Alina Marian – Strange Duality

Note Title 10/8/2006

$U(k,d)$ = semistable rank k degree d bundles on C

Rank 1 case: $Jac^{g-1}(C) \supset \Theta_1 = \{ L \text{ s.t. } h^0 = h^1 \neq 0 \}$

$Jac^d(C)$ has no canonical Θ-divisor, but if we fix a line bundle M on C with $\chi(L \otimes M) = 0$ L of deg d

$\Rightarrow \Theta_{1,M} = \{ L \text{ st } h^0(L \otimes M) \neq 0 \} < Jac^d$

Higher rank: $E \in U(k, k(g-1)) \Rightarrow \chi(E) = 0$

$\Rightarrow U(k, k(g-1)) \supset \Theta_k = \{ E : h^0(E) \neq 0 \}$

For $U(k,d) \supset \Theta_{k,E} = \{ F : h^0(E \otimes F) \neq 0 \}$

where E is chosen so that $\chi(E \otimes F) = 0$ $\forall F \in U(k,d)$

... get a divisor for <u>generic</u> E

Want E to have slope $(g-1) - \frac{d}{k}$

So minimal rank for E is $k/\gcd(k,d)$.

Now fix determinant: $SU(k,\Lambda) \subset U(k,d)$

$\mathrm{Pic}(SU)=\mathbb{Z}$. For E of minimal rank

$\Theta_{k,E}|_{SU}$ is an ample generator for $\mathrm{Pic}(SU)$.

Strange duality, degree zero case:

let
$$\begin{array}{rcl} \tau: SU(r,0)\times U(k,k(g-1)) & \longrightarrow & U(rk,rk(g-1)) \\ E,F & \longmapsto & E\otimes F \qquad \cup \\ & & \qquad\qquad \Theta_{rk} \end{array}$$

$$\tau^*\Theta_{rk} = \Theta_r^k \boxtimes \Theta_k^r \quad \text{as line bundles}$$

On level of sections:

$$H^0(SU(r,0)\times U(k,k(g-1)),\ \tau^*\Theta_{rk})$$
$$= H^0(SU(r,0),\Theta_r^k) \otimes H^0(U(k,k(g-1)),\Theta_k^r)$$

but have <u>canonical</u> section $\tau^*(\Theta_{rk})$

... namely the locus $\{(E,F) \text{ s.t. } h^0(E \otimes F) \neq 0\}$

$$\Rightarrow SD : H^0(SU(r,0), \Theta_r^k)^\vee \longrightarrow H^0(U(k, k(g-1)), \Theta_k^r)$$

$$\text{eval at } E \longmapsto \Theta_{k,E}$$

Conjecture: SD is an isomorphism
(Beauville) ... Verlinde formula ⇒ dimensions match

Proved for generic curve by Belkale
for all curves by Marian & Oprea

... Want to find enough pairs E_i, F_i $(1 \leq i \leq h^0)$
s.t. $\Theta_{k,E_i}(F_j) \neq 0$ iff $i = j$.

ie $h^0(E_i \otimes F_j) = 0$ iff $i \neq j$

---- trying to prove Θ's span!
so need a good interpretation of the Verlinde number

$$h^0(SU(r,0), \Theta_r^k) = \chi(SU(r,0), \Theta_r^k)$$

$$= \frac{r^g}{(r+k)^g} \int_{Quot_d(\mathcal{O}^{r+k}, k, C)} a_k^{top} \qquad \text{(Witten)}$$

--- intersection number on the Quot scheme with an enumerative meaning..

