# Hodge Theory · T. Pantev 2/8

$X$ smooth projective $/\mathbb{C}$. $H^\bullet_{Dol}(X,\mathbb{C}) = \bigoplus H^\bullet(X,\Omega^\bullet_X)$

$H^\bullet_{dR} = \mathbb{H}(\Omega^\bullet_X, d)$, $H^\bullet_B(X,\mathbb{C})$ = singular cohomology

Source of Hodge theory – relate these cohomologies.

Thm (de Rham): $H^\bullet_B \cong H^\bullet_{DR}$. In fact true over alg. closed field char 0, local fields etc. — comes from Riemann Hilbert.

Thm (Hodge) $H^\bullet_{DR} \cong H^\bullet_{Dol}$ (Kähler)

These come with different linear algebraic data, combine them all..

Hodge linear algebra $V$ – $2n$ dim $/\mathbb{R}$ vector space

Extra structures: $\omega \in \Lambda^2 V^\vee$ symplectic form

– metric $h \in S^2 V^\vee$, nondegenerate

– (almost) complex structure $J : V \to V$, $J^2 = -\mathrm{id}$

$$\Leftrightarrow \quad \begin{tikzcd} V\otimes_{\mathbb{R}}\mathbb{C} = V_{\mathbb{C}} \oplus \overline{V_{\mathbb{C}}} \arrow[d] \\ V \arrow[u, hook] \arrow[r, "\sim"] & V_{\mathbb{C}} \end{tikzcd}$$

Kähler structure – these three with compatibility: $V_{\mathbb{C}}$ + $H$ hermitian nondegen form on $V_{\mathbb{C}}$. The structure is *positive* if $H > 0$.

$H = h + i\omega$, $\omega(x,y) = h(ix,y)$.

On $\Lambda^\bullet V^\vee$: $J \Rightarrow$ type decomposition: $V^\vee\otimes\mathbb{C} = V^\vee_{\mathbb{C}} \oplus \overline{V^\vee_{\mathbb{C}}}$, $V^\vee_{\mathbb{C}} = V^{1,0}$, $\overline{V^\vee_{\mathbb{C}}} = V^{0,1}$

$\Lambda^{p,q}V = \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$

Kähler $\Rightarrow$ $\omega$ is of type $(1,1)$

$\omega \longrightarrow sl_2$ action on $\Lambda^\bullet V^\vee$. $N^+ \longmapsto L = \omega\wedge\cdot$

$N^- \mapsto \Lambda = i_\lambda$, contraction with Poisson structure $\lambda$: element in $\Lambda^2 V$ corresp to $\omega \in \Lambda^2 V^\vee$ via the isomorphism induced by $\omega$.

$H \longrightarrow D = (n-a)\,\mathrm{Id}$ on $\Lambda^a V^\vee$

Exercise – Check this is $sl_2$ rep and that $D$ is a derivation.

– If $V = V_1 \oplus V_2$, $\omega = \omega_1 \oplus \omega_2$ check that the $sl_2$ action is the tensor product action.

$\omega \longrightarrow$ Hodge $*$ operator: $*_\omega : \Lambda^a V^\vee \xrightarrow[\omega]{\sim} \Lambda^a V \xrightarrow[\alpha \mapsto i_\alpha(\frac{\omega^n}{n!})]{} \Lambda^{2n-a} V^\vee$

Interchanges $L$, $\Lambda$

Now let $(V, V_{\mathbb{C}}, H)$ $H > 0$ be a Kähler vspace

<u>Defn</u> Primitive $k$-forms are low-weight $k$-forms ($\Lambda\alpha = 0$).

Since $\Lambda$ is of pure bidegree $(-1,-1)$, $P^{p,q}$ makes sense.

Prop (Lefschetz decomp.)

1. $P^k = \ker L^{n-k+1} \wedge \Lambda^k V^\vee$. 2. $\Lambda^k V^\vee = \oplus L^j P^{k-2j}$

Pf - $sl_2$ decomp.

Note that $U(n) = U(V_{\mathbb{C}}, H)$ acts on $V$, hence on $\Lambda^\bullet V \otimes \mathbb{C}$

* $\boxed{\text{Exercise}}$ check that $P^{p,q}$ is an irrep of $U(n)$ for $p+q \le n$

Note also $u(n), sl_2 \subset \mathrm{End}(\Lambda^\bullet V^\vee)$ normalize each other

$\Rightarrow$ Lefschetz gives $U(n)$ decomposition

There are two natural operators from $P^k$ to $\Lambda^{2n-k} V^\vee$:

$L^{n-k}: P^k \to \Lambda^{2n-k} V^\vee$, $* : P^k \to \Lambda^{2n-k} V^\vee$.

<u>Proposition</u> Take $\alpha \in L^j P^{p,q}$. ($p+q = k$). Then

$$* \alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-2j} \bar{\alpha}$$

<u>Explanation</u> - both sides commute with $U(n)$, check on irreducibles, hence proportional by Schur, check on highest weight vector. (see Wells).

<u>Def</u> Define a hermitian inner product (polarization) on $\Lambda^k V^\vee$ by setting $\langle \alpha, \beta \rangle \, vol = i^{k^2} L^{n-k} (\alpha \wedge \bar{\beta})$

<u>Claim</u> $(-1)^p \cdot \langle , \rangle$ is positive definite on $L^j P^{p,q}$

<u>Pf</u> $\alpha \in L^j P^{p,q}$, $\langle \alpha, \alpha \rangle \, vol = i^{k^2} L^{n-2j-k} \alpha \wedge \bar{\alpha}$

$= i^{k^2} \alpha \wedge L^{n-2j+k} \bar{\alpha} = \underbrace{\alpha \wedge * \alpha}_{\langle\alpha,\alpha\rangle vol} \cdot \underbrace{(i^{k^2} (-1)^{k(k+1)} i^{\cancel{p-q}} \ldots)}_{\text{sign}}$

$i^{k^2+k-2p} = (-1)^{\frac{k(k+1)}{2} - p}$ .. all together get $(-1)^{-p}$...

$M$ $2n$-dim $C^\infty$ manifold, get $(S), (M), (C)$ varying smoothly in $M$.

$(S)$ is integrable if $\omega$ is closed. $(C)$ is integrable if $T^{1,0}_{\mathbb{C}} \subset T \otimes \mathbb{C}$ is an integrable distribution, i.e.

$[T^{1,0}, T^{1,0}] \subset T^{1,0}$.

<u>Thm</u> (Newlander-Nirenberg) If $C$ is integrable it comes from complex charts.

<u>Thm</u> (Kähler) If $M$ is Kähler, $\omega$ is tangent to second order to a flat metric

$L, \Lambda, D$ act on $\mathcal{A}^{p,q}$, $A^{p,q}_M$

Manifolds for which $L, \Lambda, D$ descend are called <u>Hodge</u> $\overset{?}{\cong}$ Kähler.

$M$ $2n$-dim $C^\infty$ manifold.

(S) $\omega \in \Gamma_{C^\infty}(M, \Lambda^2 T_M^\vee)$. (M) $h \in \Gamma_{C^\infty}(M, S^2 T_M^\vee)$

(C) $J \in \Gamma_{C^\infty}(M, \mathrm{End}\, T_M)$, $J^2 = -\mathrm{id}$. "strong"

Integrability – extra conditions to impose linearity on full fields / sheaves of functions $C^\infty(M)$. – compatibility with gluing – certain structures are locally products.

<u>Def</u> A tensor structure $t \in \Gamma_{C^\infty}(M, T_M^{\otimes k} \otimes T_M^{\vee \otimes l})$ is called integrable if for every pt $p \in M$ there exists a neighborhood $U \ni p$ & a $C^\infty$ triv. of $T_M|_U \cong U \times T_p$ so that $t$ becomes constant in this trivialization.

In symplectic, almost complex cases have easy <u>sufficient</u> conditions for integrability.

Thm (Darboux) $\omega$ integrable iff $d\omega = 0$.

Thm (Newlander-Nirenberg) $J$ integrable $\Leftrightarrow$ $T^{1,0}$ integrable distribution

Equivalently, decomposing $d = \bigoplus_{a+b=1} d^{a,b}$, suffices to look at $d: C^\infty \to T_M^\vee \otimes \mathbb{C}$, $d: T_M^\vee \otimes \mathbb{C} \to \Lambda^2 T_M^\vee \otimes \mathbb{C}$

deg $0 \to d^{1,0}, d^{0,1}$, deg $1 \rightsquigarrow d^{2,-1}, d^{-1,2}$ conjugate, these determine all.

<u>Newlander-Nirenberg</u>: $d^{2,-1} = 0$.

<u>Def</u> A manifold $M$ is equipped with a Kähler structure if it's equipped with an integrable complex structure and a hermitian metric $H = h + i\omega$, st. $d\omega = 0$.

<u>Prop</u> Let $M$ be a complex Hermitian manifold. Tfae:

1. $M$ Kähler 2. $\forall p \in M$ $\exists U \ni p$ local hol. coordinates $z_1 \dots z_n$ so that if $h_{ij} = H(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$, $h_{ij} = \delta_{ij} + O(\|z\|^2)$

Pf. Write $\omega = \sqrt{-1} \sum h_{ij}\, dz_i \wedge d\bar{z}_j$ ... $2 \Rightarrow d\omega_p = 0$.

Conversely can choose coords such that at $p$,

$h_{ij} = \delta_{ij} + \sum a_{ij}^k z_k + \sum a_{ij}^{\bar{k}} \bar{z}_k + O(\|z\|^2)$

Look for quadratic change of coords $z_i = w_i + q_i(w)$

$q_i$ homog deg 2. $\tilde{h}_{ij} = H(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j})$. what is first order

$$\frac{\partial}{\partial w_i} = \sum (\delta_{is} + \frac{\partial q_s}{\partial w_i}) \frac{\partial}{\partial z_s}$$

$$\tilde{h}_{ij} = H(\frac{\partial}{\partial z_i} + \sum \frac{\partial q_s}{\partial w_i} \frac{\partial}{\partial z_s}, \text{———}_j)$$

$$= h_{ij} + \sum \overline{\frac{\partial q_t}{\partial w_j}} h_{it} + \sum \frac{\partial q_s}{\partial w_i} h_{sj} + \dots$$

$$[\tilde{h}_{ij}]_1 = [h_{ij}]_1 + \overline{\frac{\partial q_i}{\partial w_j}} + \frac{\partial q_j}{\partial w_i}.$$

Thus $\hat{a}_{ij}^k = a_{ij}^k + \frac{\partial^2 \varepsilon_j}{\partial w_k \partial w_i}$, so set $g_{ij} = -\sum_{i,k} a_{ij}^k w_j w_k$

But $d\omega = 0 \Rightarrow a_{ij}^k = a_{kj}^i$ so this is consistent ∎

$\mathcal{A}^k(M) = \bigoplus \mathcal{A}^{p,q}(M)$. $L, \Lambda, D: \mathcal{A}^k \to \mathcal{A}^{k\pm 0,2}$

$\bar{\partial}^2 = \partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$, decompose into types.

$\Rightarrow (\mathcal{A}^{\cdot,\cdot}, \partial, \bar{\partial}) \Rightarrow$ total complex is de Rham complex.

Thm (~~deRham~~) $H^p_\beta(M, \mathbb{C}) = H^p_{DR}(M, \mathbb{C})$ (pf: all $\mathcal{A}^\cdot$ are fine, $t$-Poincaré: it is a resolution)

Thm (Dolbeault) Set $\Omega^p_M$ = sheaf of hol $p$-forms on $M$. Then $H^\cdot(\Omega^p_M) \cong H^{p,\cdot}_{\bar{\partial}}(M)$

Relationship between deRham & Dolbeault?

Filter $\mathcal{A}^\cdot$ horizontally $F^p\mathcal{A}^q = \bigoplus_{k \geq p} \mathcal{A}^{k,\cdot}$

$\Rightarrow$ spectral sequence converging to deRham,

$E_2^{p,q} = \cancel{H^q(H^p(A^{\cdot\cdot}, \bar{\partial}))} = H^q(H^p(A^\cdot, \bar{\partial}))$

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<u>Non Kähler Complex Manifolds</u>

1. Hopf manifolds: $\forall n \geq 1$ $Hopf_n = \mathbb{C}^n - \{0\}/\Gamma$,

$\Gamma = \left\{ \begin{pmatrix} a_1 & & 0 \\ 0 & \ddots & a_n \end{pmatrix} \middle| a_i = 1 \text{ or } 2 \right\}$

$Hopf_n \cong \cancel{\mathbb{C}} \; S^{2n-1} \times S^1$

2. Calabi-Eckmann $CE_{n,m} \cong S^{2n+1} \times S^{2m+1}$

$\downarrow$ Hopf

with fiber a torus; glue together $\mathbb{CP}^n \times \mathbb{CP}^m$ for a complex structure:

Choose $\tau \in$ U H plane, $E = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$

$S^{2n+1} = \{ z \in \mathbb{C}^{n+1} \mid z\bar{z} = 1 \}$ $S^{2m+1}: w\bar{w} = 1$.

