H \to \text{finite group} \to 1$

What is $H^0 \cap T^{Fw} \times T^{Fw'}$?

<u>Abstraction</u>: T a torus, with 2 Frobenii F_1, F_2

$$1 \to T^{F_1} \times T^{F_2} \to H \xrightarrow{p} T \to 1$$

$$H = \{(t_1, t_2) \in T \times T : t_1^{F_1 - 1} = t_2^{F_2 - 1}\} \qquad F_i - 1 \in \mathrm{End}\, T.$$

But over some extension $\mathbb{F}_{q^r}$ both tori split

$\Rightarrow F_1^r = F_2^r$, get maps $N : T \to H$

defined by $t \mapsto (N_1 t, N_2 t)$ $\qquad N_i = 1 + F_i + F_i^2 + \dots + F_i^{r-1} \in \mathrm{End}\, T$

$$(1 - F_i) N_i = 1 - F_i^r = 1 - F_j^r = (1 - F_j) N_j$$

$\mathrm{Im}\, N(T) \subset H$ is a torus

$p \circ N = F_i^r - 1$ (indep of i), which is surjective (Lang)

So $\mathrm{Im}\, N(T)$ has same dim as T

$\Rightarrow N(T) = H^0$ for dimension reasons.

$N_1 t \in T^{F_1} \iff t$ clear $\underbrace{N_i (F_i - 1)} = F_i^r - 1$

$\iff t \in T^{F_i^r}$.

$$\Rightarrow T^{F_1} \times T^{F_2} \cap N(T) = N(T^{F^r})$$

Consider characters $\theta_i : T^{F_i} \to \overline{\mathbb{Q}}_\ell^*$

$\Rightarrow (\theta_1, \theta_2) : T^{F_1} \times T^{F_2} \to \overline{\mathbb{Q}}_\ell^*$

This is trivial on $T^{F_1} \times T^{F_2} \cap$ {identity component}

$\iff \theta_1 \circ N_1 = \theta_2 \circ N_2 : T^{F^r} \to \overline{\mathbb{Q}}_\ell^*$

... "geometric conjugacy" for tori!

$$\begin{tikzcd} & T^{F^r} \arrow[dl, "N_1"'] \arrow[dr, "N_2"] & \\ T^{F_1} \arrow[r, "\theta_1"] & \overline{\mathbb{Q}}_\ell^* & T^{F_2} \arrow[l, "\theta_2"'] \end{tikzcd}$$

surjective by Lang

$$\begin{array}{ccc} G/N & \supset & \pi^{-1}(BwB/B) \\ \pi\downarrow & & \downarrow \\ G/B & \supset & BwB/B \simeq \mathbb{A}^{\ell(w)} \end{array}$$

principal T-bundle $\Longleftrightarrow$ ℓ line bundles over $\overline{\mathbb{F}}_q$ $T = \mathbb{G}_m$

We chose a lifting $\dot{w} \in N_G(T)$ of w. On $\mathbb{A}^{\ell(w)}$,
$\mathcal{O}_{\mathbb{A}^{\ell(w)}} = \overline{\mathbb{F}}_q[x_1, \dots x_{\ell(w)}]$

To give triv $\Longleftrightarrow$ give nowhere vanishing section — unique up to constant.
- so once we've determined lift at one point, determines section uniquely. This is correctly given by N orbits.

Let's calculate the Euler char. of Z_{w,w',w_1} (compact support)

T acts on Z_{w,w',w_1} — can calculate Euler characteristic from torus fixed points (Bialynicki-Birula Top. 12(1973) 99-103)

Euler chars $\dim H_c^\bullet(Z_{w,w',w_1}) = \dim H_c^\bullet(Z^T_{w,w',w_1})$

- in fact suffices to take 1-dim torus in general enough position — reduces to $\mathbb{G}_m$ case.
Ex $\mathbb{P}^1 \circlearrowleft \mathbb{G}_m$, fixed pts are $0, \infty$, rest is $\mathbb{G}_m$ which has Euler char. zero.
or more general $X \supset X^{\mathbb{G}_m} \cup \{$principal $\mathbb{G}_m$-bundle$\}$ - again has zero Euler char.

What is Z^T_{w,w',w_1}: sits inside $X \times X$. & $X^T \longleftrightarrow W$

we need $\{B \xrightarrow{w_1} \tilde{B}\} \subset X \times X$

$$\Rightarrow Z^T_{w,w',w_1} = \{(x,y) \in W \times W : \begin{array}{ccc} \cdot & \xrightarrow{w_1} & \cdot \\ w \downarrow & & \downarrow w' \\ x & \xrightarrow{w_1} & y \end{array}\}$$
$$= \begin{cases} * & \text{if } w = w_1 w' w_1^{-1} = w'' \\ \varnothing & \text{otherwise} \end{cases}$$

So $\dim H_c^\bullet(Z_{w,w',w_1}) = \begin{cases} 1 & w = w'' \\ \varnothing & \text{otherwise} \end{cases}$

<u>Consequences</u>

a. $\langle R^\theta_{w}, R^{\theta'}_{w'} \rangle = \#\{w_1 \in W \text{ s.t. } w = w_1 w' w_1^{-1} \text{ \& } \theta = {}^{w_1}\theta'\}$

Recall θ is character $\theta : T^{F_w} \longrightarrow \overline{\mathbb{Q}_\ell}$

$\theta' : T^{F_{w'}} \nearrow$

w_1 conjugates w' to w → ask if it conjugates θ' to θ

${}^{w_1}\theta' : T^{F_w} \longrightarrow \overline{\mathbb{Q}_\ell}$

- two conjugate w's define conjugate tori over $\mathbb{F}_q$
→ can compare characters.
.. we calculated identity component $[\,]^0$ on $T^{F_w} \times T^{F_{w'}}$

.. we're trying to decompose cohomology of $\tilde{X}_w$, T^{F_w} bundle over X_w.
- look for pieces on product which on first decomposes as θ & on other acts as θ'
- but some subgroup is being forced to act trivially from H^0 action ⟹ so get compatibility between characters being enforced.

$X_w \times X_{w'}$ ∩ $(\mathcal{O}(w_1))$ decomposition breaks up H^*_c of $X_w \times X_{w'}$ into that of the Y's: Euler char is additive — so have to count # of pieces w_1 which contribute.
(characters also must match or else get zero contribution).

Here we use that the contragredient of R^θ_w is $R^{\theta^{-1}}_w$
(need contragred in $\langle , \rangle$ vs rep itself in Künneth decomposition)
... last time we had a compatibility of $\theta, ({}^{w_1}\theta')^{-1}$.
Follows from the fact that the character of
$$\tilde{H}_c(\tilde{X}_w) = \bigoplus_\theta R^\theta_w \quad \text{of} \quad G^F \times T^{F_w}$$
has integer values (see [DL], Ann Math)

- if values are integers then rep is self-contragredient
(contragred gives complex conjugation on characters).