Without the annoying $\frac{r^g}{(r+k)^g}$, find

$$\int_{Quot} a_b^{top} = h^0\left(U(k,0),\ \Theta_{k,M}^{r} \otimes \det{}^* \Theta_{1,L}\right)$$

where M, L line bundles of deg $g-1$

$$= h^0\left(U(r,0),\ \Theta_{r,M}^{k} \otimes \det{}^* \Theta_{1,L}\right)$$

So let's reformulate the duality in terms of these bundles instead, where have nicer interpretation...

$$\Delta_{L,M} = \{ (E,F) \text{ s.t. } h^0(E \otimes F \otimes M) > 0 \ \underline{\text{or}}\ h^0(\det E^\vee \otimes \det F \otimes L^{-1} \otimes K) > 0 \}$$

$$\cap$$

$$U(r,0) \times U(k,0)$$

ie have $\sigma: U(r,0) \times U(k,0) \longrightarrow U(rk, 0) \times J^0$

$$E \times F \longmapsto E \otimes F, \det E^\vee \otimes \det F$$

so $\theta(\Delta_{L,M}) = \sigma^*(\Theta_{rk,M} \boxtimes \theta_{1,L^{-1}\otimes K})$

$D: H^0(U(r,0), \Theta^k_{r,M} \otimes \det^* \theta_{1,L})^\vee \longrightarrow$

$$H^0(U(k,0), \Theta^r_{k,M} \otimes \det^* \theta_{1,L^{-1}\otimes\omega})$$

<u>Theorem</u> D is an isomorphism

idea the same: want E_i, F_i s.t. $\Delta_{E_i}(F_j) \neq 0$ iff $i=j$

Theorem $\Longrightarrow$ strange duality: must check restriction to fixed determinant locus is surjective on Θ-fns..

. . .

$\mathrm{Quot}_d(\mathcal{O}^{r+k}, k, C)$ parameter space for

irreducible for large d $\quad 0 \to \underset{\substack{\text{rk } k \\ \deg -d}}{E} \to \mathcal{O}^{r+k} \to \underset{\text{rank } r}{F} \to 0$

Universal objects:

$$\begin{tikzcd}
\mathcal{E} \arrow[r] & \mathcal{O}^{r+k} \arrow[r] \arrow[d] & \mathcal{F} \\
 & \mathrm{Quot}_d \times C &
\end{tikzcd}$$

Class $a_k = c_k(\mathcal{E}^\vee|_{\mathrm{Quot} \times p})$ $p \in C$

assume degree $d \equiv 0 \ (k)$

What is top power $a_k^{top} = a_k^s$?

Fix $p_1, .. p_s \in C$ & generic lines in $(\mathbb{C}^{r+k})^\vee = V$

$\Rightarrow$ we're counting $\int_{\mathrm{Quot}} a_k^s =$ # exact sequences

$$0 \to E_i \to \mathcal{O}^{r+k} \to F_i \to 0$$

s.t. $L_j \otimes \mathcal{O} \hookrightarrow \mathcal{O}^{r+k\,\vee} \longrightarrow E_i^\vee$

is zero at p_j

Equivalently we're counting exact sequences

$$\begin{array}{ccccccccc} 0 & \to & E_i & \to & S & \longrightarrow & F_i' & \longrightarrow & 0 \\ & & \deg -d & & \deg s & & \deg d-s & & \end{array}$$

where S is pushout
$$\begin{tikzcd} \bigoplus L_j \otimes \mathcal{O} \arrow[r] \arrow[d, hook] & \bigoplus L_j(p_j) \arrow[d] \\ V \otimes \mathcal{O} \arrow[r] & S \end{tikzcd}$$

But this is what we want: $h^0(E_i^\vee \otimes F_j') \neq 0$ if $i \neq j$ ... $\det E_i \otimes \det F_i' = \det S$

& $h^0(E_i^\vee \otimes F_i) = 0$ since it's the Zariski tangent of our 0-dimensional intersection in the quot scheme.. $\blacksquare$

intersection theoretic interpretation of Verlinde numbers:

Witten's paper Verlinde Algebra & Quantum Cohomology of the Grassmanian

Zagier: $h^0(SU(r,0), \Theta_r^k) =$

$$= \frac{r^g}{(r+k)^g} \sum_{\substack{S \cup T = \{1, \dots, r+k\} \\ |S| = k}} \prod_{\substack{s \in S \\ t \in T}} \left(2 \sin \pi \frac{s-t}{r+k}\right)^{g-1}$$

Vafa-Intrilligator:

$\int_{Quot_d} a_k^{top}$ = sum over k-tuples of (r+k)-roots of 1

... comes from localization

- looks like RHS of Zagier formula..