Coordinate charts: $V_{kj}: (z, w) \in S^{2n+1} \times S^{2m+1}$ s.t. $z_k\, w_j \neq 0$

Coord functions: $x_{kp} = \frac{z_p}{z_k}$, $y_{jq} = \frac{w_q}{w_j}$ $t_{kj} = \frac{1}{2\pi\sqrt{-1}}(\log z_k + \tau \log w_j)$ mod $\mathbb{Z} \oplus \mathbb{Z}\tau$

$(x, y, t): V_{kj} \to \mathbb{C}^n \times \mathbb{C}^m \times E$ homeo.

Transitions: on $V_{kj} \cap V_{rs}$

$x_{rp} = \frac{x_{kp}}{x_{kr}}$, $y_{sq} = \frac{y_{jq}}{y_{js}}$, $t_{rs} = t_{kj} + \frac{1}{2\pi\sqrt{-1}}(\log x_{kr} + \tau \log y_{js})$ / lattice

Remark for $M$ compact Kähler, $b_2(M) \geq 1$ from $\omega$ ($\omega^n$ volume form)
So $CE_{n,m}$ can't be Kähler: $b_2 = 0$.
Even if we start blowing up points on $Hopf_2$, get $b_2 \neq 0$
but Hodge $\Rightarrow$ it's still not zero.

$(A^{\cdot\cdot}, \partial, \bar\partial)$: diagonal complex is de Rham $(A^\cdot, d)$,
the horizontal complex is $(A^{p,\cdot}, \bar\partial)$ Dolbeault
The fact that $(A^\cdot, d)$ comes from a double complex gives us a filtration $i$ from horizontal filtration of $A^\cdot$
$$F^p A^\cdot = \bigoplus_{i \geq p} A^{i,\cdot}$$
$$E_1^{p,q} = H^q((A^{p,\cdot}, \bar\partial)) = H^q(M, \Omega^p_M) \Rightarrow H^{p+q}(M, \mathbb{C})$$

Čech version for dR, try to substitute $(A^\cdot, d)$ with $(\Omega^\cdot, \partial)$ ... except not fine
But we have $0 \to \mathbb{C} \to \Omega^0 \xrightarrow{\partial} \Omega^1 \to \dots \Omega^n$.
Think of this as map of complexes $i: \mathbb{C} \to \Omega^\cdot_M$,
get map on cohomology $H^\cdot(M, \mathbb{C}) \xrightarrow{i_*} \mathbb{H}^\cdot((\Omega^\cdot_M, \partial))$
Hope $i_*$ is iso...
Theorem $i_*$ is an isomorphism.
Pf Must show $i$ is a quasi-iso, i.e. $\mathcal{H}^k((\Omega^\cdot_M, \partial)) = \begin{cases} \mathbb{C} & k=0 \\ 0 & k>0 \end{cases}$
– the $\partial$-Poincaré lemma ∎
Spectral sequence still makes sense: Hodge-de Rham ss.
$$E_1^{p,q} = H^q(\Omega^p) \Rightarrow \mathbb{H}^{p+q}((\Omega^\cdot, \partial)).$$
Algebraically we have
$$H^q(M, \Omega^p_{alg}) \Rightarrow \mathbb{H}^{p+q}(M, (\Omega^\cdot_{alg}, \partial))$$

Thm (Grothendieck) The natural map $\mathbb{H}^k((\Omega^\cdot_{alg}, \partial)) \to \mathbb{H}^k((\Omega^\cdot_{an}, \partial))$ is an iso.
Pf In proper case, tautological (GAGA). In general need resolution of singularities.

Let $M$ be Kähler Goal: Prove that HdR degenerates at $E_1$
Digression on harmonicity Need to construct canonical reps of dR, Dol cohomology classes. – find a way to measure sizes of elements. : take positive def. metric on $M$.
$M$ compact oriented (for integration).

$(\alpha,\beta)_M = \int_M h(\alpha,\beta)\, d\mathrm{vol}$, nondegen, bilinear on $A^k(M)$.
A form is harmonic if closed & of minimal size in its coh class.
Rewrite this in a local fashion: find formal adjoint of
$d$ wrt $(\,)_M$. Use $*$
**Lemma** $d^* = (-1)^{k(d-k)+1} * d *$ is a formal adjoint of $d$ wrt $(\,)_M$.
$\Delta = dd^* + d^*d$. $\alpha$ harmonic $\iff d\alpha = d^*\alpha = 0 \iff \Delta\alpha = 0$.
Cor The natural map $\ker \Delta \cap A^k(M) \to H^k(M)$ is injective $\alpha = d\tau \Rightarrow \|\alpha\|^2 = (d\tau, d\tau)_M = (\tau, d^*d\tau) = (\tau, d^*\alpha) = 0$.

**Theorem** (Hodge) The map $\mathcal{H}^k \to H^k_{dR}(M)$ is an isomorphism.
Weyl proof for Riemann surface — use conformal flatness, reduce to case of disc where we have Green's function & Poisson kernel — For complex Hermitian manifold $(M^n, H)$, do same for $\bar\partial$. Laplacians $\Box_\partial = \partial\partial^* + \partial^*\partial$, $\Box_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ still elliptic. A form $\alpha \in A^{p,q}$ is harmonic $\iff$
$\bar\partial\alpha = \bar\partial^*\alpha = 0 \iff \Box_{\bar\partial}\alpha = 0$
Thm (Hodge) $\bar\partial$ harmonic forms $\mathcal{H}^{p,q}_{\bar\partial} \cong H^{p,q}_{\bar\partial}(M)$

For $M$ Kähler, have Kähler identities:
1. $[\Lambda, \bar\partial] = -i\partial^*$, $[\Lambda, \partial] = i\bar\partial^*$
2. $[L, \partial^*] = i\bar\partial$, $[L, \bar\partial^*] = -i\partial$
3. $[L, \partial] = [L, \bar\partial] = 0$
4. $[\Lambda, \partial^*] = [\Lambda, \bar\partial^*] = 0$
5. $\frac{1}{2}\Delta = \Box_{\bar\partial} = \Box_\partial$
6. $[L, \Delta] = [\Lambda, \Delta] = [D, \Delta] = 0$
$\Rightarrow$ $H^n = \bigoplus H^{p,q}$ & $sl_2$ action descends to cohomology.

**Remark** On cohomology $D$ doesn't depend on anything but the class of the Kähler ~~metric~~. $L = [\omega] \wedge \cdot$ depends only on the class. $\Lambda$ seems to depend on metric but doesn't. — i.e. if we have two metrics with same $[\omega]$, get two $sl_2$ triples with $L$, $D$.
$\Rightarrow$ Lefschetz decomposition / Hard Lefschetz
Hodge-Riemann relations: $Q_k(\alpha,\beta) = i^{k^2}\int_M \omega^k \wedge \alpha \wedge \bar\beta$ hermitian.
i) $Q_k(\varphi, \psi) = 0$ for $\varphi \in H^{p,q}$, $\psi \in H^{p',q'}$ unless $p, q = q', p'$
ii) $i^{p-q} Q_k(\varphi, \bar\varphi) > 0$ if $\varphi \in L^i H^{p,q}(M)$ prim.
Hodge index $M$ of even complex dim $n$, $Q_n : H^n \times H^n \to \mathbb{C}$, $\sigma(Q_n) = \sum_{a+b=n} (-1)^a h^{a,b}$

Hodge-deRham spectral sequence:

$f: X \to S$ smooth $\Rightarrow$ $(\Omega^\bullet_{X/S}, d)$, $d$ is $f^{-1}\mathcal{O}$-linear.

$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Rightarrow H^{p+q}_{dR}(X/S) = \mathbb{H}^{p+q}((\Omega^\bullet_{X/S}, d))$

- degenerates for proper morphisms of schemes in char 0. Deligne, IHES 68

There is a limit filtration $F^p H^{p+q}_{dR}(X/S)$, Hodge filtration.

$X$ smooth proper $/\mathbb{C}$ : set $H^{p,q}(X) =$ $F^p H^{p+q}_{DR}(X) \cap \overline{F^q H^{p+q}_{DR}(X)}$.

$h^{p,q} := \dim H^q(X, \Omega^p_X)$

Remarks: 1. Complex conjugation comes from the identification $H^n_{dR}(X) = H^n(X_{an}, \mathbb{C})$ (HdR ss + GAGA + dR theorem)

2. $H^{p,q}(X) = \overline{H^{q,p}(X)}$

Prop $X$ smooth, proper $/\mathbb{C}$. Then i) HdR ss degenerates at $E_1$

ii) $\exists$ canonical isoms. $F^p H^n_{DR}(X) = \bigoplus_{\ell \geq p} H^{\ell, n-\ell}(X)$

& in particular $H^n_{DR}(X) = \bigoplus H^{p,q}(X)$. $h^{p,q} = h^{q,p} = \dim H^{p,q}$

Proof Case 1 $X$ projective: $E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q}$

$H^{p,q}_{\bar\partial}(X) \longrightarrow H^{p+1,q}_{\bar\partial}(X)$

$[\alpha] \to [\partial\alpha]$, $\alpha$ is $\bar\partial$-closed.

Pick $\alpha$ $\bar\partial$-harmonic. By Kähler identities, $\alpha$ is also $d$, $\partial$ harmonic $\Rightarrow$ $\partial, \partial^*$ closed. So $d_1 = 0$.

$d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$

On $e \in E_r^{p,q}$ this is as follows: lift $e$ to $[\alpha] \in E_1^{p,q}$ s.t. the projections to $E_2^{p,q} \ldots E_{r-1}^{p,q}$ are all in the ker $d_1$, ker $d_2 \ldots$ ker $d_{r-1}$. $d_1[\alpha] = 0 \Rightarrow \exists \alpha_2 : \partial\alpha = \bar\partial\alpha_2$ etc.

$d_2[\alpha] = 0 \Rightarrow \exists \alpha_3 : \partial\alpha_2 = \bar\partial\alpha_3$. ..

$[\alpha]$ harmonic — can choose $\alpha_2 = 0 \Rightarrow$ rest $0$....

For (ii) notice $H^{p,q}_{\bar\partial} = H^q(\Omega^p) = H^{p,q}$.

Case 2 $X$ proper: Chow's lemma + Hironaka $\Rightarrow$ $g: X' \to X$, $X'$ proj & smooth, $g$ birational morphism.

$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{DR}(X)$

$E_1^{p,q\,\prime} = H^q(X', \Omega^p_{X'}) \Rightarrow H^{p+q}_{DR}(X')$

Pull back of forms $\Rightarrow$ $g^*: E \to E'$, so suffices to show $g^*$ is an embedding.

*Lemma* Let $f: X \to Y$ be a proper birational morphism between smooth varieties. Then $f^*: H^q(Y, \Omega_Y^p) \to H^q(X, \Omega_X^p)$ is injective.

*Proof* Use the Gysin map

*Gysin maps* (digression) $f: X \to Y$ proper morphism between smooth varieties. Gysin map: $Tr_f : Rf_* \Omega^p_X \to \Omega_Y^{p-d}[-d]$

$d = \dim X - \dim Y$, map in $D^b(Y)$.

It is the dual to natural pullback $f^* \Omega_Y^{n-p} \to \Omega_X^{n-p}$

*Special cases* 1. $f: X \to Y$ a finite map.

$Tr_f : f_* \Omega_X^p \to \Omega_Y^p$. $f$ is flat so $f_* \Omega_X^p$ is locally free, so $Tr_f \in H^0(Y, Hom(f_* \Omega_X^p, \Omega_Y^p))$

$$
\begin{aligned}
Hom(f_*\Omega^p_X, \Omega^p_Y) &= (f_* \Omega_X^p)^\vee \otimes \Omega_Y^p = && \text{(relative duality)}\\
&= f_*(\Omega_X^{p\vee} \otimes \omega_{X/Y}) \otimes \Omega_Y^p = && \text{(projection formula)}\\
&= f_*(\Omega_X^{p\vee} \otimes \Omega_X^n \otimes f^*\Omega_Y^{n\vee} \otimes f^*\Omega_Y^p) = && \text{(contraction)}\\
&= f_*(\Omega_X^{n-p} \otimes f^*(\Omega_Y^{n-p})^\vee) = f_* Hom(f^*\Omega_Y^{n-p}, \Omega_X^{n-p})
\end{aligned}
$$

where we have a preferred section.

Explicitly, if $y \in Y$ is pt where $Y$ is étale

$$
\begin{tikzcd}
(f_* \Omega_X^p)_y \arrow[r, "Tr_f"] \arrow[d, equal] & (\Omega_Y^p)_y \arrow[d, "\wr"] \\
\bigoplus_{x \in f^{-1}(y)} (\Omega_X^p)_x \arrow[r, "\sum_{x \in f^{-1}(y)} (df^\vee)_x^{-1}"'] & (\Omega_Y^p)_y
\end{tikzcd}
$$

*Case 2* $f: X \to Y$ smooth with connected fibers of dim $d$.