Is there cancellation of reps in taking Euler chars?

b. $H^i_c(X_w, \mathcal{L}_\theta)$, $H^j_c(X_{w'}, \mathcal{L}_{\theta'})$
are disjoint representations of G^F unless
θ is geometrically conjugate to θ' i.e.
$\exists\, w_1 \in W$ s.t. θ is geometrically to ${}^{w_1}\theta'$:

$$\begin{tikzcd}
 & T^{F_w} \arrow[r, "\theta"] & \overline{\mathbb{Q}}_\ell \\
T^{\Phi} \arrow[ur, "N_1"] \arrow[r, "N_2"] & T^{F_{w''}} \arrow[ur, "{}^{w_1}\theta'"'] &
\end{tikzcd}
\qquad w'' = w_1 w' w_1^{-1} \qquad \Phi = F_w^r = F_{w''}^r$$

Not same as character on alg closure, only an element
of direct limit $\varinjlim_r (T^{F^r})^\vee$ via norm
maps $T^{F^r} \supset T^F$ (N) not via inverse limit from inclusions!

c. $\underset{f}{R_w^\theta} = \underset{f'}{R_{w'}^{\theta'}} \Longleftrightarrow \exists\, w_1 \in W$ s.t. $\begin{cases} w' = w_1 w w_1^{-1} \\ \theta' = {}^{w_1}\theta \end{cases}$

__Pf__ Scalar product: if equal $\Rightarrow$ scalar prod $\neq 0 \Rightarrow$
$\exists\, w_1$. Now have to go backwards, want to
show they're actually __the same__! ($\Leftarrow$ obvious)

$\langle f', f' \rangle = \langle f, f \rangle = \langle f, f' \rangle$ by the formula

$\Rightarrow \langle f - f', f - f' \rangle = 0$. Choose $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$
--- our scalar product becomes pos definite
($\langle f, f \rangle \neq 0$ unless $f = 0$ for any rep) $\Rightarrow f - f' = 0$ ∎

d. $\langle R_w^\theta, R_w^\theta \rangle = \# \operatorname{Stab}_W(w, \theta)$
So if θ is in general position (no symmetries: $\operatorname{Stab} = \{1\}$)
$\Rightarrow \langle R_w^\theta, R_w^\theta \rangle = 1$ so R_w^θ is either irreducible
or $-1 \cdot$ an irreducible.

e. Special case of b: concentrate on θ trivial

$H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ is disjoint from $H^i_c(X_{w'}, \mathcal{L}_{\theta'})$
whenever $\theta' \neq 1$. — can't be geom conjugate to __trivial__ character!

— hardest case is for θ trivial, easiest for θ general!

Def Irr. Constituents of $H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ (some i, w) are called unipotent representations.

$\iff$ nonzero inner product with (nonzero mult in) some $R_w = R_w^1$.

- i.e. not too much cancellation: occurs in Euler char if it is some constituent. -- uses fact from DL / Carter: Character of regular rep is a $\mathbb{Q}$-linear combination of the R_w^θ's. - everything occurs in some R^θ_w!

f. $\forall w, w' \in W$, $\forall i, j$ all eigenvalues of $F = $ Frob in $\left[H^i_c(X_w, \overline{\mathbb{Q}}_\ell) \underset{\overline{\mathbb{Q}}_\ell}{\otimes} H^j_c(X_{w'}, \overline{\mathbb{Q}}_\ell) \right]^{G^F}$

$\wedge$ $H^{i+j}_c(G^F \backslash X_w \times X_{w^{-1}})$ are powers of q.

- we broke up into pieces Y_{w,w',w_1} - which has same cohomology of Z_{w,w',w_1}.

Pf (sketch) that e'vals on $H^i_c(Z)$ powers of q:

$Z = \tilde{B}, \tilde{B}' \in X \times X$: $B \xrightarrow{w} \tilde{B} \xrightarrow{w_1} \tilde{B}' \xrightarrow{(w')^{-1}} B'$

$\Rightarrow$ for any $r \in \mathbb{Z}_+$, $\# Z_{w,w',w_1}(\mathbb{F}_{q^r}) = $ coefficient of T_{w_1} in $T_w T_{w_1} T_{(w')^{-1}}$ $\quad T_w \in \mathcal{H}(q^r)$

Let $I(q) = i_B = \overline{\mathbb{Q}}_\ell[X(\mathbb{F}_q)]$ Note $X(\mathbb{F}_q) = G(\mathbb{F}_q)/B(\mathbb{F}_q)$ by Lang.

- Rep of $G(\mathbb{F}_q)$ $\quad g \cdot \varphi(x) = \varphi(g^{-1}x)$

$\mathcal{H}(q) = \mathrm{End}_{\overline{\mathbb{Q}}_\ell[G(\mathbb{F}_q)]}(I(q)) = \overline{\mathbb{Q}}_\ell[B(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / B(\mathbb{F}_q)]$

- act by right convolution on $I(q)$ by reversed function $(g \to g^{-1})$

$f \in \mathcal{H}(q)$ $\quad \varphi \in I(q)$ $\quad f \cdot \varphi = \varphi * f_r$

$$(f \cdot \varphi)(x) = \sum_{y \in G(\mathbb{F}_q)/B(\mathbb{F}_q)} \varphi(xy) f(y)$$

$\boxed{f}$ We're showing $Z_{w w_1 w_2}(\mathbb{F}_{q^r})$ = coeff of T_{w_1} in $T_w T_{w_1} T_{(w)^{-1}}$ $\in \mathcal{H}(q^r)$

Let $Z = \{(B_1, B_2) : B \xrightarrow{x} B_1 \xrightarrow{y} B_2 \xrightarrow{z} B'\}$ with $B \xrightarrow{w} B'$
(w, x, y, z fixed)

$|Z(\mathbb{F}_{q^r})|$ = coeff of T_w in $T_x T_y T_z$ (same for iterated products)

Claim: this is polynomial in q^r, with integer coefficients.
$\rightsquigarrow$ suggests that all eigenvalues of Frobenius on $H^i_c(Z, \overline{\mathbb{Q}_\ell})$ are powers of q.

