Character Sheaves

Suppose we want a formula for character i_B, $B \subset G$ Borel

$i_B = \mathrm{Ind}_{B^F}^{G^F} 1$ — just a permutation rep! have a set

$X(\mathbb{F}_q) = G^F/B^F$, G^F permutes these elts.

Trace of a permutation matrix is just ~~#~~ # fix points

(non fix pts give 0 diagonal entry)

so for $g \in G^F$, $tr(g, i_B) = \left| \{ x \in X(\mathbb{F}_q) \mid gx = x \} \right|$

X^g closed subvariety of X, $\quad \overset{\shortparallel}{\left| \; X^g(\mathbb{F}_q) \; \right|}$.

& we want its number of pts, as function of g.

$$\Rightarrow \text{Consider} \quad \begin{array}{ccc} X^g & \subset & \tilde{G} = \{ (g,x) \in G \times X \mid g \cdot x = x \} \\ \downarrow & & \downarrow \pi \\ g & \in & G \end{array} \qquad \pi^{-1}(g) = X^g. \qquad \pi \ \underline{\text{proper}}: \tilde{G} \underset{\text{closed}}{\subset} G \times X$$

<u>Lefschetz</u>: $|X^g(\mathbb{F}_q)| = \sum_{i=0} (-1)^i tr(F; H_c^i(X^g, \overline{\mathbb{Q}}_\ell))$

$g \mapsto H_c^i(X^g) = (R^i\pi_! \overline{\mathbb{Q}}_\ell)_g$ by proper base change.

$\overset{\shortparallel}{H^i}(X^g) = (R^i\pi_* \overline{\mathbb{Q}}_\ell)_g$

functions	complexes of ℓ sheaves
$g \mapsto tr(g; i_B)$	$R\pi_* \overline{\mathbb{Q}}_\ell$
Other characters	?

: take $\sum (-1)^i Tr(F; R^i\pi_* \overline{\mathbb{Q}}_\ell)$

Y variety over $\mathbb{F}_q$, $\mathcal{F}$ constructible on $/\mathbb{F}_q$ $\quad y \in Y(\mathbb{F}_q)$

$F: Y \to Y$ & $F: F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

$Fy = y$ & $F^*\mathcal{F} \to \mathcal{F}$ $\Rightarrow$ endo $\mathcal{F}_y \circlearrowleft F$

If $y \in \mathbb{F}_{q^d}$ $\Rightarrow$ $F_y^d : \mathcal{F}_y \circlearrowleft$

$\Longrightarrow$ $\mathcal{F} \rightsquigarrow$ function $Y(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$,

$y \mapsto tr(F_y, \mathcal{F}_y)$

& similarly fns on $Y(\mathbb{F}_{q^d})$

- $\mathcal{F}^\bullet$ bounded complex & $H^i(\mathcal{F}^\bullet)$ constructible

$\Rightarrow \sum (-1)^i tr(F_y, H^i(\mathcal{F}^\bullet))$

$\{ g \mapsto tr(g; i_B) \} = $ ~~sum~~ linear combo of characters of the reps $\pi(E)$ $(E \in W^\vee)$ with coefficient of $\pi(E)$ is $\dim E$.

...or really $E \in \mathrm{Irrep}(H)$ in general

(For E_7, E_8 need to choose $q^{\frac{1}{2}} \in \overline{\mathbb{Q}}_\ell$ to match up reps of W & $\mathbb{H}$.)

$$i_B = \bigoplus_{\mathrm{Irr}(\mathbb{H})} \pi(M) \otimes M$$: commuting actions of group alg & $\mathbb{H}$.

Now for $g \in G^F$, $T \in \mathbb{H}$ $\Rightarrow$ look at $g \cdot T \in \mathrm{End}\, i_B$

$$\mathrm{Tr}(g \cdot T; i_B) = \sum_{M \in \mathrm{Irr}\,\mathbb{H}} \mathrm{Tr}(g; \pi(M)) \cdot \mathrm{tr}(T; M)$$

$$= \sum_E \mathrm{tr}(g; \pi(E)) \cdot \mathrm{tr}(T, \tilde{E})$$ $\qquad \underset{\mathbb{H}}{\tilde{E} = M} \longleftrightarrow \underset{W}{E}$

So for fixed T get linear combo of the irred characters, with some coefficients — vary T gets lots of such.

Obvious choice for T: take the basis elts $T_w \in \mathbb{H}$

$$\Rightarrow \mathrm{tr}(g \cdot T_w, i_B) = \sum_E \mathrm{tr}(g, \pi(E)) \cdot \mathrm{tr}(T_w, \tilde{E})$$

Let's compute $\mathrm{tr}(g \cdot T_w, i_B)$:

$\varphi \in i_B$: $\varphi : G^F/B^F \longrightarrow \overline{\mathbb{Q}}_\ell$

$(g \cdot \varphi)(x) = \varphi(g^{-1}x)$ $\qquad$ left action of G^F.

$(T_w \varphi)(x) = \sum_{h \in G^F/B^F} \varphi(xh)\, T_w(h)$ $\qquad$ convolution

$$((g \cdot T_w) \cdot \varphi)(x) = \sum_{G^F/B^F} \varphi(g^{-1}xh)\, T_w(h)$$

$h = x^{-1}gy$

$$= \sum_{y \in G^F/B^F} \varphi(y)\, T_w(x^{-1}gy)$$ — kernel function

— so trace is given by sum of diagonal elements:

$$\mathrm{tr}(gT_w; i_B) = \sum_{x \in X(\mathbb{F}_q)} T_w(x^{-1}gx)$$ — orbital integrals of T_w.

((T_w is B^F-biinvariant fn on G^F so the expression $T_w(x^{-1}gx)$ makes sense for a coset x.

$|Y_{w,g}(\mathbb{F}_q)|$, where $Y_{w,g} = \{x \in X : (x, gx) \in \mathcal{O}(w)\}$

- "Deligne Lusztig with g instead of F"
- locally closed subvariety of X,

~~[illegible]~~ Study as function of g

$\Rightarrow$ $Y_w \subset G \times X$: $\{(g,x) \mid (x, gx) \in \mathcal{O}(w)\}$

Thm $\downarrow$ not proper

G

• $G \times X = \coprod_w Y_w$

• Y_w is nonsingular of dimension $= \dim G + \ell(w)$

... same proof as for nonsingularity & dim of O-L varieties

$Y_1 = \tilde{G}$ from before, closed stratum

& singularity of $\overline{Y_w}$ loc isom to Schuberts...

To apply Lefschetz trace - must take $R\Gamma_c$

Functions	Sheaves
$G^F \ni g \mapsto tr(g T_w, i_B)$	$K_w = R\pi_{w!}(\overline{\mathbb{Q}}_\ell)$
	$\overline{K}_w = R\overline{\pi}_{w!}(IC(\overline{Y}_w))$

constant sheaf on Y_w.

shifted IC so that restriction to Y_w is $\overline{\mathbb{Q}}_\ell$ ... this is pulled back from $IC(\overline{X(w)})$

Look for related perverse sheaves

Def A unipotent character sheaf ($/\overline{\mathbb{F}}_q$ or $\mathbb{C}$) on G is an irred perverse sheaf occurring in one of the K_w s

$\Longleftrightarrow$ in one of the $\overline{K}_w$. "Occurs": appears in perverse cohomology sheaves - irred constituent of a perverse cohomology sheaves

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$\overline{\pi}_w$ projection $\overline{Y}_w \to G$.

$\overline{K}_w$: semisimple: are direct sums of shifts of irred perverse sheaves.

P parabolic $\supset B$

Let $\tilde{G}_P = \{(g,x) \in G \times G/P : gx = x\}$

partial flag version of $\tilde{G}$, so $\tilde{G}_B = \tilde{G} = Y_1$.

$$\begin{array}{c} \tilde{G}_P \\ \downarrow \pi_P \\ G \end{array}$$

Let $k_P = R\pi_{P*}\overline{\mathbb{Q}}_\ell$ (π_P proper)

<u>Analogies</u>

	R_w	K_w
Reps from IC of O-L varieties	$\overline{R}_w$	$\overline{K}_w$
	i_P	k_P

$$k_P = \bigoplus_{E \in W^\vee} E^{W_P} \otimes \mathcal{C}(E)$$

E^{W_P} multiplicity spaces

Case of Borel: $k_{\mathcal{B}} = K_1 = \bigoplus_{E \in W^\wedge} E \otimes \mathcal{C}(E)$

- gives dfn of perverse sheaves $\mathcal{C}(E)$ .. character sheaves indexed by reps of W.

<u>Claim</u> π_P is <u>small</u> ...

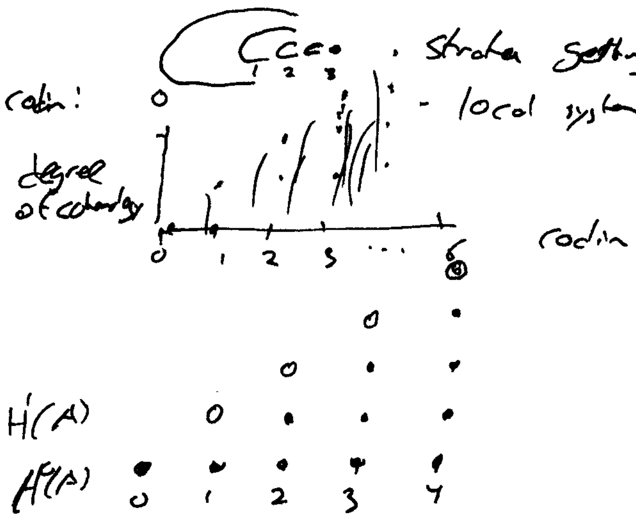
$\tilde{G}_P$ nonsingular, of same dimension as G:

$$\begin{array}{ccc} \tilde{G}_P & \xrightarrow{\alpha} & G/P \\ {\scriptstyle \pi_P}\downarrow & & \\ G & & \end{array}$$

fiber over a point in $G/P \ni x$, $\alpha^{-1}(x)$ is xPx^{-1}.