$Rf_* \Omega^p$ still locally free.

$$Tr_f : R^d f_* \Omega_X^p \to \Omega_Y^{p-d}$$

Again $Tr_f$ section of $Hom(R^d f_* \Omega_X^p, \Omega_Y^{p-d})$

$$= (R^d f_* \Omega_X^p)^\vee \otimes \Omega_Y^{p-d} = R^{d-d} f_*((\Omega_X^p)^\vee \otimes \omega_{X/Y}) \otimes \Omega_Y^{p-d}$$

$$= f_*(\Omega_X^{p\vee} \otimes \Omega_X^n \otimes f^*\Omega_Y^{m\vee} \otimes f^*\Omega_Y^{p-d})$$

$$= f_*(\Omega_X^{n-p} \otimes f^*\Omega_Y^{n-p\vee}),$$

<u>General situation</u> follow same pattern in derived category

<u>Remark on relative duality</u>: Duality is a statement about interchangeability of $f_*$, Hom. Ideally would like $f: X \to Y$ to have a right adjoint $f^!: \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ s.t. $\mathrm{Hom}(f_* F, G) = \mathrm{Hom}(F, f^! G)$, $F \to X$, $G \to Y$.

But $f_*$ is not right exact — too much to hope.

Remedy: $Rf_* : D^b(X) \longrightarrow D^b(Y)$

<u>Relative duality</u> $\Rightarrow$ $\exists\, Rf^!$ right adjoint ($f$ proper)

In previous cases $Rf^!$ is representable,

$Rf^! G = R\mathcal{H}om(G, \omega_{X/Y}[d])$

Properties
1) $Rf^! \Omega_Y^m = \Omega_X^n[+d]$
2) contraction: $R\mathcal{H}om(\Omega_Y^{m-p}, \Omega_Y^m) = \Omega_Y^p$
3) biduality formulas:

$$Rf^! R\mathcal{H}om(K, L) = R\mathcal{H}om(Lf^* K, Rf^! L)$$
$$R\mathcal{H}om(K, L) = Rf_* R\mathcal{H}om(Lf^* K, Rf^! L)$$

Gysin: Need a global section $Tr_f$ in $R\mathcal{H}om(Rf_* \Omega_X^p, \Omega_Y^{p-d}[-d])$ (3)

$= Rf_* R\mathcal{H}om(\Omega_X^p, Rf^! \Omega_Y^{p-d}[-d]) =$ (2)

$= Rf_* R\mathcal{H}om(\Omega_X^p, Rf^! R\mathcal{H}om(\Omega_Y^{n-p}, \Omega_Y^m)[-d]) =$ (3),(1)

$= Rf_* R\mathcal{H}om(\Omega_X^p, R\mathcal{H}om(Lf^* \Omega_Y^{n-p}, \Omega_X^n)) =$ (transposition)

$= Rf_* R\mathcal{H}om(Lf^* \Omega_Y^{n-p}, R\mathcal{H}om(\Omega_X^p, \Omega_X^n)) =$ (2)

$= Rf_* R\mathcal{H}om(\underbrace{Lf^* \Omega_Y^{n-p}}_{= f^* \Omega_Y^{n-p}}, \Omega_X^{n-p})$

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<u>To prove the lemma</u> $f^*: H^q(\Omega_Y^p) \to H^q(\Omega_X^p)$, show it's injective:

compose with the Gysin map, going the other way

$Tr_f : H^q(\Omega_X^p) \to H^q(\Omega_Y^p)$

$Tr_f \circ f^* : H^q(\Omega_Y^p) \to H^q(\Omega_Y^p)$. Claim: this is identity:

$$\begin{tikzcd} \Omega_Y^p \arrow[r] \arrow[rr, bend right, "Rf_* f^*"'] & Rf_* f^* \Omega_Y^p \arrow[r] & Rf_* \Omega_X^p \arrow[r, "Tr_f"] & \Omega_Y^p \end{tikzcd}$$

Restricted to regular locus of the map (birational) this map is the identity $\Rightarrow$ map between coherent sheaves of same rank which is id. on open set $\Rightarrow$ must be the identity ∎

By induction on $r$ we get $g^*: E_r \hookrightarrow E_r' \Rightarrow d_r = 0$, $E_r = E_{r+1}$ $\Longrightarrow$ part (i) ∎

Part ii): Note that $F^p H^n_{DR}(X') \cap \overline{F^{n-p+1} H^n(X')} = \{0\}$ by degeneration of $E'$. But $E$ is a sub-ss $\Rightarrow$
$F^p H^n_{DR}(X) \cap \overline{F^{n-p+1} H^n_{DR}(X)} = \{0\}$

$$\Rightarrow \dim H^n_{DR}(X) \geq \sum_{l \geq p} h^{l,n-l} + \sum_{l \geq n-p+1} h^{l,n-l}$$
$$\| $$
$$\sum_l h^{l,n-l}$$

$$\Rightarrow \sum_{l \leq n-p} h^{l,n-l} \geq \sum_{l \geq p} h^{l,n-l}.$$

But if $N = \dim X$, by Serre duality $h^{ij} = h^{N-i,N-j}$, get opposite inequality. So $H^n(X) = F^p H^n \oplus \overline{F^{n-p+1} H^n}$.
Restricting on $F^{p-1} H^n$ get $F^{p-1} H^n = F^p H^n \oplus H^{p-1,n-p+1}$ ∎

Theorem $S$ scheme in char 0, $f: X \to S$ proper & smooth.
i). $R^q f_* \Omega^p_{X/S}$ are locally free & of finite type
ii) The Hodge de Rham s.s degenerates
iii) At every point of $S$, $R^q f_* \Omega^p_{X/S}$ & $R^p f_* \Omega^q_{X/S}$ have same rank.

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Hodge-de Rham: $E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Rightarrow R^n f_* \Omega^\bullet_{X/S}$

Proof $f$ proper $\Rightarrow R^q f_* \Omega^p_{X/S}$ are coherent, & have base change property. In particular it suffices to prove iii at a point Spec $k$.
Reductions: – may assume $S$ affine (statement local in $S$).
– May assume $S$ noetherian: commutes with inductive limits.
– May assume $S$ local
– faithful flatness of completions $\Rightarrow$ assume $S$ complete.
– Since every $R$ local, complete, Noetherian is a proj. limit $\varprojlim R/m^k$
use Comparison theorem in cohomology: same holds in cohomology – $E_1$ term of H-dR is a projective limit of the $E_1$ terms of H-dR over $R/m^k$.

$R/m^k$ are finite length, & $R^q f_* \Omega^p_{X/S}$ are projective over them $\Rightarrow$ (i) holds automatically. Also $E_1^{p,q}(R/m^k)$ induces $E_1^{pq}(R/m^n)$ by reduction mod $m^n$
$\Rightarrow$ the projective limit will be a free $R$-module.

Ultimately, may assume $S = \operatorname{Spec} A$, local Artinian with residue field of char. $0$.

Use Lefschetz principle + Galois action $\Rightarrow$ assume $k = \mathbb{C}$.

Thus $A$ is a local finite $\mathbb{C}$-algebra, $S = \operatorname{Spec} A$, $m$ = max ideal.

The S.S. is $(**)$ $H^q(X, \Omega^p_{X/S}) \Longrightarrow R^{p+q}\Gamma(\Omega^\bullet_{X/S})$

Let $X^{an}$ be the underlying complex analytic manifold of $X$.

<u>Lemma</u> $\Omega^{\bullet an}_{X/S}$ considered as sheaf of $A$-modules on $X$, is a resolution of the constant sheaf $A$.

<u>Proof</u> – The differential is $f^{-1}\mathcal{O}_S$ linear $\Rightarrow$ $A$-linear.
– The $m$-adic filtration on $\Omega^{\bullet an}_{X/S}$ is a filtration by subcomplexes.

$\Rightarrow$ $Gr\, \Omega^{\bullet an}_{X/S} = Gr A \otimes_{\mathbb{C}} Gr^0 \Omega^{\bullet an}_{X/S}$.

But notice that $Gr^0 \Omega^{\bullet an}_{X/S}$ is dR complex for $X^{an}$ - reduced manifold -
$\Rightarrow$ resolves $\mathbb{C}$. $\Rightarrow$ $Gr\, \Omega^{\bullet an}_{X/S}$ resolves $Gr A$.

Now the fact that the $m$-adic filtration is filtration by subcomplexes, + claim give the lemma —

<u>Claim</u> If $A$ is a filtered Noetherian ring (exhaustively filtered), and if $M, N$ are two filtered $A$-modules, then
(a) $Gr\, M$ is a free $Gr\, A$ module iff $M$ is a free $A$-module.
(b) If $f: M \to N$ is filtration preserving morphism, then $gr\, f$ is iso iff $f$ is iso. ■

Invoke GAGA. $l$ = length.

$$l_A R^n\Gamma(\Omega^\bullet_{X/S}) = l(A) \cdot \dim_{\mathbb{C}} R^n\Gamma(\Omega^\bullet_{X_{red}})$$

On the other hand we always have

$$l_A H^q(X, \Omega^p_{X/S}) \leq l(A) \dim_{\mathbb{C}} H^q(X_{red}, \Omega^p_{X_{red}})$$

+ equality occurs iff LHS is a free $A$-module.

$$HdR \Rightarrow \sum_{p+q=n} l_A H^q(X, \Omega^p_{X/S}) \geq l_A R^n\Gamma(\Omega^\bullet_{X/S})$$

with equality iff HdR degenerates.

Combining these get $l(A) \sum_{p+q=n} \dim_{\mathbb{C}} H^q(X_{red}, \Omega^p_{X_{red}}) \geq l(A) \dim_{\mathbb{C}} H^n_{dR}(X_{red}, \mathbb{C})$.
But this is $=$ by theorem for $X_{red}$
$\Rightarrow$ get all equalities, degenerating

<u>Corollary</u> $A$ fin dim local algebra $/\mathbb{C}$, $f: X \to \operatorname{Spec} A$ smooth & proper $\Rightarrow$ $H^q(X, \Omega^p_{X/A})$ is a free $A$-module & $H^q(X, \Omega^p_{X/A}) = A \otimes_{\mathbb{C}} H^q(X_{red}, \Omega^p_{X_{red}})$

$\boxed{\text{Algebraic Proof of degeneration}}$: Deligne–Illusie, Invent (1987) vol 89.
Builds on work of Faltings, Fontaine–Messing, K. Kato, Mazur, M. Raynaud.

<u>Remark</u>: <u>Liftings to char 0</u>. $X$ variety over $k$, char $k = p > 0$.
If $k = \mathbb{Z}/p\mathbb{Z}$, natural candidate is some $\tilde{X} \to \operatorname{Spec} \mathbb{Z}$ s.t. $\tilde{X} \otimes (\mathbb{Z}/p) = X$.
Look at this locally at least, complete $\Rightarrow$ another version is a scheme over $\operatorname{Spec} \mathbb{Z}_p$ specializing to $X$.
In general, have ~~Thm-Definition~~ (Witt) If $k$ is a perfect field, char $= p > 0$. $\Rightarrow$ (1) $\forall\, n \geq 0$ $\exists\, W_n(k)$ ring, uniquely determined by (a) $W_n(k)$ flat over $\mathbb{Z}/p^n$ (b) $W_n(k)/p W_n(k) = k$.
(2) $\varprojlim W_n(k)$ is a complete DVR uniformized by $p$ with residue field $k$.

<u>Def</u> $X$ scheme over $k$ perfect $\Rightarrow$ lift of $X$ to char $0$ is $\tilde{X} \to \operatorname{Spec} W(k)$ specializing to $X$ at $\operatorname{Spec} k$.

<u>Theorem</u> (Deligne–Illusie...) $X$ smooth proper $/k$ perfect char $p$. Assume $X$ admits a lifting to $\operatorname{Spec} W_2(k)$. Then the Hodge-deRham s.s. satisfies $E_1^{i,j} = E_\infty^{i,j}$ for $i+j < p$.

<u>Corollary</u> If $X$ is proper & smooth $/k$ of char $0$ $\Rightarrow$ HdR degenerates.
<u>Proof</u> It is standard that for any $X$ as above $\exists$ an integral domain $A$ of fin type$/\mathbb{Z}$, equipped with $A \to k$.
& $\exists$ a smooth proper $f: \mathfrak{X} \to \operatorname{Spec} A$ specializing to $X$.

Since $R^q f_* \Omega^p_{\mathfrak{X}/S}$, $R^n f_* \Omega^\bullet_{\mathfrak{X}/S}$ are coherent, have base change property $\Rightarrow$ $X$ HdR is a $\otimes_A k$ of HdR for $\mathfrak{X}$.

Suffices to show $H_{dR}(\mathfrak{X})$ degenerates. May assume these coherent sheaves are loc. free by specializing further.

All we need is $\sum h^{p,q} = h^n$ (*)

This can be checked at any given point, eg point with finite residue char. $\mathfrak{X}/S$ generically smooth, get residue fields of arbitrarily high char. $\Rightarrow$ Corollary ∎

To simplify things assume $X$ scheme $/\mathbb{Z}/p$ with lift to char 0.

$F: X \to X$ Frobenius: identity on points, raises functions to $p$th power. $F^* a = a^p$, $a \in \mathcal{O}_X$.

Notice that $\Omega^\bullet_X$ has $F^*\mathcal{O}_X$-linear differential: $d(a^p) = p a^{p-1} da = 0 \ldots$ — so better to look at this complex, $F_*\Omega^\bullet_X$ has $\mathcal{O}_X$-linear differential.

We haven't done anything to cohomology since $F$ is a finite map!!