Why? Lefschetz formula for Frobenius: $F \circlearrowright H^i_c(Z)$ $(F : Z \circlearrowleft)$
$\# Z(\mathbb{F}_{q^r}) = \#$f.p. of $F^r = \sum_{i=0}^{2\dim Z} (-1)^i \operatorname{tr}(F^r ; H^i_c(Z))$
Suppose F has eigenvalues ${}_i\alpha_1, \dots, {}_i\alpha_{d_i}$ on $H^i_c(Z)$ $(d_i = h^i_c(Z))$

$\Rightarrow \# Z(\mathbb{F}_{q^r}) = \sum (-1)^i ({}_i\alpha_1^r + \dots + {}_i\alpha_{d_i}^r)$
So if LHS polynomial in q^r $\rightsquigarrow$ suggests RHS α_i are themselves powers of q -- could have some cancellation though. But if no eigenvalues occur both in odd & even degree then LHS poly in $q^r \Rightarrow$ all α_i powers of q. $\because \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} \neq 0$ for 3 distinct α_i.

Example: $Z = \mathbb{P}^1$ with two $\mathbb{F}_q$-rational points $(0, \infty)$ identified

$|Z(\mathbb{F}_{q^r})| = (q^r + 1) - 1 = q^r$

$H^0(Z)$ 1
$H^1(Z)$ 1 - comes from "combinatorial" loop, invariant under Frob
$H^2(Z)$ q
all 1-dim, eval of Frob
$\Rightarrow \# Z(\mathbb{F}_q) = 1^r - 1^r + q^r = q^r$

Back to Hecke algebra
$\mathcal{H}(q)$ = fns on $B(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / B(\mathbb{F}_q)$ ($\overline{\mathbb{Q}}_\ell$-valued)
under convolution for measure with $\mu(B(\mathbb{F}_q)) = 1$.

$(f \cdot \varphi)(x) = \sum_{y \in X(\mathbb{F}_q)} \varphi(xy) f(y)$ $X = G/B$

$w \in W$ T_w = char. fn of coset $B(\mathbb{F}_q) w B(\mathbb{F}_q)$ $\in \mathcal{H}(q)$

T_w form basis of $H(q)$.

$T_x T_y = \sum_w N_{x,y,w} \cdot T_w$. $N_{x,y,w}$ is an <u>integer</u>

$N_{x,y,w}$ = # of Borels B s.t. $\underbrace{B_0 \xrightarrow{x} B_1 \xrightarrow{y} B'}_{w}$ B, B' fixed

<u>Example</u> $G = GL_2$ $X = \mathbb{P}^1$ $W = \{1, w\}$. $T_1 = 1$ identity

$T_w^2 = \_\, T_1 + \_\, T_w$.   coeff of T_1 = $\#\mathbb{A}^1$ $= \{B_1 \neq B\}$ $= q$ $B \xrightarrow{w} B_1 \xrightarrow{w} B$

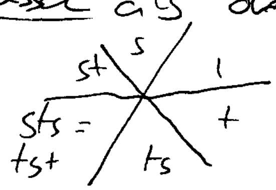
Coeff of T_w : $\underbrace{B \xrightarrow{w} B_1 \xrightarrow{w} B'}_{w}$ all three different $\Rightarrow \#\mathbb{A}^1 \setminus pt = q-1$

$T_w^2 = q T_1 + (q-1) T_w$.

Set $q = 1$ get $T_w^2 = T_1$: group algebra of W.

- have <u>assoc</u> alg over $\mathbb{Z}[q]$ q indeterminate.

In general braid relation



① $T_s^2 = q \cdot 1 + (q-1) T_s$ always holds $s \in W$ simple reflection

② braid relation $T_s T_t T_s = T_t T_s T_t$ in GL_3 ... — same as $\mathbb{C}[W]$

2′ $T_x T_y = T_{xy}$ when $\ell(xy) = \ell(x) + \ell(y)$

2′ $\Rightarrow$ $T_s T_t T_s = T_t T_s T_t$

Reference for e'vals of Frob on $H^i_c(Z_{x,y,w})$ powers of q :
- Lusztig : Reps of finite Chevalley groups , CBMS #39 1977
  Lemma 3.7

<u>Important Consequence</u> :

9. First note : Virtual rep $R_w = H^*_c(X_w)$ of G^F is self-contragredient — because its character has integer values
∴ contragredient of any unipotent rep is unipotent.

ρ unipotent rep of G^F (irred), suppose ρ occurs in both

(generalized) μ eigenspace of Frob on $H^i_c(X_w)$ &
μ' " " " $H^{i'}_c(X_{w'})$

Then $\mu\mu'$ is a power of q $\rightsquigarrow$ invariant $\mu \in \overline{\mathbb{Q}}_\ell^* / q^{\mathbb{Z}}$
(usually root of unity times power of q)

<u>Proof</u>. Let $\tilde{\rho}$ = contragredient of ρ $\Rightarrow$ unipotent
$\tilde{\rho}$ occurs in μ'' eigenspace of Frob on $H_c^{i''}(X_{w''})$

By F $\Rightarrow$ $\mu\mu''$ power of q, $\mu'\mu''$ power of q.

$$\rho\otimes\tilde{\rho} \subset H_c^i(X_w) \otimes H_c^{i''}(X_{w''})$$

1-dimensional since ρ is irreducible $\longleftarrow$ $[\rho\otimes\tilde{\rho}]^{G^F} \subset H_c^i(\quad)^{G^F}$ $\longleftarrow$ Frob acting by powers of q.

$\longrightarrow$ Frob acts by $\mu\mu''$ ∎

<u>Representations of Hecke algebras</u>

$\mathbb{H}$ as algebra over $\mathbb{Z}[q, q^{-1}] =: A$

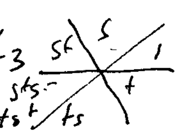
A-basis $\{T_w\}$, $w \in W$

$T_s^2 = q1 + (q-1)T_s$ $\quad$ s simple reflection

$T_x T_y = T_{xy}$ when $\ell(xy) = \ell(x) + \ell(y)$

$(T_s T_t T_s = T_t T_s T_t)$

GL_3



$$T_{st} T_{st} = T_s T_t T_s T_t = T_t T_s T_t T_t$$
$$= T_t T_s (q \cdot 1 + (q-1) T_t)$$
$$= q T_{ts} + (q-1) T_{tst}$$

More generally (arbitrary w): s simple

$$T_x T_s = \begin{cases} T_{xs} & \text{if } \ell(xs) = \ell(x) + 1 \\ q T_{xs} + (q-1) T_x & \ell(xs) = \ell(x) - 1 \end{cases}$$

<u>Reps of $\mathbb{H}$ for GL_n</u> (following general principles)

Write $X(w)$ = Schubert cell $BwB/B \subset X = G/B$, BwB/B is $\mathbb{A}^{\ell(w)}$

$X(w) \cap \overline{X(w)}$ Schubert variety.