$R\pi_{P*}\overline{\mathbb{Q}}_\ell$ is perverse up to shift & is middle extension of its restriction to regular semisimple locus $G_{rs} \subset G$.

- to see this take
$$\begin{array}{c} \tilde{G}_P \times_G \tilde{G}_P \\ \downarrow \\ G \end{array}$$



To be perverse need to be below or on diag, to be an IC complex can't have component on diagonal! - except on open stratum.

In our situation, stalks reflect cohomology of fiber:

stalk at g: $\left(H^i R\pi_{P*}\overline{\mathbb{Q}}_\ell\right)_g = H^i(\pi_P^{-1}(g), \overline{\mathbb{Q}}_\ell)$

- so need these to vanish for i not in allowed region!
- one way to arrange this is to use
$$H^i(\pi_P^{-1}(g), \overline{\mathbb{Q}}_\ell) = 0 \quad \text{for } i > 2\dim \pi_P^{-1}(g)$$
- so want stratification where dimension of fibers behaves well.

Lusztig's trick! look at fiber product $\begin{array}{c} \tilde{G}_P \times_G \tilde{G}_P \\ \downarrow \\ G \end{array}$

- exactly doubles fiber dimension:
$\dim(\text{fiber of } \tilde{G}_P \times_G \tilde{G}_P) = 2 \dim \pi_P^{-1}(g)$

For perversity: Need (dim of fiber) + (codim stratum) $\leq \dim G$
$\Leftrightarrow$ dim of fiber $\leq$ codim
Can guarantee this if dimension of whole fiber product is $\leq \dim G$!
To prove this use Bruhat decomposition:

$$\tilde{G}_P \times_G \tilde{G}_P \xrightarrow{\beta} \underset{x_1}{G/P} \times \underset{x_2}{G/P} = \coprod_{w \in W_P \backslash W / W_P} G\text{-orbit of } (1, w).$$
$\{(g, x_1, x_2) / g x_1 = x_1, g x_2 = x_2\}$

- fibers of β over each of these G-orbits has exactly dim = dim G
dim of fibers = dim of intersection = $\ell(w)$ = codim of G-orbit.

For middle extension of l.c. on open stratum: want no cohomology on diagonal for codim > 0...
suppose had H^1 on codim 1 $\Rightarrow$ contribution from a local system on a codim 1 stratum (if known to be perverse)

$\tilde{G}_P \times_G \tilde{G}_P$ not irreducible (|W| components for $P = B$ ..)
Need all top-dim irred components to map dominantly:

If we have $\dim \tilde{G}_P \times_G \tilde{G}_P = \dim G$ & every irred comp of $\tilde{G}_P \times \tilde{G}_P$ dominates G $\Rightarrow$ our sheaf is a middle extension
(small rather than semismall)
- using that $\tilde{G}_P$ is nonsingular:
need support condition both for sheaf & its Verdier dual...
But $\mathbb{D}(\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$ (shifted) on smooth $\tilde{G}_P$.

This property is proven by noting any relative position $W_P \backslash W / W_P$ can be achieved over reg s-s elements so all our irred components map dominantly.

What does $R\pi_{P*}\overline{\mathbb{Q}}_\ell$ look like over G_{rs}?

- Let $P = B$. $\tilde{G} \xrightarrow{\pi_B} G$, $g \in G$, g reg s.s.

$\pi_B^{-1}(g) = \{x \in G/B : gx = x\}$: translate of case

$g = t \in T \subset B$: $\pi_B^{-1}(t) = $ "$W \subset G/B$" standard f.p.

$$\begin{array}{c} (\tilde{G})_{rs} \\ \downarrow W \\ G_{rs} \end{array}$$ Galois cover, group W. : pullback of $$\begin{array}{c} G/T \\ \downarrow \\ G/N(T) \end{array}$$

So $\pi_1(G_{rs}) \twoheadrightarrow W$ via this covering space.

& so our local system comes from the <u>regular rep</u> of W

$R\pi_{B*}\overline{\mathbb{Q}}_\ell|_{G_{rs}}$ = loc system coming from $\overline{\mathbb{Q}}_\ell[W]$.

$$\overline{\mathbb{Q}}_\ell[W] = \bigoplus_{E \in W^\vee} E \otimes E^*$$

Right action of W commutes with π_1 action via left action

(here all self-contragredient so $E^* = E$.)

$\Rightarrow$ $R\pi_{B*}\overline{\mathbb{Q}}_\ell = \bigoplus_{E \in W^\vee} E \otimes \mathcal{C}(E)$, $\mathcal{C}(E) = (j_{rs})_{!*}$ E-component of r.ss loc system

$\mathrm{End}(R\pi_{B*}\overline{\mathbb{Q}}_\ell|_{G_{rs}}) = \overline{\mathbb{Q}}_\ell[W]$ acting on right

$\quad\quad\quad \shortparallel$

$\mathrm{End}(R\pi_{B*}\overline{\mathbb{Q}}_\ell)$ Springer representation

since $R\pi_{B*}\overline{\mathbb{Q}}_\ell = (j_{rs})_{!*}(R\pi_{B*}\overline{\mathbb{Q}}_\ell|_{G_{rs}})$

$\Rightarrow$ W acts on $R\pi_{B*}\overline{\mathbb{Q}}_\ell$, hence on all stalks of cohomology groups.

$\Rightarrow$ $W \circlearrowright H^i(X_g, \overline{\mathbb{Q}}_\ell)$ $g \in G$.

(Proper base change $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$ $(R^i f_! A)_y = H^i_c(X_y, A|_{X_y})$)

Same story for P: $W/W_P \hookrightarrow G/P$ as fixed points

Monodromy is now $W \circlearrowright \overline{\mathbb{Q}}_\ell[W/W_P]$.

$$\begin{array}{c} (\tilde{G}_P)_{rs} \\ \downarrow \\ G_{rs} \end{array}$$ is covering map but not Galois: $$\begin{array}{c} \tilde{G}_B \\ \tilde{G}_P = \cdot/W_P \\ \downarrow \\ G_{rs} \end{array}$$

$$\overline{\mathbb{Q}}_\ell[W/W_P] = \overline{\mathbb{Q}}_\ell[W]^{W_P}$$

$$\Rightarrow k_P = \bigoplus_{E \in W^\vee} E^{W_P} \otimes \mathcal{C}(E).$$

Example: GL_2

w	$\overline{K_w}$	K_w
1	$e(1)+e(\varepsilon)$	$e(1)+e(\varepsilon)$
w_0	$(1+q)e(1)$	$qe(1)-e(\varepsilon)$

in K-group of perverse sheaves, powers of q for shifting of cohomology (no grading)

$W^\vee = \{1, \varepsilon\}$ trivial & sign

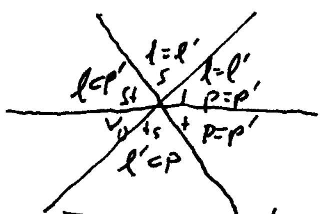
$$\overline{Y_{w_0}} = \begin{array}{c} G\times\mathbb{P}^1 \\ \downarrow \\ G \end{array} \Rightarrow R\overline{\pi}_{w_0 *}\overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell}\oplus\overline{\mathbb{Q}_\ell}[-2](-1)$$ .. Tate twist!

$H^2(\mathbb{P}^1)$ degree 2, Tate twist 1 $\rightsquigarrow$ Frob acts by q

... illustration of decomposition theorem - perverse sheaves with different shifts...

$K_1 + K_{w_0} = \overline{K_{w_0}}$ in this case: build up via distinguished triangles.

GL_3



$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $Q = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

G/P = lines G/Q = planes

$W_P = \{1, s\}$ $W_Q = \{1, t\}$

$W^\vee = \{1, \varepsilon, r\}$

w	$\overline{K_w}$	K_w
1	$j_B = e(1)+e(\varepsilon)+2e(r)$	$e(1)+e(\varepsilon)+2e(r)$
s	$(1+q)j_P = (1+q)(e(1)+e(r))$	$qe(1)-e(\varepsilon)+(q-1)e(r)$
t	$(1+q)j_Q = (1+q)(e(1)+e(r))$	$qe(1)-e(\varepsilon)+(q-1)e(r)$
st	$(1+q+q^2)e(1)+qj_Q$	$q^2e(1)+e(\varepsilon)-qe(r)$
ts	$(1+q+q^2)e(1)+qj_P$	$q^2e(1)+e(\varepsilon)-qe(r)$
w_0	$(1+q)(1+q+q^2)\cdot e(1)$	$q^3e(1)-e(\varepsilon)$

Springer sheaf j_B (perverse up to shift)

Springer for G/P: j_P

($j_B = k_B$, $j_P = k_P$ notations from last time...)

