N. Nekrasov - Physics & higher dim Langlands' 10/25/04

"Instanton partition functions"

		pure super YM	SYM w/ adjoint matter	SYM w/ fundamental matter
elliptic cohomology	6d	elliptic cohomology on $\mathcal{M}_{G,k}$	Ell $T^* \mathcal{M}_{G,k}$	ell
K-theory	5d	K on $\mathcal{M}_{G,k}$	K $T^* \mathcal{M}_{G,k}$	K
Cohomology	4d	H $\mathcal{M}_{G,k}$ framed G-bundles on $\mathbb{P}^2$: cohomology problem	H $T^* \mathcal{M}_{G,k}$	H some other bundle

G simple Lie group / $\mathbb{C}$

$\mathcal{M}_{G,k} = Bun_G^k(\mathbb{P}^2, \mathbb{P}^1_\infty)$ G-bundles on $\mathbb{P}^2$ trivialized on line at ∞ with $c_2 = k$

k: instanton charge

$\mathcal{M}_G = \coprod_{k \geq 0}^{\infty} \mathcal{M}_{G,k}$ "=" $\exp\left(\mathbb{C}^2 \times \coprod_{k=0}^{\infty} \mathcal{M}_{G,k}^{\text{centered}}\right)$

... $Aut(\mathbb{P}^2, \mathbb{P}^1) \circlearrowright \mathcal{M}_G$, in particular translations act, so can consider quotient by translations

.... look like spaces of centered instantons

Diff geometry interpretation: G_c compact group bundles on S^4 with ASD connection

Energy tends to concentrate in some regions $\Rightarrow$ instantons

[Partition fn for $\mathcal{M}_G$ will end up being exp of a "centered" one.]

Pure 5d instanton partition function:

$$Z_5^{inst}[a, \varepsilon_1, \varepsilon_2, \Lambda; \beta] = \sum_{k=0}^{\infty} \Lambda^k \underbrace{Tr_{H_{\mathcal{M}_{G,k}}}(e^{\beta a}, e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2})}_{= Z_k}$$

polynomial fns on $\mathcal{M}_{G,k}$

Fix $T \subset G$, $\mathfrak{t} = \mathrm{Lie}\, T$.

$a \in \mathfrak{t}$ $\quad \varepsilon_1, \varepsilon_2, \Lambda \in \mathbb{C}$

$\mathcal{M}_{G,k}$ acted on by $G \times \mathbb{C}^* \times \mathbb{C}^*$
G acts changing framings, $\mathbb{C}^* \times \mathbb{C}^* \subset \mathrm{Aut}(\mathbb{P}^2; \mathbb{P}^1)$
$e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2} \in \mathbb{C}^* \times \mathbb{C}^*$, $e^{\beta a} \in T \subset G$ framing
Max torus

$G \times \mathbb{C}^* \times \mathbb{C}^* \circlearrowright \mathcal{H}_{\mathcal{M}_{G,k}}$ holomorphic functions on $\mathcal{M}_{G,k}$
Want to describe character

$\mathcal{M}_{G,k}$: Uhlenbeck is affine, complement has high codim, $\mathcal{M}_{G,k}$ quasiaffine : plenty of regular functions – only polys will contribute to trace
... eigenfunctions are polynomials
(note Uhlenbeck not really compactification, only partial: analog to $\mathbb{C} \setminus 0 \to \mathbb{C}$.)

Two origins of noncompactness : $\mathbb{P}^1_\infty$, 2 pointlike instantons
Uhlenbeck deals only with latter --- high codimension
(need this to be normal to know we're calculating right thing from open piece)
Homogeneous pieces of $\mathcal{H}_{\mathcal{M}_{G,k}}$ wrt $G \times \mathbb{C}^* \times \mathbb{C}^*$ are fin dim $\Rightarrow$ character makes sense.

(for KM get dilog identities from asymptotics ???)

Asymptotics of Tr as $\beta \to 0$ (group elt $\to 1$) gives idea of size of space

$\beta \to 0$ $\quad Z_k \sim \frac{1}{\beta^{\dim \mathcal{M}_{G,k}}}$

$\dim \mathcal{M}_{G,k} = 2 \cdot k \cdot h_G^\vee$. So scaling β appropriately get limit.