$j : X(w) \hookrightarrow \overline{X(w)}$ open

On $X(w)$ take constant sheaf $\overline{\mathbb{Q}}_\ell$ & middle extend

$j_{!*}(\overline{\mathbb{Q}}_{\ell\, X(w)})$ (after shift to make perverse... by $\ell(w)$.)

$\rightarrow$ Kazhdan-Lusztig basis,

For GL_3 our $\overline{X(w)}$ s are either $\mathbb{P}^1$ or $\mathbb{P}^2$ with one point blown up — all nonsingular in this case!
$\Rightarrow$

$$j_{!*} \overline{\mathbb{Q}}_\ell{}_{X(w)} = \overline{\mathbb{Q}}_\ell{}_{\overline{X(w)}}.$$

Fonctions - Faisceaux! T_w thought of as constant sheaf $\overline{\mathbb{Q}}_\ell$ on $X(w)$, extended by zero.
So here have characteristic function of $\overline{X(w)}$

$$A_w = \sum_{X(w') \subset \overline{X(w)}} T_{w'}$$

Bruhat order: $X(w') \subset \overline{X(w)}$ $\longleftrightarrow$ all subexpressions of a reduced expression for w.

w	A_w
1	1
s	$1 + T_s$
t	$1 + T_t$
st	$1 + T_s + T_t + T_{st}$
ts	$1 + T_s + T_t + T_{ts}$
$w_0 = sts = tst$	$1 + T_s + T_t + T_{st} + T_{ts} + T_{sts}$

K-L Use A_w as basis for H. To understand action of H in this basis just need $T_s \cdot A_w$ & $T_t \cdot A_w$

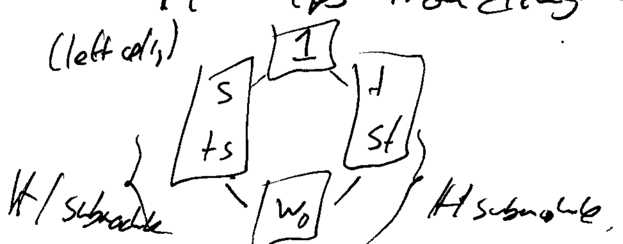
w	$T_s \cdot A_w$	$T_t A_w$
1	$-1 + A_s$	$-1 + A_t$
s	$q A_s$	$-A_s + A_{ts}$
t	$-A_t + A_{st}$	$q A_t$
st	$q A_{st}$	$q A_t - A_{st} + A_{w_0}$
ts	$q A_s - A_{ts} + A_{w_0}$	$q A_{ts}$
$w_0 = sts$	$q A_{w_0}$	$q A_{w_0}$

Check $T_s \cdot A_t = -A_t + A_{st}$:

$$T_s A_t = T_s (1 + T_t) = T_s + T_{st} = A_{st} - A_t.$$

Gen! action will be block triangular! — will get H irreps from diagonal blocks — KL cells!

(left cells)



eg A_{w_0} eigenvector for T_s, T_t
$\Rightarrow H$ subrep.

i.e. $\mathcal{A}\cdot A_{w_0} = \mathbb{Z}[q,q^{-1}]$ rank 1 $\mathcal{H}$-submodule of $\mathcal{H}$.

<u>Reps of S_3</u> Now look at $\mathcal{A}$-span of $A_s, A_{ts}, A_{w_0}, \ldots$
$\Rightarrow$ 2 1-d reps & 2 2-d reps of $\mathcal{H}$ in $\mathcal{H}$.

ε : e.g. 1-dim rep $\boxed{1}$ is just "sign rep": $T_s = -1, T_t = -1$ on here.

$\boxed{\begin{smallmatrix} s \\ ts \end{smallmatrix}}$: $T_s = \begin{pmatrix} q & q \\ 0 & -1 \end{pmatrix}$ $\quad T_t = \begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix}$

r reflection rep

$\boxed{\begin{smallmatrix} t \\ st \end{smallmatrix}}$: $T_s = \begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix}$ $\quad T_t = \begin{pmatrix} q & q \\ 0 & -1 \end{pmatrix}$

--- <u>isomorphic</u> representations

1 $\quad \boxed{w_0}$: $T_s = q \quad T_t = q$.

$T_s = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad T_t = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$

$T_s = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad T_t = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$T_s = 1 \quad T_t = 1$

set $q = 1$ $\quad \mathcal{H}/(q-1) = \mathbb{Z}[W]$

$w \backslash M$	$\boxed{1}$	$\boxed{\begin{smallmatrix} s \\ ts \end{smallmatrix}} \simeq \boxed{\begin{smallmatrix} t \\ st \end{smallmatrix}}$	$\boxed{w_0}$
1	1	2	1
s	-1	$q-1$	q
t	-1	$q-1$	q
st	1	$-q$	q^2
ts	1	$-q$	q^2
w_0	-1	0	q^3
	$(-1)^{\ell(w)}$		$q^{\ell(w)}$

q-character table: $\mathrm{trace}(T_w \text{ on } M) \in \mathcal{A}$.

$q = 1 \Rightarrow$ character table for S_3.

From calculation before on GL_3 we know $H^\bullet_c(X_w)$ all w as "virtual rep" of $G^F \times \langle F \rangle$ - we know F since all <u>closures</u> are smooth in this case, & cohomology came from $H^\bullet_c(\mathbb{A}^n)$ inside. — all in even degrees, in degree $2i$ acts by mult by q^i. Now use this info on closure to get weights on $H_c(X_w)$.

As virtual rep of G^F, $H^\bullet_c(X_w) = \mathbb{Z}$-linear combo of $\pi(1), \pi(\varepsilon), \pi(r)$ — label unipotent reps of $GL_3{}^F$.

$H^\bullet_c(X_1) = i_B = \mathrm{Ind}_{B^F}^{G^F} 1 \circlearrowleft \overline{\mathbb{Q}}_\ell[G^F] \otimes \mathcal{H}/\overline{\mathbb{Q}}_\ell$ commuting actions (each other's centralisers)

$\parallel$

$(\pi(1) \otimes \boxed{w_0}) \oplus (\pi(\varepsilon) \otimes \boxed{1}) \oplus (\pi(r) \otimes \boxed{\begin{smallmatrix} s \\ ts \end{smallmatrix}})$

We'll record the virtual rep $H^\bullet_c(X_w)$ of $G^F \times \langle F \rangle$ as $\mathbb{Z}$-linear combo of $\pi(1), \pi(\varepsilon), \pi(r)$

Now make table

$E \leftrightarrow$ irrep of W (or H)

$w \backslash E$	$\boxed{1} \leftrightarrow \varepsilon$	$\boxed{\begin{smallmatrix} s \\ ts \end{smallmatrix}} \leftrightarrow r$	$\boxed{w_0} \leftrightarrow 1$

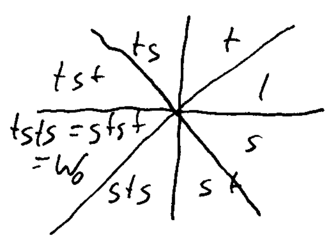
get <u>same</u> table as before if we take as table entry the coefficient of $\pi(E)$ unipotent rep in $H^\bullet_c(X_w)$ (as polys in q) !