Springer j_P: combo of $e(-)$ with multiplicity given by dim of W_P-fixed vectors

Perverse shifts: $q\cdot A := A[-2](-1)$ where things looked as eigvls of Frob before

What is $\overline{Y_s}$? $= \{(g, l, p) : gl = l\}$

$G\times$flags in rel posn w -->

$$\begin{array}{c} \overline{Y_s} \\ \downarrow \mathbb{P}^1\text{-bundle} \\ \widetilde{G_P} = \{(g,l) : gl = l\} \quad (\text{Springer}) \\ \downarrow \pi_P \\ G \end{array} \qquad \overline{\pi}_s : \overline{Y_s} \to G$$

$(1+q)j_P$: $1+q$ is cohomology of $\mathbb{P}^1$.

$$R\alpha_*(\overline{\mathbb{Q}_\ell}) = (1+q)\overline{\mathbb{Q}_\ell}$$

$\overline{K}_{ts}$: $(\overline{Y_{ts}})_g = \{ l \subset p : gl = p \}$ $\quad q^{-1}(l) = \begin{cases} pt. & \text{if } gl \neq g \\ \mathbb{P}^1 & \text{if } gl = l \end{cases}$

$\downarrow q$

$\{\text{lines } l\} = G/P = \mathbb{P}^2$

cohomology $1+q+q^2$

$e(1)$ = just constant sheaf

So $\overline{Y_{ts}}$ is $G \times \mathbb{P}^2$ blown up along ~~something to glue~~ $\tilde{G}_p$

~~$\tilde{G}_p l = \{(g,l) : gl = l\}$~~

$\overline{Y_{ts}} = \{ g, l, p : l \subset p : gl = p \}$

$\downarrow$

$\tilde{G}_p \subset G \times \mathbb{P}^2 = \{g, l\}$

dist $\mathcal{O}$ $\quad \overline{\mathbb{Q}_l} \to Rq_* \overline{\mathbb{Q}_l} \to (\overline{\mathbb{Q}_l})_{\tilde{G}_p}|_g [-2](-1)$

G/B : $1 \;\; 2 \;\; 2 \;\; 1 \;\; \Rightarrow (1+q)(1+q+q^2)$.

Coefficients for K_w give exactly character table for H of GL_3.

Exactly same for GL_n, similar behavior for general groups.

GL_n :

function on G^F	virtual combos of character sheaves
$i_P = \sum_{W^\nu} \dim E^{W_P} \cdot \pi(E)$	$j_P = \sum_{W^\nu} \dim E^{W_P} \; e(E)$
(↑ true for general G ↑)	
Special to GL_n : $\pi(E)$ ⟷	$e(E)$

should be labeled by reps of H : close $q^{\frac{1}{2}}$ in general

really labeled by reps of W (Springer)

GL_n have enough parabolics & i_P's, j_P's to separate out $\pi(E)$, $e(E)$'s

- labeled by partitions of n in both (unordered! conjugate parabolics give same i_P, j_P)
- invertible square matrix to pass from $i_P \longleftrightarrow \pi(E)$, $j_P \longleftrightarrow e(E)$.

Sp_4 : $\underline{5}$ principal series, got 4 i_P's...

$i_P \longleftrightarrow j_P$ via function-sheaf correspondence always

$\pi(E) \longleftrightarrow e(E)$ " " for GL_n thanks to above square matrix!

Deligne-Lusztig for G_2

Notation: Υ, Σ etc. reps of H corresponding to $1, \varepsilon$, etc. reps of W

Character table: $|W| = 12 = 1+1+1+1+2^2+2^2$
Traces of T_w on M

$w \backslash M$	$\tilde{1}$	$\tilde{\varepsilon}$	$\tilde{\varepsilon}_1$	$\tilde{\varepsilon}_2$	$\hat{V}$	$\tilde{V}'$
1	1	1	1	1	2	2
s	q	-1	q	-1	$q-1$	$q-1$
t	q	-1	-1	q	$q-1$	$q-1$
st	q^2	1	$-q$	$-q$	q	$-q$
ts	q^2	1	$-q$	$-q$	q	$-q$
sts	q^3	-1	$-q^2$	q	$2(q^2-q)$	0
tst	q^3	-1	q	$-q^2$	$2(q^2-q)$	0
$stst$	q^4	1	q^2	q^2	$-q^2$	$-q^2$
$tsts$	q^4	1	q^2	q^2	$-q^2$	$-q^2$
$ststs$	q^5	-1	q^3	$-q^2$	q^3-q^2	q^2-q^3
$tstst$	q^5	-1	$-q^2$	q^3	q^3-q^2	q^2-q^3
w_0	q^6	1	$-q^3$	$-q^3$	$-2q^3$	$2q^3$

Curtis & Iwahori–Kilmoyer, PubIHES

Can read off table certain combos which are true reps from geometry : e.g. q^2 suggests arising from dim 4 IC groups
--- build table for $K-L$ basis $tr(A_w, M)$
- for G_2 everything is rational homology manifold,
$A_w = \sum_{w' \le w} T_{w'}$.

-- to each $E \in H^\vee$ we assign $R_E = \frac{1}{|W|} \sum_{w \in W} tr(w,E) R_w$
& certain combos of R_E's are actual reps:
take table for A_w, separate out terms corresp to given power of q & use to form combos of R_E's.

	$\tilde{1}$	$\tilde{\varepsilon}$	$\tilde{\varepsilon}_1$	$\tilde{\varepsilon}_2$	$\tilde{V}$	$\tilde{V}'$
e.g. $tr(A_s, M)$:	$1+q$	0	$1+q$	0	$1+q$	$1+q$

$\Rightarrow R_{E=1} + R_{\varepsilon_1} + R_V + R_{V'}$ is a true rep. of G^F
$\Rightarrow R_{\varepsilon_1} + R_V + R_{V'}$ also true reps ($R_{E=1}$ is trivial rep -- so does occur!

End up with a number of linear combos of $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{V}, \tilde{V}'$ - come in one family (**) $(R_{E_1}, R_{E_2}) = (E_1, E_2)$ use orthogonality)
$\Rightarrow$ true reps :

(*) $R_{\varepsilon_2} + R_V + R_{V'} = \pi_0 + \pi_1 + \pi_2$ (use * & ** to sort out irred pieces)
$R_{\varepsilon_1} + R_V + R_{V'} = \pi_0 + \pi_1 + \pi_3$
$R_{\varepsilon_2} + R_V - R_{V'} = \pi_2 + \pi_4 + \pi_5$
$R_{\varepsilon_1} + R_V - R_{V'} = \pi_3 + \pi_4 + \pi_5$
$-R_{\varepsilon_2} + 2R_V = \alpha + \beta + \pi_0 + \pi_4 + \pi_3$
$-R_{\varepsilon_1} + 2R_V = \alpha + \beta + \pi_0 + \pi_4 + \pi_2$

- so all these + R_1, R_ε are all irred unipotents for G_2!
$\Rightarrow$ 8 + 2 unipotents
4 of the principal series
4 cuspidal unipotents!

$\pi_6, \pi_7 \longleftrightarrow \alpha, \gamma$

These 8 reps come in a natural fashion : first let's inch velden

$R_V = \frac{1}{8}[3\pi_0 + \pi_1 + 2\pi_2 + 2\pi_3 + 3\pi_4 + \pi_5 + 2\pi_6 + 2\pi_7]$

$R_{V'} = \frac{1}{2}[\pi_0 + \pi_1 - \pi_4 - \pi_5]$

$R_{\varepsilon_2} = \frac{1}{3}[\pi_1 + 2\pi_2 - \pi_3 + \pi_5 - \pi_6 - \pi_7]$

$R_{\varepsilon_1} = \frac{1}{3}[\pi_1 - \pi_2 - 2\pi_3 + \pi_5 - \pi_6 - \pi_7]$

Explanation: Nonabelian Fourier transform on group S_3 ...
component group of centralizer of a certain unipotent in G_2...

$\mathcal{G}$ finite group (e.g. S_3),
Consider pairs (x, ρ) $x \in \mathcal{G}$, $\rho \in (\mathcal{G}_x)^\vee$ irrep of centralizer
($\mathcal{G}$ abelian: just $\mathcal{G} \times \mathcal{G}^\vee$.)

Equivalence: $(x,\rho) \sim (gxg^{-1}, {}^g\rho)$ conjugacy

$\Rightarrow$ $\mathcal{M}(\mathcal{G}) = \{(x,\rho)\}/\sim$

$\mathcal{M}(S_3)$ has 8 elements

Symplectic pairing: abelian case $\tau(x)\rho(y)^{-1} \leftrightarrow \{(x,\rho),(y,\tau)\}$

- define $\{(x,\rho),(y,\tau)\} = |\mathcal{G}_x|^{-1}|\mathcal{G}_y|^{-1} \sum_{\substack{g \in \mathcal{G} \text{ s.t.} \\ x,\, gyg^{-1} \text{ commute}}} \operatorname{tr} \tau(g^{-1}x^{-1}g) \operatorname{tr} \rho(gyg^{-1})$

$\Rightarrow$ square matrix of size $|\mathcal{M}(\mathcal{G})|$

4 out of the 8 columns were our combinations above
Other 4 will correspond to other character sheaves
(above 4 came just from Springer theory) - almost characters
- cuspidal series of character sheaves.

Sp_4 Character Sheaves

$G = Sp(V, \langle,\rangle)$ $\quad G \supset P \supset B$, $G \supset Q \supset B$ $\quad W \supset W_P = \{1, s\} \supset 1$, $W \supset W_Q = \{1, t\} \supset 1$

G/P = isotropic planes in V

$\mathbb{P}^3 = G/Q$ = lines in V

Assume characteristic $\neq 2$

We got our cuspidal rep $\boxed{?}$ from G/Q.. Similarly here
we'll get everything interesting out of G/Q.

