D. Nadler · Red Groups & Geometric Langlands 10/23/04

Notation: G reductive group$/\mathbb{C}$. $K = \mathbb{C}((t)) \supset \mathbb{C}[[t]] = \mathcal{O}$

$D^\times = \operatorname{Spec} K \subset D = \operatorname{Spec} \mathcal{O}$.

Affine Grassmannian Gr ind-scheme, with $\mathbb{C}$-points $G(K)/G(\mathcal{O})$

Interpretations (1) Flag variety for loop group (quotient by parahoric)

GL_n: description as subspaces: $GL_n K \circlearrowright K^n$

$$\underset{\cup}{GL_n K} \circlearrowright \underset{\cup}{K^n}$$
$$GL_n \mathcal{O} = \operatorname{Stab} \mathcal{O} \circlearrowright \mathcal{O}^n$$

$$GL_n K / GL_n \mathcal{O} = \left\{ \begin{array}{l} \mathbb{C}\text{-subspaces } L \subset K^n : \; t^k \mathcal{O}^n \subset L \subset t^{-k}\mathcal{O}^n \quad k \gg 0 \\ \& \; t \cdot L \subset L \end{array} \right\}$$

= increasing union of fin dim projective varieties

(2) Algebraic model for ΩG_c based loop space for compact form of G

$$\Omega G_c = \{ \text{polynomial maps } \gamma : \mathbb{C}^* \to G \; : \; \gamma(1) = 1, \; \gamma|_{S^1} \subset G_c \}$$

Prop $\Omega G_c \times G(\mathcal{O}) \xrightarrow{\sim} G(K)$ homeomorphism

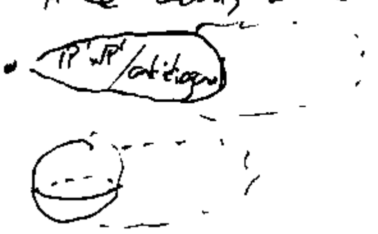
For $G = PGL_n \supset G_c = PU_n$

Fun problem! Show $\Omega G_c \underset{\text{as topological group}}{\simeq} \underset{\text{free group}}{\operatorname{Free}(\mathbb{C}P^{n-1})} \Big/ \left\langle x_1 x_2 \cdots x_n = 1 \text{ s.t. } \{x_1, \dots x_n\} \text{ form unitary basis} \right\rangle$

$n=2$: free words on $\mathbb{C}P^1$

length 0

length 1



(3). Hecke operators (modifications of bundles)

Bun_G = moduli of G-bundles on smooth complex curve C

$$\begin{tikzcd} & \mathcal{H}_C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Bun_G & & Bun_G \end{tikzcd}$$

Fibers are isomorphic to Gr up to action of $G(\mathcal{O})$ by left multiplication & of $\operatorname{Aut} \mathcal{O}$ reparametrization

Operators on $Sh(Bun_G)$ given by $G(\mathcal{O})$ equivariant sheaves on Gr.

<u>Geometric Satake correspondence</u>

Cartan decomposition: $G(\mathcal{O})\backslash Gr \simeq W_G \backslash \Lambda_G$ coweight lattice $Hom(\mathbb{C}^*, T)$

Classical Satake $\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] = K(Rep\, \check{G})$

Spherical Hecke algebra

Geometric Satake $P_{G(\mathcal{O})}(Gr) \overset{\otimes}{\simeq} Rep(\check{G})$

equivariant perverse sheaves

Consider $C_{\mathbb{R}}$ real curve (nonempty), C complexification

$G_{\mathbb{R}}$ real form of G

$Bun_{G_{\mathbb{R}}}(C_{\mathbb{R}})$ moduli of real bundles

-- Think of as bundle on C^+ with reality condition on the boundary.

2 kinds of Hecke operators on $Sh(Bun_{G_{\mathbb{R}}}(C_{\mathbb{R}}))$:

1. If $c \in C^+$ $\Rightarrow$ modifications given by $P_{G(\mathcal{O})}(Gr)$ as before
2. If $c \in C_{\mathbb{R}}$, then modifications given by sheaves on a real form $Gr_{\mathbb{R}}$ = real loop Grassmannian

$Gr_{\mathbb{R}} = G_{\mathbb{R}}(K_{\mathbb{R}})\, G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$

$K_{\mathbb{R}} = \mathbb{R}((t)) \supset \mathbb{R}[[t]] = \mathcal{O}_{\mathbb{R}}$

Interpretations: 1. Flag variety $/\!/ X_c$

2. $Gr_{\mathbb{R}} \simeq \Omega(G_c/K_{\mathbb{R}})$ $K_{\mathbb{R}} \subset G_c$ max compact

<u>explicit picture</u> for $G_{\mathbb{R}} = SO(1,n)$, $X_c = S^n$:

Fun problem! $G_{\mathbb{R}} \simeq Frame(S^{n-1}) / \langle x - antipodal(x) = 1 \rangle$

e.g. $n=3$: $SO(1,3) = SL_2\mathbb{C}$, $S^2 \simeq \mathbb{P}^1$

$n=2$ Frame $(S) / x - antip(x) = 1$

Real version of geometric Satake:

(Obvious) miracle: $G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})$-orbits in $Gr_{\mathbb{R}}$ are all even codimensional

So makes sense to consider perverse sheaves $P_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}})$

Define $Q(Gr_{\mathbb{R}})$ to be the subcategory of $P_{G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})}(Gr_{\mathbb{R}})$ arising as limits (nearby cycles) of objects of $P_{G(\mathcal{O})}(Gr) \simeq Rep(G^\vee)$

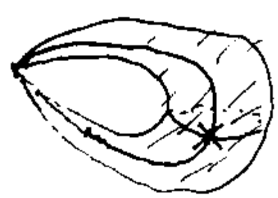
<u>Theorem</u> $Q(Gr_{\mathbb{R}})$ is a tensor category $\simeq Rep\, H^\vee$ for a reductive group $H^\vee \subset G^\vee$.

Structures:

$Q(Gr_{\mathbb{R}})$	$Rep\, H^\vee$
composition	$\otimes$
limits to boundary	restriction $Rep\, G^\vee \to Rep\, H^\vee$
total cohomology H^*	underlying vector space
	restriction $Rep\, H^\vee \to Rep(\text{max tors})$

decompose $H^* = \bigoplus F^\nu$, $\nu \in char(H^\vee)$

F^ν = cohomology of perverse sheaf along a "cell"



0-cell F_0

1-cell F_1

2-cell F_2

$F_0 + F_1 + F_2 = \int H^*(\ ) = H^*(\ )$; F_1 goes to zero here

odd dim cells don't carry cohomology

Nearby cycles preserve $\Leftrightarrow$ $G_{\mathbb{R}}$ quasisplit.