So eigenvalues of Frobenius encoded in character table for Hecke algebra!

Same holds for GL_n !

Something similar but more complicated holds in general

$\boxed{Sp_4}$



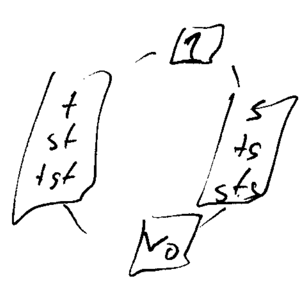
$W =$ dihedral group of order 8

w	A_w
1	1
s	$1+T_s$
t	$1+T_t$
st	$1+T_s+T_t+T_{st}$
ts	$1+T_s+T_t+T_{ts}$
sts	$1+T_s+T_t+T_{st}+T_{ts}+T_{sts}$
tst	$1+T_s+T_t+T_{st}+T_{ts}+T_{tst}$
w_0	Σ all T_w -

7 out of 8 Schuberts are nonsingular here — <u>and</u> even for singular one IC works as if smooth

-- root system not symmetric in s,t

w	$T_s A_w$	$T_t A_w$
1	$-1+A_s$	$-1+A_t$
s	qA_s	$-A_s+A_{ts}$
t	$-A_t+A_{st}$	qA_t
st	qA_{st}	$qA_t-A_{st}+A_{tst}$
ts	$qA_s-A_{ts}+A_{sts}$	qA_{ts}
sts	qA_{sts}	$qA_{ts}-A_{sts}+A_{w_0}$
tst	$qA_{st}-A_{tst}+qA_{w_0}$	qA_{tst}
w_0	qA_{w_0}	qA_{w_0}



$W = \langle s,t \rangle$ $\quad t^2 = s^2 = 1 \quad tsts = stst$

$\Rightarrow$ 4 1-dim reps: $\quad s,t \longrightarrow \pm 1$ each

$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 \rightsquigarrow$ reflection rep V

[Rmk: correction to table $T_s A_{tst} = q A_{st} - A_{tst} + A_{w_0} \ldots$]

5/1

W reps $q=1$

4 reps of $\mathbb{H}$ for Sp_4:

ε $\quad \boxed{\;}$ is 1-dim rep $\quad T_s = -1,\ T_t = -1 \quad$ ($q=1$: sign character)

$\underline{1}$ $\quad \boxed{w_0} \quad T_s = q \quad T_t = q$

$\boxed{\begin{matrix} t \\ tst \\ st \end{matrix}}$ $\quad T_s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & q & q \end{pmatrix} \quad T_t = \begin{pmatrix} q & 0 & q \\ 0 & q & 1 \\ 0 & 0 & -1 \end{pmatrix}$

$\boxed{\begin{matrix} s \\ sts \\ ts \end{matrix}}$ $\quad T_t = ? \quad T_s = ? \quad \rightsquigarrow$ suggests cell reps not irreducible

Character table for reps of $\mathbb{H}$

w \ modules	w_0	$\boxed{\;}$	t, tst, st	s, sts, ts	M_1	M_2	N
1	1	1	3	3	1	1	2
s	q	-1	$q-2$	$2q-1$	-1	q	$q-1$
t	q	-1	$2q-1$	$q-2$	q	-1	$q-1$
st	q^2	1	$-q$	$-q$	$-q$	$-q$	0
ts	q^2	1	$-q$	$-q$	$-q$	$-q$	0
sts	q^3	-1	q^2	$-q$	q	q^2	q^2-q
tst	q^3	-1	$-q$	q^2	$-q^2$	q	q^2-q
w_0	q^4	1	$-q^2$	$-q^2$	q^2	q^2	$-2q^2$

$q^{l(w)} \quad (-1)^{l(w)}$

$\mathbb{H}$ has generators $T_s, T_t \quad T_s^2 = q + (q-1)T_s \quad T_t^2 = q + (q-1)T_t$

& $T_sT_tT_sT_t = T_tT_sT_tT_s \Rightarrow$ can define 1-dim reps

by sending T_s, T_t to solns of $q + (q-1)x = x^2$

$\Rightarrow$ can assign -1 or q to either, gives

the 1-dim reps M_1, M_2 above

$\Rightarrow$ we have 4 1-dim characters $1, w_0, M_1$ & M_2

– need a 2-dim rep N left, making up

composition series for our cell reps

Indeed $\quad ch\begin{bmatrix} t \\ tst \\ st \end{bmatrix} = ch\, M_1 + ch\, N$

$\qquad\quad ch\begin{bmatrix} s \\ sts \\ ts \end{bmatrix} = ch\, M_2 + ch\, N$

Two cells 1, w_0 never interact w/ the others - but other things can mix as above.

Set $q=1 \Rightarrow$ Character table for W

$w_0 \longleftrightarrow 1$

$1 \longleftrightarrow \varepsilon$

$M_1 \longleftrightarrow \chi_1$

$M_2 \longleftrightarrow \chi_2$

$N \longleftrightarrow r$ reflection

Using this can work out what happens when we induce from Levi subgroups:

$\mathrm{Ind}_W^W 1 = 1 \qquad \mathrm{Ind}_{W_P}^W 1 = 1 + \chi_2 + r \qquad \mathrm{Ind}_{W_Q}^W 1 = 1 + \chi_1 + r$

$\mathrm{Ind}_{\{1\}}^W 1 = 1 + \varepsilon + \chi_1 + \chi_2 + 2r$ (regular rep)

Here $W_P = \{1, s\}$ Weyl group for Levi of a parabolic

$\qquad W_Q = \{1, t\}$

V 4-dim symplectic vector space, $\langle , \rangle$ alt. form $\left[\begin{array}{c|c} 0 & \begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \\ \hline \begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix} & 0 \end{array}\right]$

$G = Sp_4 = \{ g \in GL(V) : \langle gv, gu \rangle = \langle v, u \rangle \}$

$B = Sp_4 \cap \left(\begin{smallmatrix} * \\ 0 \end{smallmatrix} \right)$ - if we took different form for $\langle , \rangle$ couldn't just intersect with ∇ to get a Borel!