G-orbits on $G/Q \times G/Q$: possibilities for (ℓ, ℓ') are
- $\ell = \ell'$
- $\ell \perp \ell'$
- general pos

} orbit closures

$G \times G/Q \supset \{(g,x) : (x, gx) \in \text{one of the orbit closures}\}$

$\ell = \ell' \Rightarrow$ get fixed pts, just Springer theory

Open orbit closure $\Rightarrow$ just get $G \times G/Q$

Interesting case: closure of $\{\ell \perp \ell'\}$ orbit:

$G \times G/Q \supset \{(g,\ell) \mid g\ell \perp \ell\} =: Z$ ($\bar{\mathbb{Q}}_\ell$ is IC of this)

$\downarrow \alpha$

G

Look at $L := R\alpha_* \bar{\mathbb{Q}}_\ell$. = direct sum of shifts of character sheaves

Stalk $L_g = H^*(\{\ell \mid g\ell \perp \ell\}) = H^*(Z_g)$ fiber cohomology

$Z_g \subset \mathbb{P}^3$ closed, hypersurface or $\mathbb{P}^3$, quadric: $g\ell \perp \ell$ is quadric condition: but written in terms of skew form

Choose $v \in \ell \subset V$ nonzero: $0 = \langle gv, v\rangle = \langle v, g^{-1}v\rangle = -\langle g^{-1}v, v\rangle$ alternating

[char≠2] $= \frac{1}{2}\langle (g-g^{-1})v, v\rangle$ – symmetric bilinear form on V (quadr. form)

Classification over quadratic forms over alg closure: just dim of kernel of the symmetric bilinear form (null vectors)

$\Rightarrow$ need $\dim \ker(g-g^{-1}) = \dim\ker(g^2-1)$

– this will single out certain conjugacy classes as being special....

5 cases:

$\dim\ker(g^2-1)$	Z_g
4	$\mathbb{P}^3$
3	$\mathbb{P}^2$ (mult 2)
2	2 $\mathbb{P}^2$'s meeting in $\mathbb{P}^1$
1	cone in $\mathbb{P}^3$ with vertex, over conic in $\mathbb{P}^2$ ($\cong \mathbb{P}^1$)
0	smooth: $\mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ Segre $(u,v)\times(w,x) \mapsto (uw, ux, vw, vx)$

(a, b, c, d): $ad - bc = 0$

Cone $\setminus$ vertex is a line bundle over $\mathbb{P}^1$

$j \backslash i$	0	1	2	3	4
6					q^3
5					
4	q^2	q^2	$2q^2$	q^2	q^2
3					
2	$2q$	q	q	q	q
1					
0	1	1	1	1	1

$\Leftarrow H^j(Z_g)$ for $\dim\ker(g^2-1) = i$

Our case 1 is "fake $\mathbb{P}^2$" – same cohomology as $\mathbb{P}^2$... line bundle over $\mathbb{P}^1 \Rightarrow (q+1)q$ +1 for vertex

This has to be $\oplus$ IC sheaves of pieces...
On open subset of G get fiber $\mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow$ get local systems over rss elements : local systems of rank 1, 2, 1 in degrees 4, 2, 0.
$\to$ its middle extension will be summand of our L.
Degrees 0, 4 get constant local systems — come from cohomology of ambient $\mathbb{P}^3$ — constant l.c.'s on $G \times \mathbb{P}^3$.
... same for "half" of the $2q$ in deg 2...

Take (pt in $\mathbb{P}^1$) $\times \mathbb{P}^1 \Rightarrow$ get line in the $\mathbb{P}^3$, lines comes from homology of $\mathbb{P}^3$. So class in $\mathbb{P}^3$ goes to diagonal $pt \times \mathbb{P}^1 + \mathbb{P}^1 \times pt$ ... $\Rightarrow$ 1-dim constant sub local system.
Now subtract these constant sheaves [heavily using decomposition!]

$L = \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell[-2](-1) \oplus \overline{\mathbb{Q}}_\ell[-4](-2) \oplus \boxed{?}$ ← in derived category

(in K-group) $= C(1) \cdot (1+q+q^2) + \boxed{?}$ ← in K-groups.

Stalks of $\boxed{?}$:

	0	2	4
6			q^3
4		q^2	
2	q		

Have to be careful: "2" stratum is not irreducible, need finer decomposition. Has a component of codimension 2 in G, ξ & 3 smaller pieces..

call it ♣ ← Our q on open stratum turns out to be non-constant — comes from a 1-dim character of $W \Rightarrow$ l.c. on rss locus (not trivial or sign!)
Does its middle extension explain the degree 4 component?
Can't: go up by 2 in degree & ~~down~~ up by two in codimension — right on edge of diagram!
So this <u>is</u> perverse but <u>not</u> IC!

So $\boxed{?}$ = middle ext of ♣ $\oplus$ middle extension of a loc sys in deg 4 on codim 2 irred component ξ, $\oplus$ maybe something else?

Recall Fourier transform $\mathcal{M}(\mathcal{G})$: the linear combos that we get here for Sp_4 is $\mathcal{G} = \mathbb{Z}/2$, $\mathcal{M}(\mathcal{G}) = (\mathbb{Z}/2)^2$
- responsible for signs

$\mathcal{G}$ = comp. group of Z(subregular unipotents) in adjoint gp for Sp_4

- in general need to take a quotient of this finite group → "Lusztig's quotient group"

Cusp: actually comes from cuspidal character sheaf on SL_2!
- not unipotent, but have cuspidal char sheaves still.

Parametrization of characters on T^F

$\overline{\mathbb{F}}_q$ | $\mathbb{F}_{q^d}$ | $\mathbb{F}_q$

First case: $T = \mathbb{G}_m$. $\mathbb{G}_m(\mathbb{F}_{q^d}) = (\mathbb{F}_{q^d})^\times$

$$\mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times) \hookrightarrow \mathrm{Hom}(\varprojlim_{\text{cont } d} \mathbb{F}_{q^d}^\times, \overline{\mathbb{Q}}_\ell^\times) \qquad \mathbb{F}_{q^d}^\times \xrightarrow{F-1} \mathbb{F}_{q^d}^\times \xrightarrow{N} \mathbb{F}_q^\times \to 1 \quad (*)$$

$\alpha \mapsto \alpha^{-1}F(\alpha)$ Lang; $\theta: \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$

Norm map: $\mathbb{F}_{q^d}^\times = \mu_{q^d-1} \to \mu_{q-1} = \mathbb{F}_q^\times$

$$N(x) = x F(x) F^2(x) \cdots F^{d-1}(x) = x^{1+q+q^2+\dots+q^{d-1}} = x^{(q^d-1)/(q-1)}$$

$$\Rightarrow \varprojlim_d \mathbb{F}_{q^d}^\times = \varprojlim_N \mu_N(\overline{\mathbb{F}}_q) \underset{\substack{\text{as abelian}\\ \text{group}}}{\overset{\text{not canonical}}{=}} \varprojlim_{\substack{N\\(N,p)=1}} \mathbb{Z}/N\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell = \hat{\mathbb{Z}}^p$$

$$(*) \Rightarrow \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times) = \{ s \in \mathrm{Hom}(\varprojlim_d \mathbb{F}_{q^d}^\times, \overline{\mathbb{Q}}_\ell^\times) \mid F(s) = s \}$$

$\circlearrowleft_F$

So up to take list $\mathrm{Hom}(\varprojlim \mathbb{F}_{q^d}^\times, \overline{\mathbb{Q}}_\ell^\times) \simeq \overline{\mathbb{Q}}_\ell^\times$

$$\Rightarrow \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times) = \{ s \in \overline{\mathbb{Q}}_\ell^\times \mid s^q = s \} = \mu_{q-1}(\overline{\mathbb{Q}}_\ell^\times)$$

Now consider any torus T

$$\mathrm{Hom}(T(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times) \subset \mathrm{Hom}(\varprojlim_d T(\mathbb{F}_{q^d}), \overline{\mathbb{Q}}_\ell^\times)$$

$$\overset{\shortparallel}{\{ s \in \mathrm{Hom}(\varprojlim_d T(\mathbb{F}_{q^d}), \overline{\mathbb{Q}}_\ell^\times) : F(s) = s \}} \qquad F = F_T \text{ Frobenius for torus}$$

$$\text{(Taf)} \left(\left(\varprojlim_d T(\mathbb{F}_{q^d}) \underset{\text{(Taf)}}{\cong} \underset{F}{\overset{\circlearrowleft}{X_* T}} \otimes_{\mathbb{Z}} \underset{F=q}{\hat{\mathbb{Z}}^p} \right.\right.$$

$$\{ s \in T^\vee(\overline{\mathbb{Q}}_\ell) : F^\vee(s) = s \}$$

$T/\mathbb{F}_q \rightsquigarrow$ Langlands dual $\check{T}/\overline{\mathbb{Q}}_\ell$ so that $X_*(T) = X^*(T^\vee)$

$F: T \to T$, $F^\vee : T^\vee \to T^\vee$

$G/\mathbb{F}_q$ split $\Rightarrow$ for $w \in W$ $F_w : T \to T$, $F_w = w \circ F$.

$\mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^*) = \{ s \in T^\vee(\overline{\mathbb{Q}}_\ell) : (w^{-1} \circ F^\vee)(s) = s \}$ $F_w^\vee = w^{-1} \circ F^\vee$ contravariant

$\Longleftrightarrow$ $s^q = w(s)$

<u>Relation to rep theory over local fields</u> $F \supset \mathcal{O} \twoheadrightarrow \mathbb{F}_q$

$G(F) \supset G(\mathcal{O})$ compact open

$$\begin{array}{c} G(\mathcal{O}) \\ \downarrow \\ G(\mathbb{F}_q) \end{array}$$

So can induce up reps of $G(\mathbb{F}_q)$ to $G(F)$.