<u>Thm</u> (Cartier) $\exists$ a canonical morphism of graded $\mathcal{O}_X$-algebras

$$C_X : \bigoplus_i \Omega^i_X \longrightarrow \bigoplus_i \mathcal{H}^i F_* \Omega^\bullet_X$$

satisfying $C^1_X(da) = [a^{p-1}da] \in \mathcal{H}^1 F_*\Omega^\bullet_X$

$C^0_X(a) = a$.

If $X$ is smooth this is iso.

"Proof" Reduce to affine case, induction by dimension to affine line $\Rightarrow$ proof by hand. $\mathbb{A}^1: \Omega^0_X = k[t] \xrightarrow{d} \Omega^1_X = k[t]\,dt$

$\ker d = k[t^p]$, $\operatorname{coker} d = k[t^p]\, t^{p-1} dt$. ∎

<u>Thm</u> (Deligne-Illusie) — deRham decomposition: $X \to \operatorname{Spec} k$.

$\tilde{X} \to \operatorname{Spec} W_2(k)$ smooth lift $\Rightarrow$ canonical

$$\varphi_{\tilde X} : \bigoplus_{i<p} \Omega^i_X[-i] \longrightarrow \tau_{<p} F_* \Omega^\bullet_X \quad \text{(truncated complex)}$$

s.t. $\mathcal{H}^i \varphi_{\tilde X} = C_X$, in particular $\varphi_{\tilde X}$ is a quasi-isom.

LHS is complex with differential 0, RHS calculated dR...

<u>Sketch of proof</u>: Assume ideal situation when $F$ lifts to $\tilde{X} \circlearrowleft \tilde{F}$

Then $C_{\tilde F} : \Omega^1[-1] \longrightarrow F_*\Omega^\bullet_X$, $[da] \to \left[\frac{1}{p} d\tilde{F}^*\tilde{a} \mod p\right]$

independent of $\tilde a \in \mathcal{O}_{\tilde X}$ lift of $a$. Induces Cartier.

+ extend by multiplicativity

— need $i<p$: $\Omega^1 \otimes \cdots \otimes \Omega^1 \twoheadrightarrow \Omega^i_X$ has <u>alt</u> splitting if $i<p$: divide by $p$

Analyze how this depends on change of $F$: only depends up to homotopy, same $\varphi_X^*$ on cohomology. Lifts always exist locally, consider on level of Čech complex, which is qu-isom to sheaf cohomology..



We used the infinitesimal version of semicontinuity!
If $A$ artinian $/k$ then $\forall X \to \operatorname{Spec} A$ $\mathcal{L}$ any sheaf $F \to X$, $l_A H^i(X, F) \leq l_k(A) \dim H^i(X \otimes k, F \otimes k)$
"$=$" iff $H^i(X, F)$ is $A$-free.

<u>Crystalline</u> deRham not right in char $p$ - e.g. not nec. finite dimensional, like for an affine curve.
Grothendieck constructed example of $X$ s.t. $\dim H^1_{DR}(X/k) > 2 \dim \operatorname{Pic} X$
- $l$-adic cohomology not good - misses the prime $p$.
Monsky-Washnitzer: If we can lift our scheme over Witt vectors, can take deRham theory.. they prove that if $X \to \operatorname{Spec} k$ lifts to $\hat{X} \to \operatorname{Spec} W(k)$, then $H^i_{DR}(\hat{X}/W(k))$ depends only on $X$ and has right Betti numbers.
Grothendieck - use <u>local</u> lifts to char $0$ & patch together cohomology theory - good for smooth proper.
<u>Properties</u> i) Right Betti numbers, Poincaré duality, Künneth, cycle maps
ii) DeRham cohomology is "mod $p$ reduction" of crystalline:

$$0 \to H^i_{crys}(X) \otimes_{W(k)} k \to H^i_{DR}(X/k) \to \operatorname{Tor}_1^{W(k)}(H^{i+1}_{crys}(X)) \to 0$$

$\Rightarrow b^1_{crys} \leq b^1_{DR} \leq h^{1,0} + h^{0,1}$
iii) <u>Hodge theory</u> (Illusie) - get complex $W\Omega^\bullet_X$ - deRham Witt complex.
s.t. $H^k_{crys}(X) = \mathbb{H}^k(W\Omega^\bullet_X)$, quasi-isom to de R mod $p$.
The spectral sequence (slope s.s.) for this complex
$E_1^{ij} = H^j(W\Omega^i_X) \Rightarrow H^k_{crys}(X)$ degenerates at $E_1$ mod torsion

## Kodaira Vanishing

Serre - coherent sheaf ample $\Rightarrow$ cohomology vanishes

Thm $X$ smooth projective $/\mathbb{C}$, $L \to X$ ample line bundle.
Then $H^i(X, \omega_X \otimes L) = 0$ for $i > ~~\dim X~~ 0$

Kodaira (1953) proved that if $M$ compact complex manifold, $L \to M$ a holomorphic line bundle which possesses a Hermitian metric with everywhere $>0$ curvature $\Rightarrow H^i(M, \omega_M \otimes L) = 0$ $i > 0$.

This implies the algebraic version, but in fact they're equivalent by Kodaira embedding.

(1958) Akizuki-Nakano simplified & generalized proof, got stronger statement: under some assumptions $H^i(M, \Omega^j_M \otimes L) = 0$ for $i+j > \dim M$ — uses Hodge thm + existence of a Green operator.

There are algebraic proofs, but not known to what extent it generalizes in char. $p$.. Proofs:

- Raynaud (uses liftability to char 0) - algebraic
- Kollár (geometric proof) - separates geometry from Hodge theory..

Raynaud proof Thm: $X$ smooth, projective over $k$ perfect of char $p$, liftable to $W_2(k)$. Then if $L \to X$ is ample, then $H^j(X, \Omega^i \otimes L) = 0$ $\forall i+j > \max(\dim X, 2\dim X - p)$.

Cor Kodaira in char 0.

(Assumed) Proof From deRham decomposition theorem, we know that there is a quism $\varphi : (\Omega_X^\bullet, 0) \xrightarrow{\sim} (F_* \Omega_X^\bullet, d)$.

Also note that for any line bundle $M \to X$, $F^* M = M^{\otimes p}$.

By projection $\Rightarrow H^j(X, \Omega_X^i \otimes M^{\otimes p}) = H^j(X, F_*\Omega_X^i) \otimes M$

use $\mathcal{O}_X$-linearity of $d$ on $F_*\Omega_X^\bullet$

By decomposition, $\sum_{i+j=k} \dim H^j(\Omega^i \otimes M)$

$$= \dim \mathbb{H}^k((\Omega^\bullet \otimes M, 0)) = \dim \mathbb{H}^k((F_*\Omega_X^\bullet \otimes M, d))$$

$$\leq \sum_{i+j=k} \dim H^j(F_*\Omega^i \otimes M) \quad \text{(by spectral sequence)}$$

$$= \sum_{i+j=k} \dim H^j(\Omega^i \otimes M^{\otimes p}) \quad \text{(projection)}$$

- Serre vanishing, these eventually die ...

($H^j(\Omega_X^i \otimes L^{-\otimes p^m}) = 0$ for $m$ big enough, $i+j < d$). ■

Kollár's proof (Shafarevich map book). If a coherent cohomology has topological origin, then vanishing occurs..

The fact we need is that the natural map
$H^i(X, \mathbb{C}) \longrightarrow H^i(X, \mathcal{O}_X)$ is surjective $\forall i$ ($X$ smooth proper)

Digression on cyclic covers: For a vector bundle $E \to X$ denote by tot $(E)$ the total space of $E$

If $L \to X$ any line bundle, $s \in H^0(X, L^{\otimes n})$
$\Rightarrow$ canonical cyclic cover $X[\sqrt[n]{s}] \to X$ of deg $n$ with complete ramification along $D = \mathrm{div}(s)$.

It is the fiber product
$$\begin{array}{ccc} X[\sqrt[n]{s}] & \longrightarrow & \mathrm{tot}\, L \\ \downarrow & & \downarrow \text{mult} \\ X & \xrightarrow{\ s\ } & \mathrm{tot}\, L^{\otimes n} \end{array}$$
- special case of spectral cover

Equation for $X[\sqrt[n]{s}]$: Let $\lambda \in H^0(\mathrm{tot}\, L, p^* L)$ be the tautological section. Then $X[\sqrt[n]{s}]$ is the divisor of $\lambda^{\otimes n} - p^* s \in H^0(\mathrm{tot}\, L, p^* L^{\otimes n})$

Proposition $p_* \mathcal{O}_{X[\sqrt[n]{s}]} = \bigoplus_{i=0}^{n-1} L^{\otimes -i}$, and the $\mathcal{O}_X$-algebra structure is given by $L^{-a} \otimes L^{-b} \to L^{-a-b}$ if $a+b \leq n-1$.
$L^{-a} \otimes L^{-b} \to L^{-a-b} \xrightarrow{\otimes s} L^{-a-b+n}$, $a+b \geq n$.

Proof The short exact sequence for $X[\sqrt[n]{s}] \subset \mathrm{tot}(L)$
$$0 \to p^* L^{\otimes -n} \xrightarrow[\otimes(\lambda^{\otimes n} - p^* s)]{} \mathcal{O}_{\mathrm{tot}(L)} \to \mathcal{O}_{X[\sqrt[n]{s}]} \to 0$$
Push everything to $X$: $p_* \mathcal{O}_{\mathrm{tot}(L)} = \bigoplus_{i \geq 0} L^{-i} = \mathrm{Sym}\, L^{-1}$

$\Rightarrow$ the sheaf $p_* \mathcal{O}_{X[\sqrt[n]{s}]}$ is the cokernel $\bigoplus_{i \geq n} L^{-i} \xrightarrow{\otimes(\lambda^n - p^* s)} \bigoplus_{j \geq 0} L^{-j}$

Now $(\cdot \otimes \lambda^n) : L^{-n} \to L^{-n}$ is identity,
$(\cdot \otimes p^* s) : L^{-n} \to \mathcal{O}_X$ is the dual of
$s : \mathcal{O} \to L^{\otimes n}$ ∎

Remark: If $X, D$ smooth $\Rightarrow X[\sqrt[n]{s}]$ smooth
fiber product of two maps whose discriminant loci don't intersect is smooth.
— look in Kollár

<u>Proof of Kodaira</u> Let $s \in H^0(L^{\otimes n})$, $n$ big enough, s.t. $D = \operatorname{div} s$ is smooth & connected. Consider $Z = X[\sqrt[n]{s}] \xrightarrow{p} X$.

<u>Fact</u>: $H^i(X, L^{-1}(-bD)) \to H^i(X, L^{-1})$ ~~is~~ surjective for all $b \geq 0$. (gives Kodaira: we know LHS $= 0$ by Serre for $b$ big enough).

<u>Proof</u> Consider $Z$. Let $p_* \mathbb{C}_Z \to X$ be the pushforward of the constant sheaf. Decompose this sheaf under the characters of our cyclic Galois group.

$$p_* \mathbb{C}_Z = \bigoplus_{i=0}^{n-1} G_i \subset p_* \mathcal{O}_X = \bigoplus_{i=0}^{n-1} L^{-i}.$$

Arrange labels so that $G_i \subset L^{-i}$.

(Local systems with sings. along manifolds).

<u>Lemma</u>: $-G_0 = \mathbb{C}_X$, $-G_1 \subset L^{-1}(-bD) \subset L^{-1}$

<u>Proof</u> - <u>Later</u>.

Notice that since $Z$ is smooth, $H^i(Z, \mathbb{C}_Z) \twoheadrightarrow H^i(Z, \mathcal{O}_Z)$

- get same projection on any isotypical component.

In particular $H^i(X, G_1) \twoheadrightarrow H^i(X, L^{-1})$

By the lemma $H^i(X, G_1) \twoheadrightarrow H^i(X, L^{-1})$

$\searrow H^i(X, L^{-1}(-bD)) \nearrow$

$\Rightarrow$ Kodaira.

<u>Proof of Lemma</u>: $- G_0 = (p_* \mathbb{C}_Z)^{\mathbb{Z}/n} = \mathbb{C}_X$.

$- G_1 \subset L^{-1}$, $L^{-1}(-bD) \subset L^{-1}$.

$\Rightarrow$ local question to show $G_1 \subset L^{-1}(-bD)$.

the sheaves are different only near $D$...

Let $U \subset X$ be a connected open s.t. $U \cap D \neq \emptyset$.

What is $H^0(U, G_1|_U)$? It's zero -

local system outside $D$, nontrivial monodromy: $G_1$ has $\neq 0$ character, no invariants.

Locally on $U$ $Z = \mathbb{C}^m \to \mathbb{C}^m$,

$(x_1, \dots x_n) \to (x_1^n, x_2, \dots x_n)$

$\to$ need constants of homogeneity degree $> 0$

$\Rightarrow 0$. $\blacksquare$

General principle: If a geometric property can be expressed in terms of $H^q(X, \Omega^p)$ then it's "rigid"

Want to study moduli of $X$ smooth projective, $\omega_X$ trivial.