4d : $Z_4^{inst}[a,\epsilon_1,\epsilon_2,\Lambda] = \lim_{\beta\to 0} Z_5^{inst}[a,\epsilon_1,\epsilon_2;\beta^{2h^\vee}\Lambda,\beta]$

Λ = low energy scale

β = radius of circle $\mathbb{R}^4\times S^1_\beta$ in 5d theory

[Must assume Λ small so series converges & then analytically continue]

Z_4, Z_5 have interesting behavior when $\epsilon_1,\epsilon_2\to 0$:
("exp of species" formula)

asymptotics $Z_4^{inst}[a,\epsilon_1,\epsilon_2,\Lambda] = \exp\left[\frac{1}{\epsilon_1\epsilon_2}F^{inst}(a,\Lambda) + \text{less singular terms}\right]$

(as $\epsilon_1,\epsilon_2\to 0$ uniformly)

F^{inst} comes from totally different problem, in geometry of curves...

periodic Toda system

SYM with adjoint matter (adjoint hypermultiplet)

(elliptic CM)

Pure 5d : relativistic Toda (difference Toda: Ruijsenaars/Macdonald)

Pure 4d: periodic Toda (for dual to affinization of $\mathfrak{g}$)

~~Pure~~ adjoint 4d: elliptic CM

(7d too high for right amount of SUSY)

adjoint 5d: Λ splits into two parameters m, q

$$Z_5^{inst}[a,\epsilon_1,\epsilon_2,m,q;\beta] = \sum_{k=0}^{\infty} q^k \,\mathrm{Tr}_{H_{\mathbb{T}}\mathcal{M}_{G,k}}\left(e^{\beta a}\times e^{\beta\epsilon_1}\times e^{\beta\epsilon_2}\times(-e^{\beta m})\right)$$

odd tangent bundle

$\mathcal{H}$ = functions on odd tangent bundle ; holomorphic diff forms

multiply odd coord by extra $\mathbb{C}^*$ $(e^{\beta m})$

-- $G\times\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^* \circlearrowright \Pi T M_{G,k}$

$$Z_4^{inst}[a,\varepsilon_1,\varepsilon_2,m,q] = \lim_{\beta\to 0} Z_5^{inst}[a,\varepsilon_1,\varepsilon_2,q,m;\beta]$$

$m=0$ (trace on -1) : cancel ordinary bare by odd part :
$\to$ has well defined limit as $\beta\to 0$, don't need to rescale.

[$m=0 \Rightarrow$ more SUSY.]

Z^{inst} are no correct physical partition functions : need correction

(pure case) $Z_4[a,\varepsilon_1,\varepsilon_2,\Lambda] = Z_4^{pert} \times Z_4^{inst}$

cf.

$\boxed{\text{L-function part! asympt. of representation = local IC}}$

Perturbative part $Z_4^{pert} = \exp \sum_{\alpha\in\Delta_+} \gamma\left(\frac{\langle\alpha,a\rangle}{\Lambda},\varepsilon_1,\varepsilon_2\right)$

Δ_+ pos. roots of G

quantum dilogarithm
$$\gamma(x,\varepsilon_1,\varepsilon_2) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t}\, t^s \frac{e^{-tx}}{(e^{t\varepsilon_1}-1)(e^{t\varepsilon_2}-1)}$$

Satisfies $\gamma(x)-\gamma(x+\varepsilon_1)-\gamma(x+\varepsilon_2)+\gamma(x+\varepsilon_1+\varepsilon_2)$
$$= \log\frac{1}{x}$$

bundles on surfaces

Z world (parameter space) $\xrightarrow{\varepsilon_1,\varepsilon_2\to}$ integrable systems on curves

ideal sheaves on 3d varieties on $CY3_G$

GW invariants on $CY3_G$ (univ def of ALE)

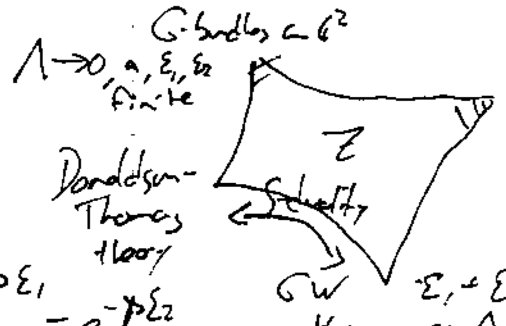
$G^\vee$ natural in q-parameter picture NOT in pure gauge theory