For cuspidal reps of $G(F)$ the compact induced rep will be an <u>irreducible</u> of $G(F)$!

Basic conjecture: irreps of $G(F)$ (on $\overline{\mathbb{Q}}_\ell$ vector spaces) parameterized by homomorphisms

$W_F' \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$ + additional data

Weil-Deligne group $= W_F \ltimes (\ldots)$

$$\begin{tikzcd} W_F \arrow[r, hook] \arrow[d] & \mathrm{Gal}\,\overline{F}/F \ni \sigma \arrow[d] \\ \{\mathrm{Frob}^{\mathbb{Z}}\} \arrow[r, hook] & \mathrm{Gal}\,\overline{\mathbb{F}}_q/\mathbb{F}_q \ni \bar{\sigma} \end{tikzcd}$$

I

$$\begin{tikzcd} 1 \arrow[r] & I \arrow[r] \arrow[dr] & \mathrm{Gal}\,\overline{F}/F \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ & & W_F \arrow[r] & \mathbb{Z} \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] \overline{F} \arrow[d, dash, "\text{inertia } I"'] \\ F^{un} \arrow[d, dash, "\mathrm{Frob}", "\hat{\mathbb{Z}}"'] \\ F \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] \overline{F} \arrow[d, dash, "P"'] \\ F^{tame} \arrow[d, dash, "I/P"'] \\ F^{un} \arrow[d, dash] \\ F \end{tikzcd}$$

$1 \longrightarrow \underset{\text{wild inertia}}{P} \longrightarrow I \longrightarrow \underset{\text{tame inertia}}{I/P} \longrightarrow 1$

F^{un} contains all p-prime power roots of unity: get from residue field extensions.

Tame inertia: can take $\sqrt[N]{\alpha}$ $\alpha \in F^{un} \setminus 0$, $(N, p) = 1$.

$\tau \in I/P$, $\tau(\sqrt[N]{\alpha})$ = some other $\sqrt[N]{\alpha}$ = N^{th} root of unity $\cdot \sqrt[N]{\alpha}$.

$\Rightarrow$ $\boxed{I/P = \varprojlim_d \mu_d(\overline{\mathbb{F}}_q)}$ (lift uniquely to roots of unity in F)

So $\mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^*) \overset{\text{canonical}}{=} \{ s \in \mathrm{Hom}_{cont}(I/P, T^\vee(\overline{\mathbb{Q}}_\ell)) : Fr^\vee(s) = s \}$

Fr acts on I/P & as <u>finite order</u> automorphism of $T^\vee(\overline{\mathbb{Q}}_\ell)$ (from $X_* T$)

w	$\overline{K_w}$
1	$k_B = C_1 + C_\varepsilon + C_{\chi_1} + C_{\chi_2} + C_r$
s	$(q+1)\,[C_1 + C_{\chi_2} + C_r] = (q+1)k_P$
t	$(q+1)\,[C_1 + C_{\chi_1} + C_r] = (q+1)k_Q$
st	$L' + q k_P$ = Same as ts ?! ⇒ can solve for L'
ts	$L + q k_Q = (1+2q+q^2)C_1 + qC_{\chi_1} + C_{\chi_2} + qC_r + q^2 \mathrm{cusp}$
sts	$(1+q)L' = (1+q)[(1+q+q^2)C_1 + qC_{\chi_1} + q^2\cdot \mathrm{cusp}]$
tst	$(1+q)L = (1+q)[(1+q+q^2)C_1 + qC_{\chi_2} + q^2\cdot \mathrm{cusp}]$
w_0	$(1+q)(1+q+q^2+q^3)C_1$

⇒ can now work out the K_w's.

Recall from D-L Theory : $R_E := \frac{1}{|W|}\sum_{w\in W} tr(w,E)R_w$.

Analog: $K_E := \frac{1}{|W|}\sum_{w\in W} tr(w,E)K_w$, formal $\mathbb{Q}$-linear combo of shifts of character sheaves.

Miracle before: $R_w = \sum_E \underbrace{(\dots)}_{\text{character table of } H}\cdot R_E$ ----> as rep of G^F
as rep of $G^F \times \langle F^d\rangle$ for some d; $tr(T_w, \tilde{E})$

[K_w : have to use purity for character sheaves]

Equivalently: $\overline{R_w} = \sum_E tr(A_w,\tilde{E})\, R_E$ $\tilde{E}$ = H irrep corresponding to E

A_w : K-L basis - express $IC(\overline{R_w})$ in terms of R_w's) co'homology

Similar statement for character sheaves $\overline{K_w}$ .... seems to be better to use $q\cdot\mathrm{cusp}$ as basis elt instead of cusp
⇒ get $\overline{K_w} = \sum tr(A_w,\tilde{E})\, K_E$

$K_{E=1} = C_1$ (recall $R_{E=1} = \Pi(1)$ $R_\varepsilon = \Pi(\varepsilon)$ ...)
$K_\varepsilon = C_\varepsilon$ $K_{\chi_1} = \frac{1}{2}[C_{\chi_1} - C_{\chi_2} + C_r - q\,\mathrm{cusp}]$
$K_{\chi_2} = \frac{1}{2}[-C_{\chi_1} + C_{\chi_2} + C_r - q\,\mathrm{cusp}]$
$K_r = \frac{1}{2}[C_{\chi_1} + C_{\chi_2} + C_r + q\,\mathrm{cusp}]$
$\boxed{} = \frac{1}{2}[-C_{\chi_1} - C_{\chi_2} + C_r + q\,\mathrm{cusp}]$: suggested by pattern above

Function associated to sheaf K_E gives character of principal series rep — not irreducible any more as perverse sheaves, but give irreducible character

$\boxed{}$ gives character of cuspidal unipotent of Sp_4.

$\mathcal{C}(\xi) = k_G + k_B - k_P - k_Q$, comes from Steinberg rep
– so we're writing character of Steinberg rep – but only for conjugacy classes that split over our field already (those listed are all classes over the alg. closure!)

~~Regular but nonsemisimple~~ Nonsemisimple elts have ~~vanishing~~ Steinberg character!

ξ is actually a single conjugacy class

$$\begin{array}{cc|c} 1 & & x \\ -1 & x & \\ & -1 & \\ \hline & & 1 \end{array}$$

& the middle extension of this piece is actually just extension by zero!

$\boxed{?} = q^2 \mathrm{Cusp} \oplus D$. (Cusp is IC of loc system on ξ)

D is actually an IC complex! once we account for piece from ξ everything left satisfies inequality to be an IC complex...

D = middle extension of ✿.
$= \mathcal{C}(?)$

$k_P = \mathcal{C}(1) + \mathcal{C}(\chi_2) + \mathcal{C}(r)$ $\Rightarrow k_P - k_Q = \mathcal{C}(\chi_2) - \mathcal{C}(\chi_1)$
$k_Q = \mathcal{C}(1) + \mathcal{C}(\chi_1) + \mathcal{C}(r)$

✿ comes from local system on reg ss set & in fact comes from a character of Weyl group (though $\pi_1(rss)$ is bigger)
$\Rightarrow$ so it's a $q\mathcal{C}(?)$. ... check its $q\,\mathcal{C}(\chi_2)$.

Recap: $G = Sp_4$ G/Q = lines G/P = planes

$\{(g,\ell) : g\ell \perp \ell\} = Z \subset G \times G/Q$
$\downarrow \xi$
G

$R\xi_*(\overline{\mathbb{Q}}_\ell = IC(Z)) = L$

We wrote L as direct sum of twisted perverse sheaves!

$L = (1+q+q^2)\,\mathcal{C}(1) + q^2 \cdot \mathrm{cusp} + q\,\mathcal{C}(\chi_2)$