$P = Sp_4 \cap \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \qquad Q = Sp_4 \cap \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix} \right)$

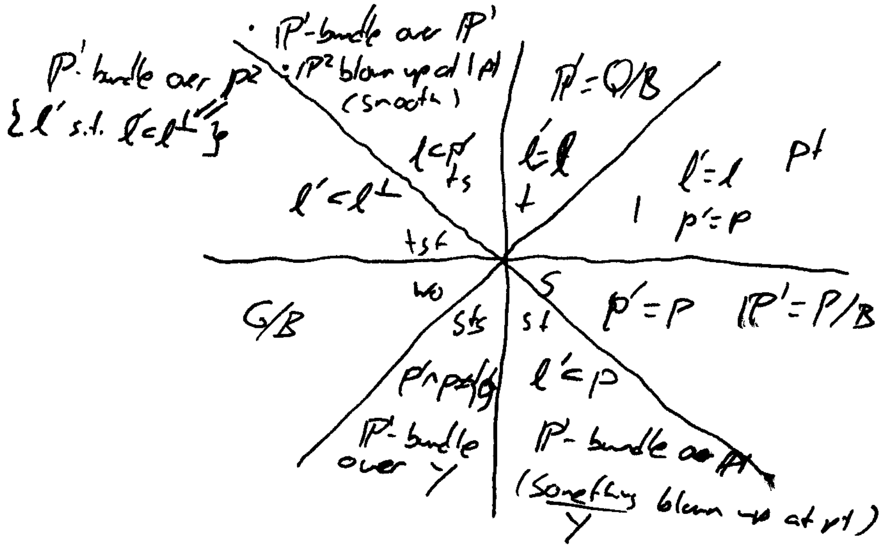
P, Q different: W is symmetric wrt $s \longleftrightarrow t$, but root system isn't.

$G/B = \{ V \supset V^3 \supset V^2 \supset V^1 \supset \{0\} \text{ s.t. } V^{4-i} = (V^i)^\perp \}$

$\quad = \{ p \supset l \text{ plane} \supset \text{line s.t. } p = p^\perp \text{ isotropic} \}$

Now write down Schubert varieties $\overline{X(w)}$

Fix standard $l, p \qquad \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$



$Y = \{$ isotropic planes p' s.t. $p' \cap p \neq 0 \}$
singular surface, with distinguished point $p' = p$.
$Y - \{p\}$ is nonsingular

Write $P = L \cdot U$ $\quad L \cong GL_2$ $\quad \left(\begin{array}{c|c} L & U \\ \hline 0 & (L^t)^{-1} \end{array}\right) = P$

Take affine orbit of $\bar{U}$, nbhd of $p \in G/P$ $\quad \bar{U} = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & 1 \end{array}\right)$

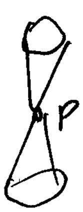
$p \in \bar{U} \cdot p \subset G/P =$ Lagrangian Grassmannian

$\begin{bmatrix} 1 & \\ A & 1 \end{bmatrix}$ $\quad A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ (symplectic condition)

$Y \cap \bar{U} \cdot p$ affine open nbhd of p in Y:

$Y_0 = \{ (a,b,c) \mid a^2 + bc = 0 \}$: $a^2 + bc = \det(p' \longrightarrow V/p)$

$\longrightarrow$ quadratic cone.



But this is just a quotient singularity

$Y_0 \cong \mathbb{A}^2 / \mathbb{Z}/2$ $\quad \mathbb{Z}/2 : \{1, \tau\}$ $\quad \tau : (x,y) \longmapsto (-x,-y)$

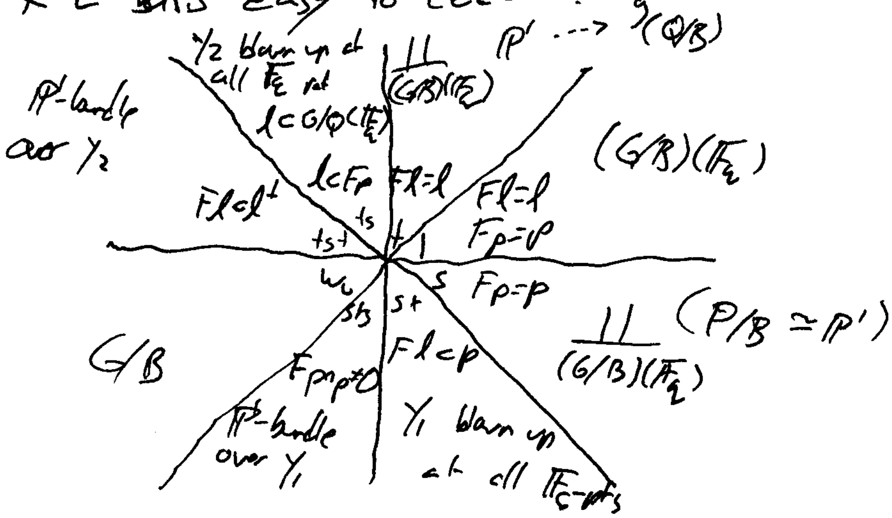
since $k[x,y]^\tau = k[x^2, y^2, xy]$

-- rational cohomology manifold .....

Claim : $\mathbb{D}(\bar{\mathbb{Q}}_\ell{}_Y) = \bar{\mathbb{Q}}_\ell[2d]$ like manifold.

Stack quotient is nice, its cohomology is $\mathbb{Z}/2$.
equivariant cohomology - group cohomology for $\mathbb{Z}/2$
But rational group cohomology for finite groups is just H^0.... so pt/G has same cohology as just a pt, rationally.

$\Rightarrow$ local IC of $\overline{X}(w)$ same as ordinary cohomology, K-L basis easy to calculate.



Deligne-Lusztigs X_w

$Y_1 = \{$isotropic planes p : $Fp \cap p \neq \{0\}\}$
$Y_2 = \{$lines l s.t. $Fl \subset l^\perp\}$
Y_1 singular at $\{p : Fp = p\}$
Y_2 non sing.

$$\dashrightarrow \begin{vmatrix} a^q - a & b^q - b \\ c^q - \cancel{c} & -(z^q - a) \end{vmatrix} = 0$$

$$= -(a^2 + bc) + \text{higher order}$$

Look at cohomology of Y_2:
$Y_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{P}^3 \mid \langle (x_1 \ldots x_4), (x_1^q, \ldots x_4^q) \rangle = 0\}$
- hypersurface of degree $q+1$ in $\mathbb{P}^3$, nonsingular

By Lefschetz hyperplane thm can calculate $H^\cdot(Y_2, \overline{\mathbb{Q}_\ell})$:

i	$H^i(Y_2, \overline{\mathbb{Q}_\ell})$
0	$\overline{\mathbb{Q}_\ell}$ = triv rep of $G^F \times F$
1	0
2	$\overline{\mathbb{Q}_\ell} \oplus \boxed{?}$
3	0 ↖ triv rep of G^F, F acts by q
4	$\overline{\mathbb{Q}_\ell}$ ← triv rep of G^F, F acts by q^2.