Z functions are Weyl group invariant (up to trivial perturbative piece)
but exhibit unexpected analytic properties:
the discriminant locus in Z-space with ~~mo~~ monodromy bigger than Weyl:
roughly $Sp(2r, \mathbb{Z})$ $r = \mathrm{rk}\, G$

(modulo So far not much success writing diff eqs for Z^{inst}
but some cases obey nonlinear difference eqns (infinite order!)
related to blowup rules

proportional to ϵ
$\parallel$
diff eqs in asymptotic case

$G = SL_n$: can replace $\mathcal{M}_{G,k}$
by Grassmannian, calculate trace
by fixed points $\longleftrightarrow$ partitions
In $\epsilon \to 0$ limit: saturated by a single partition
(common in theory of random partitions)
$\to$ see curve.



integrable system (spectral curve) -- leading term in $\epsilon \to 0$ asymptotics
a, Λ finite
$\epsilon_1, \epsilon_2 \to 0$

$\epsilon_1 + \epsilon_2 = 0$
theory a, Λ, β finite leading term is $g=0$ GW

integrable systems misleading in general! just families of periods of abelian varieties

For G of ADE type $\longleftrightarrow \Gamma_G \subset SU(2)$ McKay

$CY_G = \left((\mathcal{O}(-1) \oplus \mathcal{O}(-1)) / \Gamma_G \right)$, resolve singularities: ADE resolution bundle $/\mathbb{P}^1$
$\downarrow$
$\mathbb{P}^1$

BC, F_4, G_2 cases: Take $\tilde{G}$ s.t. Dynkin diagram$(\tilde{G})/\mathbb{Z}_n$ $(n=2$ or $3)$ is Dynkin diagram (G)

$\Rightarrow CY_G = CY_{\tilde{G}} / \mathbb{Z}_n$: acts on $\mathbb{P}^1$ with fixed points & on fibers: lifts to permute exceptional cycles

– get orbifold CY

Integrable System side: $\varepsilon_1, \varepsilon_2 \to 0$

$$Z_4[a, \varepsilon_1, \varepsilon_2, \Lambda] = \exp\left[\frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F}(a, \Lambda) + \text{lower order terms}\right]$$

Prepotential for special Kähler Geometry.

Consider a family $\mathcal{U} = \mathbb{Z}/w$ of curves : $C_u \subset \underset{x}{\mathbb{C}} \times \underset{z}{\mathbb{C}^*}$ $u \in \mathcal{U}$

$$z + \frac{\Lambda^{2h^\vee}}{z} + P_{h^\vee}(x, u) = 0 \qquad P \text{ poly deg } h^\vee.$$

$$P_{h^\vee}(x, u) = x^{h^\vee} + u_1 x^{h^\vee - d_1} + u_2 x^{h^\vee - d_2} + \dots + u_r$$

– for each elementary invariant polynomial of $\mathfrak{g}$

[$\mathfrak{sl}_n$: P_n, $n-1$ coeffs. get double cover of $\mathbb{P}^1$
... family of hyperelliptic curves]

Meromorphic differential on these curve $x \frac{dz}{z} = \lambda$, residue $= 0$
variation is holomorphic. Its periods give hol coordinates on base

Periods of λ give embedding $\varphi: \mathcal{U} \setminus \text{Discriminant} \to H^1(C_{u_0}, \mathbb{C})$

Image is a Lagrangian submanifold

[$C = \det (L(z) - x)$ spectral curve]
for Toda system
Curve of degree of ~~maps~~ fibers is Coxeter # ..