Let $Z' = \{(g,p) : gp \cap p \neq \{0\}\}$
$\downarrow \xi'$
G

$L' = R\xi'_*(\overline{\mathbb{Q}}_\ell = IC(Z'))$

CS 13

um of 1)	codim stratum	typical elt. g in stratum $a,b \neq \pm 1$, $* \neq 0$	$k_Q = C(1)+C(x_1)+C(x)$	$k_P = C_1 + C_{x_2} + C_x$	k_B	$C(1) = k_G$	$C(\epsilon) = k_B - k_P - k_Q + k_G$	$L = (1+q+q^2)\,C(1) + \boxed{n}$	$\boxed{n} = q^2 \cdot \text{cusp} + D$	cusp	$\therefore D = q\, C(x_2)$	$k_P - k_Q = C(x_2) - C(x_1)$	$C(x_2)$	$C(x_1)$	$C(x)$
0	0	a b b^{-1} a^{-1} ($a \neq b, b^{-1}$)	4	4	8	1	1	q^2+2q+1	q		q		1		2
1	1	a $1\ *$ 1 a^{-1}	3	2	4	1	0	q^2+q+1	0			-1		1	1
1	1	a $-1\ *$ -1 a^{-1}	3	2	4	1	0	q^2+q+1	0			-1		1	1
0	1	$a\ *$ a $a^{-1}\ *$ a^{-1}	2	3	4	1	0	q^2+2q+1	q		q	1	1		1
2	2	$1\ \ \ *$ $-1\ *$ -1 1	2	1	2	1	0	$2q^2+q+1$	q^2	1		-1		1	
2	3	a 1 1 a^{-1}	$q+3$	$2q+2$	$4q+4$	1	q	$2q^2+q+1$	q^2		q^2	$q-1$	q	1	$q+1$
2	3	a -1 -1 a^{-1}	$q+3$	$2q+2$	$4q+4$	1	q	$2q^2+q+1$	q^2		q^2	$q-1$	q	1	$q+1$
0	3	a a a^{-1} a^{-1}	$2q+2$	$q+3$	$4q+4$	1	q	q^2+2q+1	q		q	$-q+1$	1	q	$q+1$
3	4	1 $-1\ *$ -1 1	$q+2$	$q+1$	$2q+2$	1	0	q^2+q+1	0			-1		1	q
3	4	-1 $1\ *$ 1 -1	$q+2$	$q+1$	$2q+2$	1	0	q^2+q+1	0			-1		1	q
4	6	1 -1 -1 1	$2q+2$	q^2+2q+1	$2q^2+4q+2$	1	q^2	q^3+q^2+q+1	q^3		q^3	q^2-1	q^2	1	$2q$
1	2	$\pm\ 4$	1	1	1	1	0	q^2+q+1	0			0			0
2	4	$\pm\ 2+2$	$q+1$	$2q+1$	$3q+1$	1	0	$2q^2+q+1$	q^2		q^2	q	q	0	q
3	6	$\pm\ 2+1+1$	q^2+q+1	$q+1$	q^2+2q+1	1	0	q^2+q+1	0		0	$-q^2$	0	q^2	q
4	10	$\pm\ 1+1+1+1$	$1+q+q^2+q^3$	$1+q+q^2+q^3$	$1+2q+2q^2+2q^3+q^4$	1	q^4	$1+q+q^2+q^3$	q^3		q^3	0	q^2	q^2	q^3+q
		last 4 are $\pm$ unip. cl. (sizes of													

Given $W_F \longrightarrow T^\vee(\overline{\mathbb{Q}}_\ell)$ tame $\rightsquigarrow$ character on $T(F)$

$\Rightarrow$ $I/P \longrightarrow T^\vee(\overline{\mathbb{Q}}_\ell)$

$\Rightarrow$ character θ of T^F

$$T(F) \supset T(\mathcal{O}) \longrightarrow T(\mathbb{F}_q)$$

$$\begin{array}{l} T(F) \longrightarrow \overline{\mathbb{Q}}_\ell^\times \\ \cup \\ T(\mathcal{O}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times \\ \downarrow \qquad \nearrow \theta \\ T^F(\mathbb{F}_q) \end{array}$$

Langlands for tori

For G: $\varphi : W_F \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$ tame

$\Rightarrow \varphi_0 : I/P \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$

$F_r \in W_F/I$ acts by conjugation on I/P, raising to q^{th} power

φ_0 is equivalent to $\varphi_0 \circ$(raising to q^{th} power)

... equivalence is conjugacy under $G^\vee(\overline{\mathbb{Q}}_\ell)$ $\Longleftrightarrow$ $\underline{s^q = w(s)}$.

(s = image of $1 \in \hat{\mathbb{Z}}^{p} \cong I/P$).

Given $\theta : T_w^{F_w} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ $\rightsquigarrow$ $s \in T^\vee(\overline{\mathbb{Q}}_\ell)$ $s^q = w(s)$

T split

F_w nonsplit frob

$\Rightarrow$ s^q conjugate to s.

$\Longrightarrow$ defines Langlands parameter $\varphi_0 : I/P \longrightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$

also — for Langlands parameter need to send Frobenius somewhere else. — many ways to extend this.

As we saw for Sp_4 (keeping track of shifts)

(*) $K_w = \sum_{E \in W^\vee} tr(T_w, \tilde{E}) K_E$ (Lusztig: Char. Sheaves III Cor. 14.11 p.299)

$K_E := \frac{1}{|W|} \sum_W tr(w, E) \cdot K_w$

The computation we did in beginning of char. sheaf discussion shows

(**) f^n assoc to K_w is $\sum_{E \in W^\vee} tr(T_w, \tilde{E})$ (char of $\pi(E)$) ---> principal series.

$\Rightarrow$ fn associated to K_E is $\pi(E)$.

Last time: $G = SL_2 \supset \mathcal{U} = \{\text{unipotents}\} \cdot \simeq G/\mu_2 \cdot \mathbb{G}_a$

- Stabilizer of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $\pm \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$.

char $\neq 2$: μ_2 has 2 irreps $\{\text{triv}, \chi\} \Rightarrow$ 2 G-equivariant local systems on $\mathcal{U}$, $\overline{\mathbb{Q}}_\ell$ & $\mathcal{L} = \mathcal{L}_\chi$.

- Extension by 0 of $\mathcal{L}_\chi$ to G is its middle extension

$k : \mathcal{U} \hookrightarrow G \Rightarrow k_! \mathcal{L} \xrightarrow{\sim} Rk_* \mathcal{L}$

Claim: this extension is a character sheaf on SL_2.

$W = \{1, w_0\}$ $\quad G \xleftarrow{\pi} Y_{w_0} \to T/T^2$ $\quad$ [general: $G \leftarrow Y_w \to T/(w-1)T$

- stack quotient of T by T, acting on itself by map $t \mapsto wt/t$.

$N >> 0$ prime to characteristic $\Rightarrow$ take local system which is T^N & $(w-1)T$ equivariant - pull up & push forward, get loc sys on $G \Longrightarrow$ char. sheaves.]

In our case $w_0 - 1 = 2 \Rightarrow T/T^2$, $\exists!$ 1-dim nontrivial T^2-equiv loc sys on T.

Over $\mathbb{C}$: $\pi_1(T) = \mathbb{Z}$, our loc sys is 1-dim rep $\mathbb{Z} \ni 1 \mapsto -1 \in \overline{\mathbb{Q}}_\ell^*$

$R\pi_! \mathcal{L}_Y \quad G \xleftarrow{\pi} Y_{w_0} \xrightarrow{\alpha} T/T^2$, $\quad \mathcal{L}_Y = \alpha^* \mathcal{L}_T$

g	$(R\pi_! \mathcal{L}_Y)_g = H_c^*(\pi^{-1}(g), \mathcal{L}_Y)$	$\pi^{-1}(g)$
± 1	0	empty
$\pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	our loc sys is trivial on $\mathbb{A}^1$! (though $\mathbb{A}^1$ not simply connected) $\Rightarrow q$ (dim $H_c^* = 1$)	$\mathbb{P}^1 \setminus$ f.p. of elt $\mathbb{P}^1 \setminus \infty = \mathbb{A}^1$
$\begin{bmatrix} a & \\ & b \end{bmatrix}$ $a \neq b$	$\Rightarrow$ our nontriv loc sys on torus has 0 cohomology $\Rightarrow 0$	$\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$

$\Rightarrow$ we get exactly our local system on unipotent set middle extended : $R\pi_! \mathcal{L}_Y = k_{!*}(\mathcal{L}_Z[-2](-1))$

- Relate this cuspidal character sheaf to $G = Sp_4$:

conjugacy class $s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$ $\quad g = \left[\begin{array}{c|cc|c} 1 & & & * \\ \hline & -1 & * & \\ & & -1 & \\ \hline & & & 1 \end{array}\right] = s \cdot u$ Jordan, $*$'s nonzero

Centralizer $G_s = \left[\begin{array}{c|cc|c} * & & & * \\ \hline & * & * & \\ & * & * & \\ \hline * & & & * \end{array}\right] = SL_2 \times SL_2$

$u \in G_s$ is reg unipotent $\times$ reg unipotent.

Write $\mathcal{O}(g) = \{x^{-1} g x \mid x \in G\}$ conjugacy class

What is $\overline{\mathcal{O}(g)} \subset G$? $\qquad G \times_{G_s} \overline{\mathcal{O}_{G_s}(u)} \xrightarrow{\sim} \overline{\mathcal{O}}(g)$

$x s u' x^{-1}$ $\longleftarrow$ (x, u') $\qquad$ orbit of u in stabilizer of s

$\sim (xy^{-1}, y u' y^{-1})$ equivalence relation.

- $\mathcal{O}(s)$ is closed

In general this describes closure of orbits of elements using Jordan form! $\quad G \times_{G_s} \overline{\mathcal{O}_{G_s}(u)} \xrightarrow{\sim} \overline{\mathcal{O}_G(su)}$

G-equiv $\left\langle \begin{array}{l} \text{loc sys on } \mathcal{O}(g) \\ \text{perv sh. on } \overline{\mathcal{O}(g)} \end{array} \right.$ $\iff$ G_s-equiv $\left\langle \begin{array}{l} \text{loc sys on } \mathcal{O}_{G_s}(u) \\ \text{perv sh on } \overline{\mathcal{O}_{G_s}(u)} \end{array} \right.$

So reduces our question to $SL_2 \times SL_2$:

$(\mathcal{L} \times \mathcal{L})$ on $\mathcal{U} \times \mathcal{U}$ for $SL_2 \times SL_2 \iff$ cuspidal ch. sh. for Sp_4.