$\boxed{?}$ will contain a cuspidal irrep of $Sp_4(\mathbb{Q})$ - not describable in any simpler way!

$$H^\cdot(Y_2, \overline{\mathbb{Q}_\ell}) = (1 + q + q^2) \cdot \underset{\substack{\uparrow \\ \text{triv rep of } G^F.}}{\pi(1)} + \boxed{?}$$

From general principles, we get $\dim \boxed{?} = q^3 - q^2 + q$

(smooth deg $q+1$ hypersurface)

$i_B = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} 1$ etc

$i_G = \pi(1)$ triv etc

w	$IH^\cdot(\overline{X}_w) = H^\cdot(\overline{X}_w)$ in our case
1	i_B
s	$(q+1)\, i_P$
t	$(q+1)\, i_Q$
st	$(1+q+q^2)\, i_G + \boxed{?} + q \cdot i_Q$
ts	$(1+q+q^2)\, i_G + \boxed{?} + q \cdot i_Q$
sts	$(q+1)\,[\,(1+q+q^2)\, i_G + q \cdot i_Q - q \cdot i_P + \boxed{?}\,]$
tst	$(q+1)\,[\,(1+q+q^2)\, i_G + \boxed{?}\,]$
w_0	$(1+2q+2q^2+2q^3+q^4)\, i_G$

Poincaré poly for G/B, trivial rep

simple reflections: always get induced from $H^\cdot$ of projective line $= q+1$ ..

Blow up! replace 1 by $\mathbb{P}^1 \sim 1+q$, difference is a q ($\mathbb{A}^1$)

Y_1 & Y_2 closely related: $\overline{X}_{ts} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \overline{X}_{st}$

st & ts conjugate & of same length! conjugate by simple reflection!

So on open varieties get maps $\varphi \circ \psi = F$, $\psi \circ \varphi = F$

F isomorphism on cohomology

$(l, p) \longmapsto (l, Fp)$

$(Fl, p) \longleftarrow (l, p)$

$\Rightarrow \quad H^\cdot(\overline{X}_{ts}) \cong H^\cdot(\overline{X}_{st})$ as $G^F \times \langle F \rangle$ modules.

$\leadsto$ coh of Y_1 & Y_2 closely related as well.

$I\overline{H}^{\cdot}_{c}(Y_1) = H^\cdot(Y_1) = (1+q+q^2)\, i_G + q \cdot i_Q - q \cdot i_P + \boxed{?}$

as rep of $G_F \times F$

w	$H_c^*(X_w)$	$H_c^*(X_w)$ without Frobenius ($q=1$)
1	i_B	i_B - calculate by subtracting off orbit closures from previous list
s	$(q+1)i_P - i_B$	$2i_P - i_B$
t	$(q+1)i_Q - i_B$	$2i_Q - i_B$
st	$\boxed{?} + (1+q+q^2)i_G - (1+q)i_P - i_Q + i_B$	$\boxed{?} + 3i_G - 2i_P - i_Q + i_B$
ts	ditto	''
sts	$(q-1)\boxed{?} + (q+1)(1+q+q^2)i_G - (q^2-1)i_P + (q^2+1)i_Q - i_B$	$2i_Q - i_B$
tst	'' + '' $+ (1+q)i_P - (q-1)i_Q - i_B$	$2i_P - i_B$
w_0	$-2q\boxed{?} + (1+q^4)i_G + (q^2\cancel{+1})i_P - (q^2+1)i_Q + i_B$	$-2\boxed{?} + 2i_G - 2i_Q + i_B$

s & tst conjugate ⟹ give same rep for $q=1$, but different if take Frob into account.

E : reps of Weyl group, R_E linear combo with q-coeffs of characters

$\pi(1)$: principal series rep corresp to trivial rep of W

E	R_E
1	$i_G = \pi(1)$
ε	$i_G - i_P - i_Q + i_B = St = \pi(\varepsilon)$
$M_1 \longleftrightarrow \chi_1$	$\frac{1}{2}[-\boxed{?} - i_G + i_Q]$
$M_2 \longleftrightarrow \chi_2$	$\frac{1}{2}[-\boxed{?} - i_G + 2i_P - i_Q]$
$N \longleftrightarrow r$	$\frac{1}{2}[\boxed{?} - i_G + i_Q]$
$r+\chi_1 \longleftrightarrow M_1 + N$	$-i_G + i_Q$
$r+\chi_2 \longleftrightarrow M_2 + N$	$-i_G + i_P$
$r-\chi_1 \longleftrightarrow N - M_1$	$\boxed{?}$
$r-\chi_2 \longleftrightarrow N - M_2$	$\boxed{?} - i_P + i_Q$

$\left.\begin{array}{l} R_1 = \pi(1) \\ R_\varepsilon = \pi(\varepsilon) \end{array}\right\}$ in general

actual reps (not virtual): no minuses & no fractions, true characters

Characters are fns on Weyl group - use their values & divide by $8 = |W|$, not quite character
- add up all our virtual characters $H_c^*(X_w)$ over W with coeffs given by character of W & divide by 8. .

For GL_n we would find $R_E = \pi(E)$ always, all irreducibles, just parametrizing the irreps again.
So R_E are generalizations of the irreps - but some are irreps some aren't - but generally pretty close to irreducibles, combos of "small #s of irreps"

<u>Def</u> $R_E = \frac{1}{|W|} \cdot \sum_W tr(w,E) \cdot R_w$, R_w = D-L rep $H_c^*(X_w)$.

$\boxed{GL_n}$: $R_E = \pi(E)$ principal series rep & all unipotents arise this way : <u>all</u> <u>unipotents</u> are constituents of i_B here

R_E's not irreducible for same reason cell reps are not irreducible - we got M_1+N & M_2+N instead.
So cells are slightly out different ones.. but don't separate irreps! 2 cell reps for three reps M_1, M_2, N
Consider N as special (special rep of W)
& introduce combos $N-M_1$, $N-M_2$

We get from these actual reps, not virtual fractional combos..
How to prove? IC is better than $H_c^\cdot$ ---
We know purity for IC : know eigenvalues of Frobenius on IC. In IC table powers of q & deg of cohomology matches up. - so can tell which parts come from which cohomology! & these are actual not virtual reps. $\Rightarrow$ can check our R_E's are actual reps.
Remark: $\langle R_{E_1}, R_{E_2}\rangle \in \mathbb{Q}$ & our formula holds:
$\boxed{\langle R_{E_1}, R_{E_2}\rangle = \langle E_1, E_2\rangle}$ inner product of W characters.