$\Rightarrow \exists s \in H^0(X, \omega_X)$ nowhere vanishing, gives

is: $T_X \xrightarrow{\sim} \Omega_X^{n-1}$ ($n = \dim X$)

$\Rightarrow H^1(X, T_X) \simeq H^1(X, \Omega_X^{n-1})$ infinitesimal deformations $\Rightarrow$

Thm (Bogomolov-Tian-Todorov) The moduli space of such $X$ is smooth.

Deformation Theory / $\mathbb{C}$

Examples: • In diff. geometry: - moduli of Riemannian metrics - too big & general - Einstein metrics: finite dim, maybe singular. (hard to show nonempty)

- complex structures (fin dim, exist ...)
- Kähler structures (good but uninteresting) ← Kodaira proved any deformation of Kähler as complex is Kähler.
- Extremal Kähler metrics (very interesting) - minimize a functional on a Kähler class, if you can. Subclasses of Ricci flat, constant scalar curvature. In general existence not governed by topology - Futaki geometric examples.

• In algebra - Associative algebras, Lie alg., reps
- If an object (algebra) has deformations, then representations don't have deformations & conversely.

• In alg. geometry - Schemes, sheaves etc.

Given $X$ geometric object $\Rightarrow$ Moduli $(X)$ $M$. If interested in local structure of $(M, 0)$ $\Rightarrow$ replace by $\hat{\mathcal{O}}_{M,0}$.

Try to describe it or rather $\hat{\mathcal{O}}_{M,0}^*$ - dual topological coalgebra

— Doran - annotated bibliography of deformation theory

ftp://www.math.harvard.edu/mazur-handout/ann2.ps.

Functors on Artin algebras

Art - the category of local Artin $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Set - the category of sets.

Def A deformation functor is any covariant functor $D: \mathrm{Art} \to \mathrm{Set}$. s.t. $D(\mathbb{C})$ is a one point set.

Basic construction: $\mathcal{C}$ a class of geometric objects.
$\sim$ an equivalence relation in $\mathcal{C}$, $X \in \mathcal{C}$ distinguished object
$\Rightarrow$ $D(\mathcal{C}, \sim, X)$: Art $\to$ set, $A \mapsto \{$equiv classes wrt $\sim$ of families $\tilde{X} \to \operatorname{Spec} A$ of objects in $\mathcal{C}$ s.t. $\tilde{X}/\operatorname{spec} \mathbb{C} = X$.
ProRepresentability = existence of formal moduli.
Ex X compact analytic space, $\mathcal{C}$ class of all complex manifolds, $\sim$ biholomorphism. $D$ : Art $\to$ Set
$A \to \{$set of families $\tilde{X} \to \operatorname{spec} A$, $\tilde{X}/\operatorname{spec} \mathbb{C} = X\}$
$= \{$coherent sheaves $\mathcal{O}_X^{\sim} \to X$ equipped with $A$-algebra structures s.t. $\mathcal{O}_X^{\sim}/m\,\mathcal{O}_X = \mathcal{O}_X\}$
Remarks 1) Same construction $\Rightarrow$ moduli functor
$D$: Sch $\to$ Set
2) Grothendieck general definition of a moduli functor:
In practice $(\mathcal{C}, \sim)$ comes from a category $\mathcal{G}$ of geometric objects, $Ob(\mathcal{G}) = \mathcal{C}$, $A, B \in Ob(\mathcal{G})$ are equivalent iff $\exists f: A \to B$ iso in $\mathcal{G}$
- all we need are the isomorphisms, so forget the rest of morphisms, and assume $\mathcal{G}$ is a groupoid.
Given any groupoid can recover set of equiv classes of objects in it - get moduli space with Galois group of automorphisms, isotropy group at any point.
Moduli functor in general is a functor
$M$: Sch $\to$ Set that factors through the 2-category of groupoids.
$\to$ Groupoids
Representability: could come from a scheme, or more generally a category $\mathcal{M} \xrightarrow{\pi}$ Sch s.t.
$M(S) = \pi^{-1}(S)$. $\mathcal{M}$ with topology is a stack

Kontsevich's approach to formal deformation theory.
$D$: Art $\to$ Groupoids. What kind of object can represent $D$?
Feigin's idea look for a D.G.L.A. representing $D$.
Def A DGLA is a $\mathbb{Z}$-graded complex vector space $\mathfrak{g}^\bullet = \oplus\, \mathfrak{g}^n$ equipped with graded bracket
$[\mathfrak{g}^n, \mathfrak{g}^m] \hookrightarrow \mathfrak{g}^{n+m}$ with axioms
- graded skew symmetry $[Y, X] = (-1)^{\deg X \deg Y + 1} [X, Y]$
- graded Jacobi: $[X, [Y, Z]] + (-1)^{\deg Z(\deg X + \deg Y)} [Z, [X, Y]] + (-1)^{\deg Y(\deg Z + \deg X)} [Y, [Z, X]] = 0$

$- \quad d[X,Y] = [dX,Y] + (-1)^{\deg X}[X,dY]$

To any $\mathfrak{g}^\bullet$ DGLA $\Rightarrow$ $D: \mathrm{Art} \to \mathrm{Groupoids}$

Start with $A = \mathbb{C} \oplus m$. Tensor with $\mathfrak{g}^\bullet$

$\mathfrak{g}^\bullet \otimes m \subset \mathfrak{g}^\bullet \otimes A$ inclusion of DGLA's.

The objects of the groupoid $D(A)$ will be all elements in $\mathfrak{g}^1 \otimes m$ satisfying Maurer-Cartan

$Z = \{ \gamma \in \mathfrak{g}^1 \otimes m \mid d\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \}$

Morphisms: look first at $\mathfrak{g}^0 \otimes m$, nilpotent Lie algebra. This acts on $\mathfrak{g}^1 \otimes m$ preserving $Z$, by affine vector fields $\mathfrak{g}^0 \otimes m \xrightarrow{v} \Gamma(\mathfrak{g}^1 \otimes m, T\mathfrak{g}^1 \otimes m)$

$v(x)_\gamma = dx + [\gamma, x]$ — gauge action

Lemma (1) $v$ is a Lie algebra homomorphism

(2) The infinitesimal action of $\mathfrak{g}^0 \otimes m$ on $\mathfrak{g}^1 \otimes m$ preserves $Z$.

Pf (2) $Z$ is zero scheme of $K: \mathfrak{g}^1 \otimes m \to \mathfrak{g}^2 \otimes m$

$\gamma \mapsto d\gamma + \frac{1}{2}[\gamma,\gamma]$. Must check that for any $\gamma \in \ker K$, $x \in \mathfrak{g}^0 \otimes m$, we have

$(L_{v(x)} K)(\gamma) \overset{?}{=} 0$.

$= d(v(x)_\gamma) + [v(x)_\gamma, \gamma]$

$= d(dx + [\gamma,x]) + [dx + [\gamma,x],\gamma]$

$= [d\gamma, x] - \cancel{[\gamma, dx]} + \cancel{[dx,\gamma]} + [[\gamma,x],\gamma] =$

$= -\frac{1}{2}[[\gamma,\gamma],x] + \frac{1}{2}([\gamma,x],\gamma] + \frac{1}{2}([\gamma,x],\gamma] = 0$ ∎

Recall $n$ nilpotent Lie algebra — can exponentiate it always to a group $\exp(n)$ = set of all $\exp(x)$, $x \in n$, with BCH multiplication.

Claim $\exp(\mathfrak{g}^0 \otimes m)$ acts on $Z$, action given by $\varphi \in \exp(\mathfrak{g}^0 \otimes m)$

takes $\gamma \in Z \Rightarrow \gamma \mapsto \varphi\gamma\varphi^{-1} - d\varphi\,\varphi^{-1}$

where if $\varphi = \exp x$, $\varphi\gamma\varphi^{-1} = \sum_{n \geq 0} \frac{1}{n!}(ad_x)^n \gamma$

$d(\exp x) = \int_0^1 [\exp(tx)\, dx\, \exp(1-t)x]\, dt$

$\Rightarrow d(\exp x)\exp(-x) = \sum \frac{1}{(n+1)!}(ad_x)^n(dx)$

$\Rightarrow$ Groupoid: $\gamma_1, \gamma_2 \in Ob(D(A)) := Z$, $\mathrm{Hom}(\gamma_1,\gamma_2) = \{\varphi \in \exp(\mathfrak{g}^0 \otimes m) \mid \varphi(\gamma_1) = \gamma_2\}$

Examples i. $X$ complex manifold, $Def_X$ deform complex structure.
$\mathfrak{g}^k = \Gamma(X, T^{1,0} \otimes \Lambda^k(T^{0,1})^*)$ — $(0,k)$ forms with coeffs $(1,0)$ vector fields.
Lie bracket on v. fields and $\wedge$ on forms differential $= \bar\partial$.
ii. $X$ $C^\infty$ manifold, $G$ Lie group, $V \to X$ principal $G$-bundle, $\nabla$ flat connection on $V$.
$\mathfrak{g}^k = ad_V \otimes \mathcal{A}^k_X$, obvious bracket.
$d = d^\nabla$ covariant derivative.
$d\gamma + \frac{1}{2}[\gamma,\gamma]$ usual Maurer-Cartan equation, $V(X)$ is gauge action.

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Theorem If $\mathfrak{g}^\bullet$ DGLA, $H^i(\mathfrak{g})$ its cohomology - $\mathbb{Z}$-graded
• Assume $\mathfrak{g}^{<0} = 0$, $H^0(\mathfrak{g}^\bullet) = 0$. Then $\exp(\mathfrak{g}^0 \otimes m)$ acts freely on $Z(A)$ for any $A = \mathbb{C} \oplus m$
• The deformation functor associated to $\mathfrak{g}$ is representable by a coalgebra $C$, i.e. $D(A) = Hom(A^*, C)$, $A^*$ topological dual coalg.
Moreover $C = H_0(\mathfrak{g}; \mathbb{C})$, Lie algebra homology
• If in addition $H^2(\mathfrak{g}^\bullet) = 0$, $C$ is (non canonically) isomorphic to $Sym^\bullet(H^1(\mathfrak{g}))$ — smoothness.

Meaning of the theorem: roughly it says, if there aren't any infinitesimal automorphisms $\Rightarrow$ the formal moduli exists. If ~~there aren't any obstructions to smoothness~~ the space where obstructions to smoothness live vanishes, the formal moduli is smooth. $\mathfrak{g}^\bullet$ quism to free Lie algebra which is Chevalley complex ... ?

Pro-representability & smoothness for deformations
$D: Art \to Sets$ deformation functor - complete to get formal schemes.
$\widehat{Art}$ - category of complete local noetherian $\mathbb{C}$-algebras $A$ s.t. $A/m^r \in Art$ $\forall r$
Def A couple for a deformation functor $D$ is a pair $(A, \xi)$, $A \in \widehat{Art}$, $\xi \in D(A)$. Morphism of couples $u: (A,\xi) \to (A',\xi')$ is a morphism $u: A \to A'$ s.t. $D(u)(\xi) = \xi'$

Remark If we extend $D$ to a functor $\hat{D}$ on $\widehat{Art}$ by $\hat{D}(A) = \varprojlim D(A/m^n) \Rightarrow$ pro-couples & morphisms.

If $R \in \widehat{Art}$ then we have $h_R : Art \to Set$, $A \mapsto Hom(R, A)$

If $(R, \xi)$ is a pro-couple $\Rightarrow$ morphism of functors $h_R \to D$:

$\xi \in \hat{D}(R) \Rightarrow \xi = \varprojlim \xi_n$, $\xi_n \in D(R/m^n)$

Let $u \in Hom(R, A)$. $A$ Artin $\Rightarrow \exists n$ s.t. $u : R \to A$, $R \to R/m^n \xrightarrow{u_n} A$

factor $u$ through quotient.

$\Rightarrow$ take $D(u_n)(\xi_n) \in D(A)$

Def $(R, \xi)$ pro-couple for $D$ pro-represents $D \iff$ $h_R \to D$ is iso.

Geometrically, if $D = D_{(\mathcal{C}, \sim, X)}$, pro-representability of $D$ by $(R, \xi)$ says $\hat{M} = Spec\, R$ formal moduli for $(\mathcal{C}, \sim, X)$ and a family of objects in $\mathcal{C}$ $\xi \xrightarrow{f} \hat{M}$

s.t. a) $f$ is complete, i.e. for any other family $\eta \to (S, s_0)$ pointed local scheme, we have $\eta$ is a pullback via a map $S \xrightarrow{\pi} \hat{M}$ locally around $s_0$.

b) $f$ is universal, i.e. the completion of $\pi$ at $s_0$ is uniquely determined.

Miniversal - complete universal family exists but not quite uniquely determined - only its differential is.

Sufficient conditions for pro-representability:

- Grothendieck general nec. & suff condition - left-exactness of the functor.
- Schlessinger conditions $\underbrace{(H1)(H2)(H3)}$, (H4), (H5)
  sufficient for hull / miniversal deformation. H1-H4 sufficient for pro-rep. H5 (Mark Gross) - has to do with smoothness. Used this to prove that if $X$ proper scheme $/\mathbb{C} \Rightarrow Def_X$ has a miniversal deformation (Kuranishi's theorem in complex geometry)
- If $X$ has $H^0(X, T_X) = 0 \Rightarrow Def_X$ is pro-representable.