This Lagrangian is locally the differential of a generating function $F(a, \Lambda)$: Image of φ is graph of dF

SL_N case: $P_N(x,u) = x^N + u_1 x^{N-2} + \dots + u_{N-1}$

$\Lambda \to 0$ C_u degenerates into two lines glued together along roots of this characteristic polynomial

For theory with adjoint matter:

$$\exp\left[\frac{1}{\varepsilon_1 \varepsilon_2} F(a, m, q) + \dots\right] = Z_4 [a, \varepsilon_1, \varepsilon_2, m, q] \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0$$

$G = SL_N \Rightarrow$ elliptic Calogero-Moser

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - m^2 \sum_{i \neq j} \wp(q_i - q_j) \qquad \sum q_i = 0$$

Phase space $\mathcal{X} = (T^* E)^{N-1} / S_N$ (resolve singularities)
E elliptic curve $q = e^{2\pi i \tau}$

$$\Phi(z) = \left\| \begin{matrix} p_i \delta_{ij} \text{ diagonal part} \\ \dfrac{\theta_{11}(q_i - q_j + z)\,\theta_{11}'(0)}{\theta_{11}(q_i - q_j)\,\theta_{11}(z)} \text{ off diagonal} \end{matrix} \right\|_{N \times N}$$

$q_i \in \mathrm{Jac}\, E \Rightarrow \mathcal{E} = \oplus \mathcal{L}_{q_i}$, $\Phi(z)$ is Higgs field with first order pole proportional to m.

Spectral curve $\det(\Phi(z) - x) = 0 \subset T^* E$

2-parametric deformation of elliptic CM : related somehow to $\varepsilon_1, \varepsilon_2$?

- nonstationary deformation : our moduli of elliptic curve modular parameter τ one out of N deformation directions

$$\frac{\partial H_l}{\partial t_m} - \frac{\partial H_m}{\partial t_l} + \{H_l, H_m\} = 0$$

- quantization (KZB) $\frac{\partial}{\partial t_l}\psi + \{H_l, \psi\} = 0$

Modified version of Z where we consider G-bundles with parabolic structure $\Longrightarrow$ more parameters expect to see KZB direction.

(Frenkel) <u>1-dim Langlands</u> : $\mathrm{Conn}_{G^\vee}(X)$ $\quad$ $\mathcal{D}\text{-mod}(\mathrm{Bun}_G X)$

$$\mathcal{O}\text{-mod}(\mathrm{Bun}_{G^\vee} X) \overset{?}{\longleftrightarrow} \mathcal{D}\text{-mod}(\mathrm{Bun}_G X$$

More symmetric

$$T^*\mathrm{Bun}_G X \longrightarrow T^*_\lambda \mathrm{Bun}_G X = \mathrm{Conn}^\lambda_G X$$

Adjoint matter: $\quad Z[\varepsilon_1, \varepsilon_2] \qquad \varepsilon_i \longleftrightarrow (\mathbb{C}^*)^2 \curvearrowright \mathbb{P}^2$

Expansion $\varepsilon_1, \varepsilon_2 \to 0$ get $e^{\frac{1}{\varepsilon_1\varepsilon_2}F} + \ldots$

corresponds to Hitchin: $T^*_{0,0}\mathrm{Bun}_G$

$\varepsilon_1, \varepsilon_2 \longleftrightarrow \lambda, \hbar$ defs of $T^*\mathrm{Bun}_G$?

"Can consider integrals of arbitrary characteristic classes of G-bundles on

Gauge theory on $\mathbb{P}^2$ vs Integrable Systems on Curves?

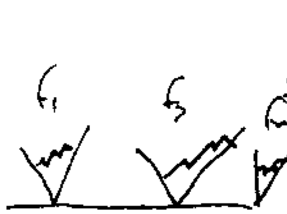
- 6d theory ("related to nonabelian gerbes") compactified on curve gives 4d gauge theory; compactified on 4-manifold ⇝ Hitchin theory on curves... So we're just reinterpreting 6d theory in two ways.

$\boxed{\text{Calculating partition functions}}$

Compactifications of $\mathcal{M}_{G,k}$:

- Uhlenbeck compactification: simple to describe, hard to calculate Tr: only one fixed pt of torus action: where all points are sitting at origin... very singular point $\Longrightarrow$ want finer compactifications
- $G = SL_N$ Gieseker (torsion-free sheaves) $\tilde{\mathcal{M}}_{G,k}$ - sufficiently fine for fixed pt formula: $T \times \mathbb{C}^* \times \mathbb{C}^*$ acts w/ isolated fixed pts