$\boxed{\text{Springer theory}}$ $\qquad \tilde{G} = \{(g, x) \in G \times G/B : g x = x\}$

$$\begin{array}{c} \tilde{G} \\ \pi \downarrow \\ G \end{array}$$

$$K_1 = R\pi_* \overline{\mathbb{Q}_\ell} = R\pi_! \overline{\mathbb{Q}_\ell} = \bigoplus_{E \in W^\vee} E \otimes C(E)$$
middle ext of loc sys on $G_{rs} \subset G$ determined by monodromy rep E.

$$\begin{array}{ccccc} \tilde{\mathcal{U}} & \subset & \tilde{G} & \supset & \tilde{G}_{rs} \\ \pi_{\mathcal{U}} \downarrow & & \pi \downarrow & & \downarrow \pi_{rs} \ W \text{ Galois} \\ \mathcal{U} & \subset & G & \supset & G_{rs} \end{array}$$

$\tilde{G}$ nonsingular & π small

$R\pi_* \overline{\mathbb{Q}_\ell}|_{G_{rs}} = R\pi_{rs*} \overline{\mathbb{Q}_\ell}$ = loc system $\overline{\mathbb{Q}_\ell}[W]$ (regular rep)

$\tilde{\mathcal{U}}$ nonsingular ($\simeq T^* G/B$) $\qquad \dim \tilde{\mathcal{U}} = 2 \dim G/B$

- affine bundle assoc to cotangent bundle ~~(...)~~

$\pi_{\mathcal{U}}$ is semismall: look at $\tilde{\mathcal{U}} \times_{\mathcal{U}} \tilde{\mathcal{U}}$, stratify,

$|W|$ irred components all of same dimension

— but not small! irred components don't map dominantly!

- Only one component dominates $\mathcal{U}$! but all comps same dim enough for semismall.

$R\pi_{\mathcal{U}*}\overline{\mathbb{Q}}_\ell|_{\mathcal{U}_{reg}} = \overline{\mathbb{Q}}_\ell$! $\exists !$ Borel containing any reg unipotent.

$\begin{array}{c} Y \\ \downarrow F \\ X \end{array}$ Proper semismall map from Y something nonsingular (Y, Y irreducible)

$\Rightarrow$ in derived category $\mathrm{End}(Rf_*\overline{\mathbb{Q}}_\ell) = \mathbb{D}H_c^{\text{top}}(Y\times_X Y, \overline{\mathbb{Q}}_\ell)$

$d = \dim X = \dim Y$: top cohomology (thanks to semismallness)

$A, B \in D_c^b(Y)$

$$\begin{tikzcd} & Y\times_X Y \arrow[dl] \arrow[dr] \arrow[d, "h"] & \\ Y \arrow[dr, "f"'] & X & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

$A \boxtimes B$, A, A

- Künneth: $Rh_! A\boxtimes B = Rf_!A \otimes Rf_!B$
- $\mathbb{D}(A\boxtimes B) = R\mathcal{H}om(A, \mathbb{D}B)$

$\Rightarrow R\mathcal{H}om(Rf_!A, Rf_*\mathbb{D}B) = \mathbb{D}Rh_!A\boxtimes B$

f proper: $f_! = f_*$. $R\mathcal{H}om(Rf_*A, Rf_*A) = \mathbb{D}Rh_!A\boxtimes \mathbb{D}A$

$\mathrm{Hom}(Rf_*A, Rf_*A) = \mathbb{D}H_c^0(Y\times_X Y, A\boxtimes\mathbb{D}A)$

Nonsingular: $\mathbb{D}\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell[\text{shift}]$

Typically get $\mathbb{D}H_c^{2d}(Y\times_X Y, \overline{\mathbb{Q}}_\ell)$ middle cohomology, but here semismallness $\Rightarrow \dim X = \dim Y$, the middle is actually the top! (0 rel dim) $\Rightarrow$ just count components

---

$\Rightarrow \dim \mathrm{End}\, R(\pi_{\mathcal{U}})_*\overline{\mathbb{Q}}_\ell = |W|$ unipotent locus

In fact $\mathrm{End}\, R\pi_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell[W]$ using reg ss locus!

But $R\pi_*\overline{\mathbb{Q}}_\ell|_{\mathcal{U}} = R\pi_{\mathcal{U}*}\overline{\mathbb{Q}}_\ell$ (proper base change)

So $\overline{\mathbb{Q}}_\ell[W] = \mathrm{End}(K_1) \xrightarrow[i^*]{\text{restrict}} \mathrm{End}\, R\pi_{\mathcal{U}*}\overline{\mathbb{Q}}_\ell$ ($\dim = |W|$)

$i: \mathcal{U} \hookrightarrow G$

Claim i^* is an isomorphism. (enough to show injective!)

- in fact enough to look at stalks over identity $1\in\mathcal{U}$

... maps to $\mathrm{End}(\text{stalk of } 1) = \mathrm{End}(H^*G/B)$

- get faithful rep of W

(over $\mathbb{C}$ $G/B = K/T$ compact real gp)

So have perverse sheaf $R\pi_{\mathcal{U}*}\overline{\mathbb{Q}}_\ell[\dim\mathcal{U}] = \bigoplus$ Irred IC complexes by decomp theorem (over alg closure), with $\mathrm{Endo} = \overline{\mathbb{Q}}_\ell[W]$

But $K_1|_{\mathcal{U}} = \bigoplus_E \dim E \; \mathcal{C}(E)|_{\mathcal{U}}$, each component can't break up further since endomorphisms are the same! (End doesn't get bigger on $\mathcal{U}$) $\Rightarrow$ $\mathcal{C}(E)|_{\mathcal{U}}$ irred perverse sheaf (up to shift). & Pairwise nonisomorphic (again dim bounds)

$\mathcal{U}$ = finite union of unipotent classes, & $K_1|_{\mathcal{U}}$ G-equivariant
So $\mathcal{C}(E)|_{\mathcal{U}}$ = middle extension of local systems on unipotent orbits.

$$\Rightarrow \text{Springer map} \quad W^\vee \hookrightarrow \{(\mathcal{C}_u, \mathcal{L}) \mid \mathcal{C}_u \text{ unip class}, \mathcal{L} \text{ loc sys on } \mathcal{C}_u \text{ irred } G\text{-equivariant}\}$$
$$E \longmapsto \mathcal{C}(E)|_{\mathcal{U}} = j_{!*}\mathcal{L}$$

and $\mathcal{L} \longleftrightarrow$ irrep of stabilizer group $\pi_0(G_u)$

Miraculous theorem on characters of Hecke

(Lusztig's version of pf by Digne Michel)

$H_c^\cdot(X_w)$ as $G^F \rtimes \langle F \rangle$ linear combo of R_E with coeffs characters of Hecke...

Ann Math Stud Thm 3.8

First study $(g \in G^F)$ $\quad tr(g \cdot F^r, H_c^\cdot(X_w)) = \quad (r \geq 1)$
$$= \#\{h \in G^{F^r}/B^{F^r} : h^{-1} g^\# F(h) \in BwB\} \quad \text{(twisted orbital integral)}$$

– use Lefschetz formula: count fix points of $g^{-1}F^r$ on X_w (left G-module, cohomology contravariant) + Lang's thm for B, F^r + Lang's thm for G, F^r ... – need $r \gg 1$ to control local contributions!

Here $g^\# \in G^{F^r}$ is defined as follows: choose (by Lang) $x \in G$ s.t. $g = F^r(x) \cdot x^{-1}$ (not unique...)
$g^\# = x^{-1} F(x)$. Ambiguity in x is something fixed by F^r, $\Rightarrow$ $g^\#$ well defined up to F-conjugacy
$g^\# \sim y^{-1} g^\# F(y)$, $y \in G^{F^r}$

1. $g^\# \in G^{F^r}$: work in $G^{F^r} \rtimes \langle F \rangle$,
$g^\# F = x^{-1} \cdot F \cdot x$, $\quad F^r = x^{-1}(g^{-1}F^r)x$
ie pair $g^\# F, F^r$ are taken to $F, g^{-1}F^r$ by x conjugation. So $g^\# F, F^r$ commute iff $F, g^{-1}F^r$ commute $\Rightarrow$
$g^\#$ commutes with $F^r \Longleftrightarrow g$ commutes with F!
but we know $g \in G^F$. $\square$

2. The F-conjugacy class of $g^\#$ is well defined
$\Longrightarrow$

get bijection $G^F/\text{conj} \xrightarrow{\sim} G^{F^r}/F\text{-conj}$ (Shintani)
$$g \longmapsto g^\#$$

3. Moreover, cardinality $\dfrac{|\text{conj class of } g|}{|G^F|} = \dfrac{|F\text{-conj cl of } g^\#|}{|G^{F^r}|}$

(comes up when computing inner products of class functions).

Consider induced rep $I(q^r) = \mathbb{C}[G^{F^r}/B^{F^r}] \ni \varphi$

- rep of $G^{F^r} \rtimes \langle F \rangle$:

For $g^\# \in G^{F^r}$, acts by $(g^\# \varphi)(x) = \varphi(g^{\#^{-1}} x)$

$(F\varphi)(x) = \varphi(F^{-1}x)$

$H(q^r) = \mathbb{C}[B^{F^r} \backslash G^{F^r}/B^{F^r}] = \mathrm{End}_{G^{F^r}} I(q^r)$

$f : \varphi \mapsto f \cdot \varphi = \varphi * f_r$, f_r : f reversed $f_r(x) = f(x^{-1})$

$$(T_w \cdot \varphi)(x) = \sum_{\substack{y \in (G/B)^{F^r} \\ x^{-1}y \in BwB}} \varphi(y)$$

Calculate: $\mathrm{tr}(g^\# F \cdot T_w, I(q^r)) = \#\{x \in (G/B)^{F^r} : x^{-1} g^\# F(x) \in BwB\}$

Note: Frobenius acts trivially on $W \Rightarrow$ hence on H : $F = id$.

so $g^\# F, T_w$ commute.