How about smoothness? Assume $D$ has a hull. $(R, \xi)$

Want to test Spec $R$ for smoothness:

Reduce to a question that can be handled inductively.

$A_n = \mathbb{C}[t]/t^{n+1}$, $n \geq 1$, $S_n = \operatorname{Spec} A_n$, $\alpha_n : A_{n+1} \to A_n$.

Def $D : Art \to Set$ is called unobstructed if $\forall n \geq 0$ the map $D(\alpha_n) : D(A_{n+1}) \to D(A_n)$ is surjective

Prop Let $D$ be a deformation functor, pro-rep by $\hat{M}$. Then $D$ is unobstructed $\Leftrightarrow$ $\hat{M}$ is smooth

Proof Let $(R, \xi)$ be the pro-couple representing $D$. $\hat{M} = \operatorname{Spec} R$, $\xi \in \hat{D}(R)$. Inf lifting property $\Rightarrow$ $h_R(\alpha_{n+1})$ is onto for $\hat{M}$ smooth $\Rightarrow$ $D$ unobstructed.

$D$ unobstructed & $\hat{M}$ not smooth : $\hat{M} \subset \hat{A}$ = Zariski tangent at Spec $\mathbb{C}$ to $\hat{M}$. $= D(A_1)$

$\hat{M}$ not smooth $\Rightarrow$ $\exists$ formal line $L \subset \hat{A}$ not contained in the tangent cone to $\hat{M}$. But $D$ is unobstructed $\Rightarrow$ can lift $L$ to a formal curve $C \subset \hat{M}$.... contradiction ■

Remark Deformation problems in practice always give same objects: geometric object $X$ we want to deform $\leadsto$ get a sheaf or complex of sheaves $\mathcal{F}_X$ on a suitable space, that "linearizes" the geometry of $X$, i.e. $H^0(\mathcal{F}_X)$ = inf. automorphisms of $X$, $H^1(\mathcal{F}_X)$ = inf. deformations of $X$. $H^2(\mathcal{F}_X)$ = obstructions

Examples 1). $X$ proper smooth $/\mathbb{C}$, $\mathcal{F}_X = T_X$.

2) $X$ is a vector bundle on scheme $S$, $\mathcal{F}_X = \operatorname{End}_{\mathcal{O}_S} X$

Kontsevich If $D$ comes from a DGLA, then $H^i(\mathfrak{g}^\bullet)$, $i = 0,1,2$ give us above objects.

Def The tangent space of a def. functor $D$ is the set $t_D = D(\mathbb{C}[t]/t^2)$

Exercise Hartshorne III Ex 4.10: $t_{\operatorname{Def} X} = H^1(X, T_X)$ for smooth scheme.

Lemma If $D$ is pro-rep, $t_D$ has natural structure of $\mathbb{C}$-vector space

Pf : $\alpha, \beta \in t_D = D(A_1) \Rightarrow \alpha \times \beta \in D(A_1) \times_{D(\mathbb{C})} D(A_1)$

For any $D$ there is a map $\boxed{D(A_1 \otimes_{\mathbb{C}} A_1) \longrightarrow D(A_1) \times_{D(\mathbb{C})} D(A_1)}$ $*$

If $D$ is pro rep $\Rightarrow$ this is a bijection. Compose with diagonal map $\Rightarrow$ $\alpha + \beta$. ■

Remark Suffices $*$ to be bijection $\Rightarrow$ essentially Schlessinger's H2 (for any $A$.)

<u>Def</u> A deformation functor $D: Art \to Set$ has an obstruction space if $\exists$ a complex vector space $t^2_D$ s.t. for any surjection of Artin algebras $p: B \to A$ s.t. $\ker p \cdot m_B = 0$, there exists an functorial obstruction map $\delta_p : D(A) \to \ker p \otimes t^2_D$ which fits in an exact sequence $D(B) \xrightarrow{D(p)} D(A) \xrightarrow{\delta_p} \ker p \otimes t^2_D$
in the sense that $\delta_p(\alpha) = 0 \iff \alpha \in \operatorname{Im} D(p)$

– i.e. map to linear space measuring problem with lifting deformations from $A$ to $B$.

<u>Prop</u> Assume $D$ is pro-rep., then $D$ has an obstruction space

<u>Proof</u> · realize as quotient of polynomial ring where we can do it

Let $R$ be an algebra pro-representing $D$, $R = \mathbb{C}[[x_1 \dots x_r]]/J$ where $J \subset (x_1, \dots x_r)^2$ (to make $R$ local).

Set $t^2_D = \left(J/(x_1, \dots x_r)J\right)^\vee$.

To describe $\delta_p : D(A) \to \ker p \otimes t^2_D$, take $f \in D(A) = \operatorname{Hom}(R, A)$

Build $\delta_p(f)$ in stages:

- $\alpha_i := f(x_i)$ $i = 1, \dots r$
- lift $\alpha_i$ to $\beta_i \in B$
- use them to define $g: \mathbb{C}[[x_1, \dots x_r]] \to B$
- If $s \in J \Rightarrow f(s) = 0$ in $A \Rightarrow g(s) \in \ker p$. Moreover, if $s \in (x_1, \dots x_r)J$, then $g(s) = 0$ since $g$ is a local homomorphism and $\ker p \cdot m_B = 0$ $\Rightarrow$ $g$ induces a $\mathbb{C}$-linear map $J/(x_1 \dots x_r)J \to \ker p$ i.e. an element in $t^2_D \otimes \ker p$.

<u>Exercise</u> $X$ smooth proper $/\mathbb{C}$, $H^0(X, T_X) = 0$, show $H^2(X, T_X)$ is the obstruction space for $Def_X$.

<u>Kawamata-Ran Criterion</u>

<u>Def</u> (tangent functor) : $D: Art \to Set$ deformation functor

Then the inf deformations of $D$ give a functor assigning to every $D$-couple $(A, \xi)$ the set
$T^1_D(A, \xi) = \{ \eta \in D(A \otimes_{\mathbb{C}} A_1) \mid D(p_A)(\eta) = \xi$ where $p_A : A \otimes A_1 \to A \}$

– infinitesimal extensions of our given family over dual numbers or

<u>Remarks</u> i. If $D$ is prorep, the same constr. as before gives us a canonical structure of $A$-module on $T^1_D(A, \xi)$

ii) Kawamata-Ran reformulates unobstructedness of $D$ as a question about $T^1_D$, which is a linear object.

— i.e. this is a linearization procedure

Let $A_n = \mathbb{C}[t]/(t^{n+1})$, $S_n = \operatorname{Spec} A_n$, $\alpha_n : A_{n+1} \to A_n$

<u>Def</u> We say a functor $D : \mathrm{Art} \to \mathrm{Set}$ satisfies the $T^1$ lifting property if for any $n \in \mathbb{Z}_{>0}$, $\xi_{n+1} \in D(A_{n+1})$, the natural linear morphism $T^1_D(\alpha_n) : T^1_D(A_{n+1}, \xi_{n+1}) \to T^1_D(A_n, \xi_n)$ is surjective.

Geometrically this means $\forall n \geq 1$, any family $\xi_{n+1} \to S_{n+1}$, any inf. deformation of the projection family $\xi_n \to S_n$ can be lifted to an inf. def of $\xi_{n+1}$

— linear problem — first order extension of linear object ...

<u>Theorem</u> If $D$ is prorep and satisfies $T^1$-lifting, then $D$ is unobstructed.

<u>Proof</u> Want to check that $\forall n \geq 1$ the map $D(\alpha_n) : D(A_{n+1}) \to D(A_n)$ is surjective. Set $B_n = A_n \otimes A_1 = \mathbb{C}[x,y]/(x^{n+1}, y^2)$

$C_n = B_n \otimes_{A_{n+1}} A_n = \mathbb{C}[x,y]/(x^{n+1}, yx^n, y^2)$

Natural maps $\alpha_n : A_{n+1} \twoheadrightarrow A_n$ $\quad\quad \beta_n : B_n \twoheadrightarrow A_n$

$\gamma_n : B_n \to C_n$

$\varepsilon_n : A_{n+1} \to B_n \quad t \mapsto x+y$

$\varepsilon_n' : A_n \to C_n \quad t \mapsto x+y$

Let $t^2_D$ = obstruction space $\Rightarrow$ commutative diagram

$$\left.\begin{array}{ccccc}
D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) & \xrightarrow{\delta_{\alpha_n}} & t^2_D \otimes (t^{n+1}) \\
\downarrow D(\varepsilon_n) & & \downarrow D(\varepsilon_n') & & \downarrow \mathrm{id} \otimes \varepsilon_n \\
D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) & \xrightarrow{\delta_{\gamma_n}} & t^2_D \otimes (x^n y)
\end{array}\right\} \text{Auxiliary extension.}$$

But $\varepsilon_n$ sends $(t^{n+1})$ to $(x^n y)$ isomorphically as $\mathbb{C}$-vector spaces:

$\varepsilon_n(t^{n+1}) = (x+y)^{n+1} \bmod (x^{n+1}, y^2) = (n+1)x^n y \bmod (x^{n+1}, y^2)$

— only place where we need char $= 0$.

$\Rightarrow \delta_{\gamma_n} = 0 \Leftrightarrow \delta_{\alpha_n} = 0$

To prove $\delta_{\gamma_n} = 0$ use $T^1$ lifting property: consider

$p_n : C_n \to A_n$, $\quad x \mapsto t$, $y \mapsto 0$.

$q_n : C_n \to B_{n-1}$, $\quad x \mapsto x$, $y \mapsto y$

Let $\xi \in D(C_n)$. $\quad D(p_n)\xi \in D(A_n)$, $D(q_n)\xi \in D(B_{n+1})$

By def $C_n = B_{n+1} \otimes_{A_{n+1}} A_n$

$\Rightarrow$ by pro rep of $D$, $\quad \xi = D(p_n)\xi \underset{D(A_{n+1})}{\times} D(q_n)\xi$

But $D(q_n)\xi \in T^1_D(A_{n-1}, D(\alpha_{n-1} \circ D(p_n)\xi)$

By $T^1$ lifting, can find $\eta \in T^1_D(A_n, D(p_n)(\xi)) \subset D(B_n)$
s.t. $\eta$ maps to $D(q_n)(\xi)$ in $D(B_{n+1})$

By pro rep $D(k_n)(\eta) = (\text{image of } \eta \text{ in } D(B_{n+1})) \times_{D(A_{n+1})} \left(\begin{array}{c}\text{image of } \eta \\ \text{in } D(A_n)\end{array}\right)$
$= D(q_n)\xi \times D(p_n)\xi = \xi$. ■

<u>Remarks</u> i) Don't need $D$ pro-rep: need a hull, + linear structure of tangent functor. First comes from $H_1$–$H_3$, latter either by $H_4$ or weaker $H_5$:

ii) <u>Condition H5</u> Recall that $\forall A' \to A$, $A'' \to A$ in Art, have canonical map $\psi_{A',A'',A} : D(A' \times_A A'') \to D(A') \times_{D(A)} D(A'')$

The condition (H5) is: for any surjections $A' \to A$, $A'' \to A$ $\exists$ a map $\varphi_{A',A'',A} : D(A') \times_{D(A)} D(A'') \to D(A' \times_A A'')$
s.t. $\psi \circ \varphi = id$, and $\varphi$ is universal - i.e.

$\forall$
$$\begin{tikzcd} B \arrow[r] \arrow[d] & A'' \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$$
the maps
$$\begin{tikzcd} D(B) \arrow[r, "t_1"] \arrow[d, "t_2"] & D(A' \times_A A'') \\ D(A') \times_{D(A)} D(A'') \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

<u>Exercise</u> ~~Fact (Gross)~~ If $D$ has a hull & satisfies H5 then $D$ is unobstructed iff $T^1$ lifting holds.

<u>Fact (Gross)</u> If $X$ is any scheme $\Rightarrow$ $Def_X$ satisfies $H_5$

## $\boxed{\text{Applications}}$

Theorem (Bogomolov–Tian–Todorov): $X$ is a smooth proj variety, with torsion $[\omega_X]$ $\Rightarrow$ $Def_X$ is unobstructed

<u>Proof</u> (Ran) The same calculation as for $t^1_{Def\,X}$ gives
$\forall A \in Art$, $Y \in Def_X(A)$ $\Rightarrow$ $T^1_D(A, Y) = H^1(Y, T_{Y/A})$

Now assume $\omega_X$ is trivial.

<u>Fact 1</u> For any family $X_{n+1} \in D(A_{n+1})$, $\omega_{X_{n+1}/S_{n+1}}$ is also trivial.