Fixed pt: splits as $\oplus$ rank 1 t-f sheaves — ideal sheaves, for 0-dim subschemes supported at 0 $\longleftrightarrow$ N partitions $\underline{k}_1, \underline{k}_2, \dots \underline{k}_N$



$$\mathrm{Tr}_{\mathcal{H}_{G,k}} = \sum_{\underline{k}_1, \dots, \underline{k}_N} m_{\underline{k}}(a, \varepsilon, \beta)$$

Draw partitions $\underline{k}_1, \dots \underline{k}_N$ sitting at points $a_1, \dots a_N$ in Cartan

write i^{th} partition as graph of f_i

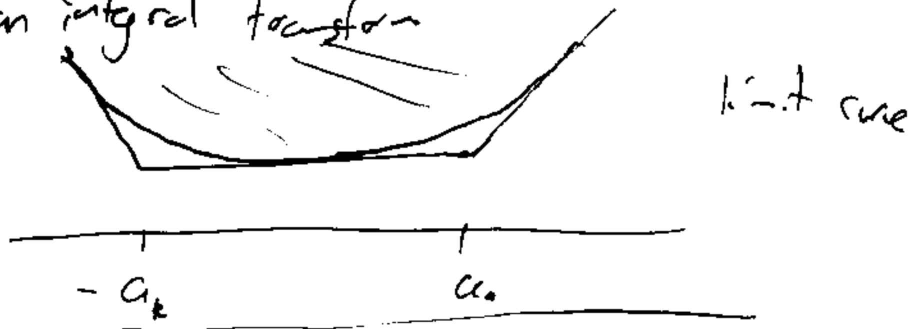
$f(x) = \sum_{i=1}^{N} f_i(x)$ functions: f_i piecewise linear functions associated to the partitions

Show $\Lambda^{\Sigma |k_i|} \cdot m_k(a, \varepsilon, \beta) = \iint dx\,dy\, f''(x) f''(y) \gamma(x-y)$

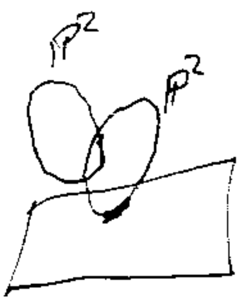
- actually finite sum, since f'' is a delta-fn (f only PL ...)

$\varepsilon_1, \varepsilon_2 \to 0$ get limit shape of partitions
described by a curve, related to spectral
curve by an integral transform

e.g. SL_2



- $\mathcal{M}_{G,k} = \mathrm{Maps}\big((\mathbb{P}^1, \infty) \longrightarrow (LG/L_+G, 1)\big)$
  has quasi-map compactification
  ... for splitting $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$ ... must vary splitting
  $\Rightarrow$ "relativistic version"
  Don't want relativistic version!
- So consider (following Givental)
  Stable maps $(\mathbb{P}^1, \infty) \to (\mathbb{P}^1 \times LG/L_+G, \text{const})$?
  $\mathcal{M}_{0,0}(\mathbb{P}^1 \times LG/L_+G, \infty, 1)$ degree $(1, k)$
  – compactification of space of graphs
  Have isolated & non-isolated f. points here, but latter
  don't contribute ;(products of Deligne-Mumford spaces)
  -- add much more stuff than Gresetor



- "Stable" compactification (w/ Kontsevich)
  When bundle degenerates, blow up that point,
  replace with $(\mathbb{P}^2, \mathbb{P}^1_\infty)$
  along that blowup $\mathbb{P}^1$, get G-bundles there
  on singular surface

f function! way to package $\mathrm{ch}\, E$ chern character
of vector bundle E which is fixed point of $G\times\mathbb{C}^*\times\mathbb{C}^*$

$$Z_5(\vec{a}, \Lambda, \cancel{\beta}\ \epsilon_1, \epsilon_2, \beta) = \sum_{\substack{k=0 \\ n\in\mathbb{Z}}}^{\infty} q^k \sum_{\vec{m}\in\mathbb{Z}^{N-1}} e^{-\vec{m}\cdot\vec{a}\beta} \Lambda^n \int_{[\mathcal{M}^{v.r}_{\vec{p},k}]} 1$$

$$\mathbb{Z}^{N-1} \oplus \mathbb{Z} = H_2(CY)$$

count ideal sheaves ($rk = 1$, $\det \cong \mathcal{O}$) on CY_{SL_N}

(Donaldson - Thomas corner)