- follows from formula for $\langle R_w, R_{w'}\rangle$, plus fact that every irrep of W is self-contragredient - characters are in fact rational numbers.
In general R_E's not irreps though!
$\langle R_E, R_E\rangle = 1$ does not imply irrep since have $\mathbb{Q}$-coeffs not $\mathbb{Z}$-coeffs...

Now can calculate inner products of our new R_E's with themselves
$\Rightarrow$ $\underline{-i_G + i_Q}$, $\underline{-i_G + i_P}$, $\underline{\boxed{?}}$ & $\underline{\boxed{?} - i_P + i_Q}$
are each sums of two irreps.
- can read off also their pairwise scalar products
Upshot
$R_{r+\chi_1} = -i_G + i_Q = \pi_0 + \pi_1$
$R_{r+\chi_2} = -i_G + i_P = \pi_0 + \pi_2$
$R_{r-\chi_1} = \boxed{?} = \pi_3 + \pi_2$
$R_{r-\chi_2} = \boxed{?} - i_P + i_Q = \pi_3 + \pi_1$

$\pi_0, \pi_1, \pi_2, \pi_3$ unipotent irreps.

In fact $\pi_0 = \pi(r)$, $\pi_1 = \pi(\chi_1)$ $\pi_2 = \pi(\chi_2)$
& π_3 is new - cuspidal unipotent.

Unipotent reps $\longleftrightarrow$ irreps occurring in some R_w. But R_E's form another basis - so unipotent $\Leftrightarrow$ occurring in R_E's. So these are all unipotents.

almost characters

$$R_r = \tfrac{1}{2}\left[\pi_0 + \pi_1 + \pi_2 + \pi_3\right]$$

add all up! (reflecting fact that r is special)

$$R_{\chi_1} = \tfrac{1}{2}\left[\pi_0 + \pi_1 - \pi_2 - \pi_3\right]$$

"Fourier transform" of characters

$$R_{\chi_2} = \tfrac{1}{2}\left[\pi_0 - \pi_1 + \pi_2 - \pi_3\right]$$

$$\square \quad \tfrac{1}{2}\left[\pi_0 - \pi_1 - \pi_2 + \pi_3\right]$$

- consider this one just by symmetry

almost characters: class fns on groups coming from character sheaves. - kind of Fourier duality between irreducibles & almost characters.

These are the irred perverse sheaves among characters - so more natural in that world.

Special one corresponds to a cuspidal character sheaves, while others are cokers of principal series character sheaves.

$\square$ is virtual character not in span of R_E's

Not true that R_w's span everything: one for every conjugacy class - only 5 of them & 6 unipotents - have to decompose individual cohomology groups

General reps reduce to unipotent reps on subgroups - unipotents on centralizers of semisimple - like Jordan decomp. Here we're doing hardest part!

Centralizer of identity $\longleftrightarrow$ unipotents on all G.

General position case reduces to unipotents on max torus - only one such. ..

write same table in "smarter" basis of R_E's.

w	$H_c(X_w)$
1	$R_{E=1} + R_\varepsilon + R_{\chi_1} + R_{\chi_2} + 2R_r$
s	$qR_{E=1} - R_\varepsilon - R_{\chi_1} + qR_{\chi_2} + (q-1)R_r$
t	$qR_{E=1} - R_\varepsilon - qR_{\chi_1} - R_{\chi_2} + (q-1)R_r$
st	$q^2R_{E=1} + R_\varepsilon - qR_{\chi_1} - qR_{\chi_2}$
ts	$q^2R_{E=1} + R_\varepsilon - qR_{\chi_1} - qR_{\chi_2}$
tst	$q^3R_{E=1} - R_\varepsilon + qR_{\chi_1} - q^2R_{\chi_2} + (q^2-q)R_r$
sts	$q^3R_{E=1} - R_\varepsilon - q^2R_{\chi_1} + qR_{\chi_2} + (q^2-q)R_r$
w_0	$q^4R_{E=1} + R_\varepsilon + q^2R_{\chi_1} + q^2R_{\chi_2} - 2q^2R_r$

as virtual rep of $G_F \times \langle F^2 \rangle$

- only holds for F^2 not F!

- problem with $F \circlearrowright \boxed{?}$

- pure, so have $1/{-q}$

If we pretend F eigenvalue on $\boxed{?}$ is q, this table is correct for F — but it's not!

$\boxed{?} = \pi_3 + \pi_2$ ← e-value q — every principal series always gets power of q

turns out to be $-q$

Theorem in general: always see power of F eigenvalue is a power of q (E_7, E_8 get $\sqrt{q}$, sometimes root of unity $\cdot\, q$)

Miracle: this is exactly the character table for IH for Sp_4.

<u>G_2 case</u>: get nonabelian Fourier transform for group S_3

Don't need to know explicitly the flags for G_2.

1st point: What does $IC^\cdot(\overline{X_w})$ look like?

G_2 $\overline{X(w)}$ are still always rational homology manifolds ($\mathbb{R}$ of them)

$\Rightarrow$ same for $\overline{X_w}$.

Why is $\dim X_w = \dim X(w)$, nonsingular, same IC ...?

Suppose
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \circlearrowleft & & \circlearrowleft \\ H & & G \end{array}$$
$H \xrightarrow{\alpha} G$ hom. of alg groups & f is H-equivariant

$\Rightarrow$ get $X/\!/H \xrightarrow{\bar{f}} Y/\!/G$ on stack quotients,

& if f is smooth of rel dim d, then $\bar{f}$ smooth of rel dim $d + \dim G - \dim H$.

— being a smooth map is local in smooth topology: can test on smooth covers.

<u>Our case</u>: f = Lang map $f(g) = g^{-1}F(g)$

$G \xrightarrow{f} G$ smooth rel dim zero.

$B \xrightarrow{\alpha} B \times B$ $\quad \alpha(b) = (b, Fb)$

$G \circlearrowleft B$ right translation, $B \times B$ acts by left & right translation.

$$\Rightarrow \quad G/B \xrightarrow{\varphi = \bar{f}} B\backslash G/B$$
$$gB \longmapsto Bg^{-1}F(g)B$$

φ smooth surjective, rel dim $= \dim B$.

Inverse image of w Schubert cell $B\backslash X(w)$ is X_w. $\quad g^* i_{!*} \mathbb{1}$

- so IC complexes on $X(w)$ pulled back by φ give ~~Schubert~~ D-L IC complexes, after shift. & X_w smooth of dim $X(w) = \ell(w)$