By the main corollary from degen of Hodge–deRham,
$H^0(X_{n+1}, \omega_{X_{n+1}/S_{n+1}})$ is a free $A_{n+1}$ module
$= \underset{\mathbb{C}}{\underset{\|}{H^0(X, \omega_X)}} \otimes A_{n+1}$

**Fact 2** $\forall X_{n+1} \in D(A_{n+1})$ by relative duality & fact 1 we have
$\Omega^{d-1}_{X_{n+1}/S_{n+1}} = (\Omega^1_{X_{n+1}/S_{n+1}})^\vee \otimes \omega_{X_{n+1}/S_{n+1}} = T_{X_{n+1}/S_{n+1}}$

$T^1$ lifting says $H^1(\Omega^{d-1}_{X^{n+1}/S^{n+1}}) \xrightarrow{r_n} H^1(\Omega^{d-1}_{X_n/S_n})$ is onto:

use Hodge theory $\Rightarrow H^1(\Omega^{d-1}_{X^{n+1}/S^{n+1}}) = H^1(X, \Omega^{d-1}_X) \otimes A_{n+1}$, (trivial bundle)
same over $S_n \Rightarrow$ surjection.
Torsion case $\Rightarrow$ get $\pi: \tilde{X} \to$ cyclic étale cover,
s.t. $\pi^* \omega_X = \omega_{\tilde{X}}$ is trivial
$T_{\mathrm{Def}\,X} \hookrightarrow T_{\mathrm{Def}\,\tilde{X}}$ $\Rightarrow$ $T^1$ lifting for $X$ $\square$

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<u>Deformations of maps & subvarieties</u>
Setup: $X, Y$ varieties, $f: X \to Y$ morphism. study several def functors:
- $D_{(X,f,Y)}$ - local moduli of the triple $X \xrightarrow{f} Y$
- $D_{(X,f)}$ - " " " " " with $Y$ fixed
- $D_{(X \subset Y)}$ - " " " " a pair $X \subset Y$, both moving
- $D_{(X \subset)}$ - local embedded deformations of $X$ in fixed $Y$
- $D_{(Y)}$ - deformations of $Y$

Remark - all these are specializations of $D(X,f,Y)$ $D(X,f) \to D(X,f,Y)$ $\downarrow$ $D(Y)$
we can use fibering to study $D_{(X,f,Y)}$ by reducing it to others
In particular can try to infer the obstruction complex for $D_{(X,f,Y)}$ from those of others.

<u>Sufficient conditions for pro-rep:</u>
- If $Y$ is smooth & proper, then $D_{(Y)}$ has a hull, and if $H^0(T_Y) = 0 \Rightarrow D_{(Y)}$ is pro-rep (Kuranishi)
- If $Y$ proper $\Rightarrow D_{(Y)}$ has a hull. If $\tau_Y$ - derivations of $\mathcal{O}_Y$, and if $H^0(\tau_Y) = 0 \to$ pro-rep (Schlessinger).
- If $Y$ projective, $X$ arbitrary $\Rightarrow D_{(X \subset)}$ is pro-rep (Grothendieck - Hilbert scheme).
- If $X, Y$ smooth, proper then $D_{(X,f,Y)}$ has a hull, and if $f: X \to Y$ has no inf. automorphisms $\Rightarrow$ prorep. (Horikawa)
- $X, Y$ proper same is true (Grothendieck)

A list of linearizing complexes for X,Y smooth (up to quism):

| Functor | Complex |
| --- | --- |
| $D(Y)$ | $T_Y$ |
| $D(X_c)$ | $N_{X/Y}[-1]$ |
| $D(X\subset Y)$ | $T_Y \to f_* N_{X/Y}$ |
| $D(X,f)$ | $T_X \xrightarrow{df} f^* T_Y$ |
| $D(X,f,Y)$ | $f_* T_X \oplus T_Y \longrightarrow f_* f^* T_Y$ |

- works for $H^1, H^2$, don't always get right inf automorphisms.

<u>Remarks</u> (1) $T_X \to T_{Y|X} \to N_{X/Y}$ exact $\Rightarrow$ so $N_{X/Y}[-1]$ is quism to $T_X \to f^* T_Y$ when $X \subset Y$

(2) Another check is look at $D(X_c) \to D(X\subset Y)$
$\downarrow$
$D(Y)$

- comes from natural short exact sequence

$$0 \to \begin{bmatrix} 0 \\ \downarrow \\ f_* N_{X/Y} \end{bmatrix} \longrightarrow \begin{bmatrix} T_Y \\ \downarrow \\ f_* N_{X/Y} \end{bmatrix} \longrightarrow \begin{bmatrix} T_Y \\ \downarrow \\ 0 \end{bmatrix} \to 0$$

Ziv Ran's approach to deformations of general $f: X \to Y$

Take $A$ - $\mathcal{O}_X$ mod, $\mathcal{B}$ - $\mathcal{O}_Y$ mod. Put

$\mathrm{Hom}_f(\mathcal{B}, A) = \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{B}, A) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{B}, f_* A)$

$f$-linear maps

<u>Goal</u> given two $f$-linear morphisms $\delta_i \in \mathrm{Hom}_f(\mathcal{B}_i, A_i)$ $i = 0,1$ define functorial groups $\mathrm{Ext}^i(\delta_1, \delta_0)$ satisfying:

1) $\mathrm{Ext}^0(\delta_1, \delta_0)$ is pairs $\alpha: A_1 \to A_0$, $\beta: \mathcal{B}_1 \to \mathcal{B}_0$ fitting in the commutative diagram

$$\begin{array}{ccc} f^*\mathcal{B}_1 & \xrightarrow{f^*\beta} & f^*\mathcal{B}_0 \\ \delta_1 \downarrow & & \downarrow \delta_0 \\ A_1 & \xrightarrow{\alpha} & A_0 \end{array}$$

2) $\mathrm{Ext}^1(\delta_1, \delta_0)$ is the set of all $A_2, \mathcal{B}_2, \delta_2$ fitting in short exact $0 \to [0] \to [2] \to [1] \to 0$

3) There's a long exact

$0 \to \mathrm{Hom}(\delta_1, \delta_0) \to \mathrm{Hom}(A_1, A_0) \oplus \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_0) \to \mathrm{Hom}_f(\mathcal{B}_1, A_0) \to$

$\to \mathrm{Ext}^1(\delta_1, \delta_0) \to \mathrm{Ext}^1(A_1, A_0) \oplus \mathrm{Ext}^1(\mathcal{B}_1, \mathcal{B}_0) \to \mathrm{Ext}^1_f(\mathcal{B}_1, A_0) \to$

$\to \dots$

where $\mathrm{Ext}^i_f(\mathcal{B}, A)$ is the derived functor of $\mathrm{Hom}_f$ in either variable. $\Rightarrow$ s.s. $E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_X}(L^q f^* \mathcal{B}, A) \Rightarrow \mathrm{Ext}^{p+q}_f(\mathcal{B}, A)$

${}'E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_Y}(\mathcal{B}, R^q f_* A) \Rightarrow \mathrm{Ext}^{p+q}_f(\mathcal{B}, A)$

4) If $f$ inclusion and $\delta_1: \mathcal{B}_1 \twoheadrightarrow f_* A_1$ with kernel $K$ $\Rightarrow$ exact sequence

$0 \to \mathrm{Hom}(\delta_1, \delta_0) \to \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_0) \to \mathrm{Hom}(K, \mathcal{B}_0) \to$

$\to \mathrm{Ext}^1(\delta_1, \delta_0) \to \dots$

To construct $Ext^i(\delta_1, \delta_0)$ first define a Grothendieck topology $T$ associated with $f$:

Open sets in $T$ will be pairs $(U, V)$, $U \subset X$ Zariski open, $V \subset Y$ Z. open, $f(U) \subset V$. The coverings of $(U, V)$ will be collections $\{(U_\gamma, V_\gamma)\}$ s.t. $U_\gamma$ cover $U$, $V_\gamma$ cover $V$.

Define $\mathcal{O}_T$ - structure sheaf on noncommutative rings

$\mathcal{O}_T((U,V)) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},\ a \in \mathcal{O}_Y(V),\ b, c \in \mathcal{O}_X(U) \right\}$

with multiplication $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ f^*a' b + b'c & cc' \end{pmatrix}$

Lemma The category of $f$-linear maps is equivalent to the category of left $\mathcal{O}_T$-modules.

Pf The two mutually inverse functors

$\{f\text{-linear maps}\} \xrightarrow{\sigma} \{\mathcal{O}_T\text{-modules}\}$

$\sigma$ : $\delta : f^*B \to A$ to $B \oplus A \to T$ trivial sheaf with $\mathcal{O}_T$ action $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} ay \\ b\delta(y) + cx \end{pmatrix}$

Inverse functor $\tau$:

$\mathcal{O}_T$ module $E \mapsto$ triple $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot E$, $\mathcal{O}_X$-module,

$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot E$ $\mathcal{O}_Y$-module, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\square$

Define $Ext^i(\delta_1, \delta_0) := Ext^i_{\mathcal{O}_T}(\sigma(\delta_1), \sigma(\delta_0))$ Yoneda Ext.

Thm If $D = \rho_{(X, f, Y)}$ and $\delta_1 : f^*\Omega^1_Y \to \Omega^1_X$, $\delta_0 : \mathcal{O}_Y \to f_* \mathcal{O}_X$, then $t^i_D = Ext^i(\delta_1, \delta_0)$ $i = 0, 1, 2 ..$

Moral : in general deformation theory need closed model categories & derived categories of additive categories ..

<u>Bloch semiregularity map</u> Want to study $\mathcal{O}_{X \in S}$

$Y$ smooth, projective of dim $n$, $X$ local complete intersection of codimension $p$ - check if $X \in Hilb(Y)$ is a smooth point.

Bloch introduced a special linear map whose injectivity $\Rightarrow$ smoothness in $X \in Hilb(Y)$

Notation - $I$ ideal of $X$ in $Y$.

$N_{X/Y} = Hom_{\mathcal{O}_X}(I_X / I_X^2, \mathcal{O}_X)$

L.C.I $\Rightarrow$ $N_{X/Y}$ is a vector bundle of rank $p$.

Standard exact sequence $0 \to N^\vee_{X/Y} \xrightarrow{\iota} \Omega^1_{Y|X} \to \Omega^1_X \to 0$

$\omega_Y$ canonical bundle of $Y$, $\omega_X$ dualizing sheaf of $X$ (line bundle here)

$\omega_{X/Y} = \omega_X \otimes \omega^{-1}_{Y|X} = \Lambda^p N^\vee_{X/Y}$ relative dualizing sheaf

**Def** $X \subset Y$ as before is called semiregular if $\pi_X$ is injective.

**Example** Hilbert scheme of a curve in its Jacobian:
$C$ smooth genus $g \geq 2$, $J_C$ = Jacobian (degree 1)
In general $Hilb(J)$ is very bad. $Hilb(C,J)$
Set theoretically can identify $Hilb(C,J)$ with $J^0$ (as reduced schemes). More precisely let $C$ be a very general pt in $M_g$, s.t. rank $NS(J) = 1$, then $Hilb(J)_{red} = J^0$.
Indeed $J^0$ acts on $J$, hence on $Hilb(C,J)$, suffices to check this action is simply transitive on closed points. If $X \subset J$ a curve, $[X] \in Hilb(J)$ then normalization $\tilde{X} \to J$, but arithmetic genus of $X = g \Rightarrow$ smooth curve $\tilde{X}$ of genus $\leq g$ maps to $J$. If gen $\tilde{X} < g$, then get map $\neq 0$ $Jac\,\tilde{X} \to Jac\,X$ — contradiction since rank $(NS) = 1 \Rightarrow J$ is indecomposable. $\Rightarrow X$ is smooth & moreover $J(X) \cong J(C)$, only one polarization — isomorphic as principally polarized Ab. varieties
$\Rightarrow$ by Torelli: $C \cong X$.
$\Rightarrow$ The isomorphism $C \xrightarrow{\sim} X$ induces an automorphism of $J$ $\Rightarrow J^0$ acts transitively on $Hilb(C,J)_{red}$.
If $T_\alpha$ (translation by $\alpha$) stabilizes $C \Rightarrow \alpha$ is torsion, $C/\langle\alpha\rangle \subset J/\langle\alpha\rangle$, curve of lower genus sitting in isogenous A.V., contradiction with irreducibility of $J$. ∎

**Lemma** Let $C$ smooth of genus 3, nonhyperelliptic.
$\Rightarrow C \subset J$ is semiregular.
**Proof** $\pi : H^1(N_{C/J}) \to H^3(J, \Omega^1_J)$
By def this is dual to $*: H^0(J, \Omega^3_J) \to H^0(C, \Omega^2_J|_C)$
$\varphi \to H^0(C, N^\vee \otimes \omega_C)$, suffices to show $\varphi$ is surjective.
$\varphi$ defined as follows
$$0 \to N^\vee_{C/J} \to \Omega^1_{J|C} \to \omega_C \to 0$$
Take $\Lambda^2$:
$$0 \to \Lambda^2 N^\vee \to \Omega^2_{J|C} \to N^\vee \otimes \omega_C \to 0$$ (part of Koszul complex)

In cohomology
$$0 \to H^0(\Lambda^2 N^\vee) \to H^0(\Omega^2_{J|C}) \xrightarrow{\varphi} H^0(N^\vee \otimes \omega_C) \to$$
$$\to H^1(\Lambda^2 N^\vee) \to H^1(\ ) \to H^1(N^\vee \otimes \omega_C) \to 0$$

$\omega_{X/Y} = \omega_X \otimes \omega_{Y/X}^{*} = \Lambda^p N_{X/Y}$

We have contraction maps $(\Omega^{n-k}_Y)^\vee \otimes \omega_Y \xrightarrow{\sim} \Omega^k Y$

$N^\vee_{X/Y} \otimes \omega_{X/Y} \xrightarrow{\sim} \Lambda^{p-1} N_{X/Y}$

$\varepsilon$ gives $\Lambda^{p-1}\varepsilon : \Lambda^{p-1} N^\vee_{X/Y} \to \Omega^{p-1}_{Y|X}$

$\Rightarrow \Omega^{n-p+1}_{Y|X} \longrightarrow N^\vee_{X/Y} \otimes \omega_X$ inducing a map on cohomology

(✱) $H^{n-p-1}(Y, \Omega^{n-p-1}_Y) \longrightarrow H^{n-p-1}(X, N^\vee \otimes \omega_X)$

<u>Def</u> The semiregularity map $\pi$ of $X$ is the dual to ✱ :

$\pi : H^1(X, N_{X/Y}) \longrightarrow H^{p+1}(Y, \Omega^{p-1}_Y)$

Q: Is the Hilbert point $[X] \in \mathrm{Hilb}(X/Y)$ smooth? <u>3/17</u>

($Y$ smooth proj, $X \subset Y$ l.c.i)

$\Leftrightarrow$ is $D_{(X\subset)}$ unobstructed?