$\Downarrow$

Same formula we saw for trace on $H_c^\circ(X_w)$ ...

$$\left[\begin{array}{l} \underline{\text{Upshot}} \quad r \geq 1, \ g, g^\# \text{ as above} \\ \mathrm{tr}(g \cdot F^r, H_c^\circ(X_w)) = \mathrm{tr}(g^\# F \cdot T_w, I(q^r)) \end{array}\right.$$

<u>Lemma</u> G any finite group, σ automorphism of G, $\tilde{G} := G \rtimes \langle \sigma \rangle$
$\langle\sigma\rangle$ is $\mathbb{Z}/n\mathbb{Z}$

V f.d. rep/$\mathbb{C}$ of $\tilde{G}$ which is <u>irred as G-module</u>

Θ = char of V : $G \rtimes \langle \sigma \rangle \longrightarrow \mathbb{C}$.

Then $\langle \Theta, \Theta \rangle_{G\sigma} = 1$

Strategy: f_1, f_2 fns on G, $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$.

$G\sigma \subset G \rtimes \langle \sigma \rangle$ coset,

f_1, f_2 on $G\sigma \Rightarrow \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g\sigma) \overline{f_2(g\sigma)}$

<u>Hint for proof</u>: Choose injective character $\langle \sigma \rangle = \mathbb{Z}/n\mathbb{Z} \xhookrightarrow{\chi} \mathbb{C}^*$

Get homomorphisms $G \rtimes \langle \sigma \rangle \twoheadrightarrow \langle \sigma \rangle \xrightarrow{\chi^i} \mathbb{C}^*$ $\quad i = 1, \dots n$.

$V \otimes \chi_i \not\cong V \otimes \chi_j$ $i \neq j$ mod n - since V irred as G-module,

use usual orthogonality relations on $G \rtimes \langle \sigma \rangle$. ...

[Fix $\sqrt{q}$]

Goal: express trace of $H^i_c(X_w)$ as $\sum_E tr(T_w, \tilde{E}_{\tilde{q}}) \, tr(g; R_E)$ for some F. (following Digne-Michel)

$\forall w \in W$, $tr(g \cdot F^r; H^i_c(X_w)) = tr(g^* F \cdot T_w; I(q^r))$ (*)

$= \sum_{E \in W^\vee} tr(g^* F, \Pi_E(q^r)) \cdot tr(T_w, E(q^r))$

$I(q^r) = \bigoplus_{E \in W} \Pi_E(q^r) \otimes \tilde{E}(q^r)$ as G^{F^r}-module: also true for F action!

System of equations labeled by w $\Rightarrow$ invert for $tr(g^* F)$ using orthogonality relation $\Downarrow$

- since $End_{G^{F^r} \rtimes \langle F \rangle} I(q^r) = [End_{G^{F^r}} I(q^r)]^F = \mathbb{H}(q^r)^F = \mathbb{H}(q^r)$ – endo ring didn't change when we restrict from large group to smaller group, so decomposition into irreps is same, so $\Pi_E(q^r)$ extend canonically to reps of $G^{F^r} \rtimes F$: principal series reps carry canonical F action.

- Have q-analog for $\mathbb{H}$ of orthogonality relation for W: $\exists$ rational fns $f_{w,E} \in \mathbb{Q}$ (one indeterminate) s.t.

$\forall E' \in W^\vee$, $\sum_{w \in W} f_{w,E}(\sqrt{q}^r) \, tr(T_w, E'(q^r)) = \begin{cases} 1 & E' \cong E \\ 0 & \text{otherwise} \end{cases}$ (**)

- lets us "invert" (nonsquare) matrix of character table.

- Multiply * by $f_{w,E'}$ and sum over w $\Rightarrow$ get

$tr(g^* F, \Pi_E(q^r)) = \sum_{w \in W} f_{w,E}(\sqrt{q}^r) \, tr(gF^r, H^i_c(X_w))$

- this is some linear combo of class fns on G, must be combo (for fixed r) a linear combo of the unipotent characters,

$\Rightarrow$ (***) $tr(g^* F, \Pi_E(q^r)) = \sum_{\substack{\text{unipotent} \\ \text{reps } \rho \\ \text{of } G^F}} c_{E,\rho}(r) \cdot tr(g, \rho)$

Will show the $c_{E,\rho}(r)$ are very simple fns of r..
We know e'vals of F on unipotent reps appearing in different cohomologies always same up to powers of q.

$\Rightarrow$ $c_{E,\rho}(r) = \lambda_\rho^r \cdot$ (rat fn, $\mathbb{Q}$ coeffs, of $\sqrt{q}^r$)

$\lambda_\rho \in \overline{\mathbb{Q}}_\ell^*$: all e-vals of F on ρ isotypic parts of all $H^i_c(X_w)$ are in $\lambda_\rho \cdot q^{\frac{1}{2}\mathbb{Z}}$.

2. (normalize λ_ρ) using <u>Riemann hypothesis</u> : compare $H^i_c(X_w)$ to IC : all these will also appear in some IC, so e'vals of F are algebraic numbers $\lambda_\rho \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_\ell}$ with $|\iota(\lambda_\rho)| = 1$ for $\forall \iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$: all abs numbers <u>are same and are $\in q^{\frac{1}{2}\mathbb{Z}}$.</u>

- $c_{E,\rho}(r) \in \mathbb{Q}(\lambda_\rho, \sqrt{q})$ number field
- $|G^F| \cdot c_{E,\rho}(r)$ is an algebraic <u>integer</u> : use LHS of $***$: write class fn $tr(g^* F, \Pi_E(q^r))$ in basis of characters : get algebraic integer divided by order of group : traces are alg integers etc.
- Fix $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Take $\langle\,\rangle$ on G^F ~~of~~ eqn $***$:

$\Rightarrow \quad \sum_\rho |c_{E,\rho}(r)|^2 = 1$

apply (3) plus lemma about abstract group : irrep of $G \rtimes \langle F \rangle$ - inner product of $tr(g^* F, \Pi_E(q^r)) = 1$.

Now fix E, ρ & consider $\{ c_{E,\rho}(r) \text{ for } r = 1, 2, 3, \ldots \}$: all in same number field, alg integers $/|G^F|$, and all their complex absolute values $|c_{E,\rho}(r)|_\iota \le 1 \quad \forall \iota, r$.

So by number theory this set is finite ! $\dashrightarrow$ <u>real</u> $\#$

but these are of form $\lambda_\rho^r \cdot$ (rat fn of $\sqrt{q}^{\,r}$)

$\Rightarrow$ if not identically zero : must have different values of r where these numbers are same, so some power of λ_ρ is <u>real</u>

$\Rightarrow$ some power of λ_ρ is ± 1. : we know $|\iota(\lambda_\rho)| = 1$.

So λ_ρ is a root of unity (if $c_{E,\rho}(r) \neq 0$)

.... for given ρ need to show $\exists E$ where this $\neq 0$ — would contradict fact that unipotent reps all appear in some cohomology .... $\exists E \in W$ $c_{E,\rho}(r) \neq 0$

$\Rightarrow \lambda_\rho$ <u>is</u> root of unity.

So now vary over r's where $\lambda_\rho^r = 1 \Rightarrow$ one rational fn taking same value ∞ many times : so this rat fn is const a rational <u>number</u> $\therefore c_{E,\rho}(r) = a_{E,\rho} \lambda_\rho^r$, $a_{E,\rho} \in \mathbb{Q}$

We now get $\operatorname{tr}(gF^r; H_c^\cdot(X_w)) =$

$$\sum_{E \in W^\vee} \sum_{\rho} a_{E,\rho} \lambda_\rho^r \operatorname{tr}(g,\rho) \cdot \operatorname{tr}(T_w, E(q^r))$$

$r \geq 1$

both sides depend coherently on r via r^{th} powers of eigenvalues --- so by linear algebra this formula must also be valid for $r=0$! (even though we couldn't apply Lefschetz at $r=0$)

$$\therefore \operatorname{tr}(g, H_c^\cdot(X_w)) = \sum_E \sum_\rho a_{E,\rho} \operatorname{tr}(g,\rho) \cdot \operatorname{tr}(w, E)$$

($q=1$ $T_w \rightsquigarrow w$)

apply orthogonality relations for W $\Rightarrow$

$$\sum_\rho a_{E,\rho} \operatorname{tr} \rho(g) = \operatorname{tr}(g, R_E)$$

So $a_{E,\rho}$ are just multiplicities $a_{E,\rho} = \langle \rho, R_E \rangle$.

Taking r that kills the root of unity λ_ρ, set $(\sum \langle \rho, R_E\rangle \rho = R_E)$

$$\boxed{\operatorname{tr}(gF^r; H_c^\cdot(X_w)) = \sum_{E \in W^\vee} \operatorname{tr}(T_w, E(q^r)) \cdot \operatorname{tr}(g, R_E)}$$

We also know now $\ast\ast$: $\operatorname{tr}(g^\# F, \pi_E(q^r)) = \sum_\rho \langle \rho, R_E \rangle \lambda_\rho^r \operatorname{tr} \rho(g)$

so twisted character (F-class fn) on LHS can be written in terms of unipotent characters :

GL_n $\lambda_\rho = 1$ $\langle \rho, R_E \rangle = \delta_{\rho, E}$

$\Rightarrow$ Shintani descent for GL_n : $\boxed{\operatorname{tr}(g^\# F, \pi_E(q^r)) = \operatorname{tr}(g, R_E)}$

(more complicated in general.)