Bloch: this implied by injectivity of semiregularity map.

$\Lambda^{p-1}\varepsilon \in H^0(\Lambda^{p-1} N_{X/Y} \otimes \Omega^{p-1}_{Y/X})$

$\Lambda^{p-1} N \otimes \Omega^{p-1}_{Y/X} = N^\vee \otimes \omega_{X/Y} \otimes \omega^{p-1}_{Y/X}$

$= N^\vee \otimes \omega_X \otimes (\Omega^{n-p-1}_{Y/X})^\vee$

$\Rightarrow \Lambda^{p-1}\varepsilon : \Omega^{n-p+1}_{Y/X} \longrightarrow N^\vee \otimes \omega_X$

induces ✱ above.

$\pi : H^1(X, N) \longrightarrow H^{p+1}(Y, \Omega^{p-1}_Y)$ map defined on the obstruction space!

<u>Example</u> If $X \subset Y$ is a divisor, $\pi : H^1(X, N) \to H^2(Y, \mathcal{O})$

- comes from edge hom of $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(X) \to N \to 0$, $N = \mathcal{O}_X(X)$
- discovered by Kodaira-Spencer, who proved $\pi$ injective $\Rightarrow$ $X$ smooth pt of $\mathrm{Hilb}(Y)$

<u>Def</u> Let $X \subset Y$ be as before, then the inf. Abel-Jacobi map for $X$ is the morphism

$\sigma : H^0(X, N) \longrightarrow H^p(Y, \Omega^{p-1}_Y)$,

which is dual to

$\Lambda^p\varepsilon : H^{n-p}(\Omega^{n-p+1}_Y) \longrightarrow H^{n-p}(N^\vee \otimes \omega_X)$

- introduced by Clemens, differential of A-J map to Griffiths intermediate Jacobian.

Thm (S. Bloch) If $X \subset Y$ as before & the semiregularity map $\pi$ for $X$ is injective $\Rightarrow$ $D_{(X \subset)}$ is unobstructed.

Proof (Kawamata)

As before have $A_n$, $S_n = \operatorname{Spec} A_n$, $f_n (: S_n \to S_{n+1}$ ? )

say $Y_n = Y \times S_n$. We'll check $T'$ lifting holds, $\Rightarrow$ the theorem.

suppose we are given a flat deformation

$$\begin{array}{ccc} X_n & \subset & Y_n \\ & \searrow & \downarrow \\ & & S_n \end{array}$$

$T'((X_n, S_n)) = H^0(X_n, N_{X_n/Y_n})$

Need to check that $H^0(N_{X_n/Y_n}) \to H^0(N_{X_{n-1}/Y_{n-1}})$ is surjective

Denote by $\sigma_n : H^0(N_{X_n/Y_n}) \to H^p(\Omega^{p-1}_{Y_n})$

the relative inf A-J map

Standard exact sequences:

Restriction $\quad 0 \to \Omega^{p-1}_Y \otimes (t^{n+1}) \to \Omega^{p-1}_{Y_n/S_n} \otimes \mathcal{O}_{n-1} \to \Omega^{p-1}_{Y_{n-1}/S_{n-1}} \to 0$

Inflation $\quad N_{X_{n-1}/Y_{n-1}}$ can be inflated to $N_{X_n/Y_n}$ as an $\mathcal{O}_{X_{n-1}}$ module

$0 \to N_{X/Y} \otimes (t^{n+1}) \to N_{X_n/Y_n} \to N_{X_{n-1}/Y_{n-1}} \to 0$

Remark If we're looking at usual deformations of a variety $X$ we know that a ring extension $0 \to \mathcal{O}_X \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_X \to 0$ (square 0 ideal)

$\Longleftrightarrow$ ext of $\mathcal{O}_X$ modules $\quad 0 \to \mathcal{O}_X \to T_{X/S_1} \to T_X \to 0$

So if $X$ is smooth e.g., then $f : X \to Y$

$$\begin{array}{ccccc} \mathcal{O}_{X_n} & \longrightarrow & T_{X_n} & \longrightarrow & T_{X_{n-1}} \\ \downarrow & & \downarrow df & & \downarrow df \\ f^*\mathcal{O}_{Y_n} & \longrightarrow & f^*T_{Y_n} & \longrightarrow & f^*T_{Y_{n-1}} \end{array}$$

Using these sequences we get a commutative diagram

$$\begin{array}{ccccc} H^0(N_{X_n/Y_n}) & \xrightarrow{*} & H^0(N_{X_{n-1}/Y_{n-1}}) & \xrightarrow{\beta} & H^1(N_{X/Y}) \otimes (t^{n+1}) \\ \sigma_n \downarrow & & \sigma_{n-1} \downarrow & & \downarrow \pi \\ H^p(\Omega^{p-1}_{Y_n/Y_{n-1}}) & \xrightarrow{\alpha} & H^p(\Omega^{p-1}_{Y_{n-1}}) & \longrightarrow & H^{p+1}(\Omega^{p-1}_Y) \otimes (t^{n+1}) \end{array}$$

We want to show $*$ surjective $\to$ show $\beta = 0$.

$Y$ doesn't deform, $Y_n = Y \times S_n$, $Y_{n-1} = Y \times S_{n-1}$

$\Rightarrow \alpha$ surjective, $\Rightarrow \pi \circ \beta = 0$.

But $\pi$ is injective $\Rightarrow$ $\beta = 0$

■

$\varphi$ is surjective $\Leftrightarrow h^1(\Lambda^2 N^\vee) + h^1(N^\vee \otimes \omega_C) = h^1(\Omega^2_{J|C})$

$$\overset{\shortparallel}{h^1(C, \Omega^{\oplus \binom{3}{2}})} = 9$$

$h^1(\Lambda^2 N^\vee)$: $\Lambda^2 N^\vee = \omega^\vee_{C/J} = \omega_C^{-1}$

" $h^1(T_C) = 6$

To calculate $h^1(N^\vee \otimes \omega_C)$ tensor $0 \to N^\vee_{C/J} \to \Omega^1_{J|C} \to \omega_C \to 0$ with $\omega_C$

$$0 \to N^\vee \otimes \omega_C \to \Omega^1_{J|C} \otimes \omega_C \to \omega_C^{\otimes 2} \to 0$$

The map $H^0(\Omega^1_{J|C} \otimes \omega_C) \to H^0(\omega_C^{\otimes 2})$

$$\overset{\shortparallel}{H^0(\omega_C) \otimes H^0(\Omega^1_J)} \xrightarrow{\mu} H^0(\omega_C)^{\otimes 2}$$

nonhyperelliptic $\Rightarrow$ $\mu$ is surjective (Noether)

$\Rightarrow h^0(N^\vee \otimes \omega_C) = 3$, $h^1(\omega_C^{\otimes 2}) = 0$

$h^1(\Omega^1_{J|C} \otimes \omega_C) = 0 \Rightarrow h^1(N^\vee \otimes \omega_C) = 3$

But note $h^1(N) \neq 0$ $\Longrightarrow$, obstruction space doesn't vanish

I. Thm (Ziv Ran) $Y$ smooth CY, $X \overset{f}{\subset} Y$ smooth divisor $\Rightarrow D(X, f, Y)$ smooth & $D(X,f,Y) \to D(Y)$ is smooth.

Thm $X$ variety st. $T^1$ lifting holds for $D(X)$, $C \hookrightarrow X$ l.c.i. curve, $H^1(C, N_{C/X}) = 0$ $\Rightarrow D(C \subset X)$ is unobstructed

Cor. $X$ 3-dim CY, $C \subset X$ rigid l.c.i (in embedded sense) $\Rightarrow D(C \subset X)$ unobstructed.

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Proof of Thm I (Ziv Ran)

Definition Given a deformation functor $D$: Art $\to$ Set a functor $F$: $D$-couples $\to$ modules/Artin algebras is called deformation invariant if $F$ has the base-change property & if $F(A, \xi)$ is a free $A$-module (Like cohomology of a relative sheaf of forms ..)

Note Given $D \rightsquigarrow T^1_D$. If $T^1_D$ is deformation invariant then $T^1$ lifting holds (Nakayama

Example: If $X_A \to \operatorname{Spec} A$, $D = \operatorname{Def} X$ $D$-couple then $R^q f_* \Omega^p_{X_A}$ is deformation invariant [for $X$ proper, smooth]

$A$ - artin local algebra. $\begin{array}{c} X_A \xrightarrow{f_A} Y_A \\ \searrow \ \swarrow \\ S_A \end{array} \in D_{(X,f,Y)}(A)$

$T^1_D(X_A, f_A, Y_A) = \mathbb{H}^1(T_{Y_A/S_A} \to f_{A*} N_{X_A/Y_A})$

$0 \to \begin{bmatrix} 0 \\ \downarrow \\ f_{A*} N_{X_A/Y_A} \end{bmatrix} \to \begin{bmatrix} T_{Y_A/S_A} \\ \downarrow \\ f_{A*} N_{X_A/Y_A} \end{bmatrix} \to \begin{bmatrix} T_{Y_A/S_A} \\ \downarrow \\ 0 \end{bmatrix} \to 0$

We saw that $Y$ CY $\Rightarrow \omega_{Y_A/S_A} = \mathcal{O}_{Y_A}$

$\Rightarrow T_{Y_A/S_A} = T_{Y_A/S_A} \otimes \omega_{Y_A/S_A} = \Omega^{n-1}_{Y_A/S_A}$

$N_{X_A/Y_A} = \omega_{X_A/S_A}$ Also not $H^0(T_{Y_A/S_A}) = 0$ for CY ($Y_{A_n}$)

In cohomology: $0 \to H^0(X_A, \omega_{X_A/S_A}) \to \mathbb{H}^1(T_{Y_A/S_A} \to f_{A*} N_{X_A/Y_A}) \to$

$\to H^1(\Omega^{n-1}_{Y_A/S_A}) \xrightarrow{f_A^*} H^1(X_A, \omega_{X_A/S_A})$

Now $H^0(X_A, \omega_{X_A/S_A})$ is deformation invariant $\Rightarrow$ map smooth

$\ker(H^1(\Omega^{n-1}_{Y_A/S_A}) \xrightarrow{f_A^*} H^1(\omega_{X_A/S_A}))$ is deformation invariant as well - both sides free, look at closed point - map of vector spaces

$\Longrightarrow T^1_D$ def invariant : total space smooth. ■

<u>Proposition</u> $X$ variety s.t. $T^1$ lifting holds for Def $X$. $C \hookrightarrow X$ lci of dim one, & assume $H^1(N_{C/X}) = 0$

$\Rightarrow D(C \subset X)$ is unobstructed.

<u>Proof</u> Start with $\begin{array}{c} C_n \hookrightarrow X_n \\ \pi \searrow \ \downarrow \\ S_n \end{array}$ flat deformation of $C \subset X$

Let $C_{n-1} \overset{f_{n-1}}{\subset} X_{n-1}$ be the restriction on $S_{n-1}$.

$T^1_{D(C \subset X)}((A_n, (C_n \subset X))) = \mathbb{H}^1(T_{X_n/S_n} \to f_{n*} N_{C_n/X_n})$

$$\begin{array}{ccccccc} H^0(C_n, N_{C_n/X_n}) & \xrightarrow{i_n} & \mathbb{H}^1(\cdots_n) & \to & H^1(T_{X_n/S_n}) & \to & H^1(N_{C_n/X_n}) \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\ H^0(C_{n-1}, N_{C_{n-1}/X_{n-1}}) & \xrightarrow{i_{n-1}} & \mathbb{H}^1(\cdots_{n-1}) & \to & H^1(T_{X_{n-1}/S_{n-1}}) & \to & H^1(N_{C_{n-1}/X_{n-1}}) \end{array}$$

$T^1$ lifting for $X \Rightarrow \alpha$ is surjective. We want to show $\beta$ surjective. $H^1(N_{C/X}) = 0 \Rightarrow T^1$ lifting holds for $D_{(C \subset)} \Rightarrow \gamma$ surjective

If we can show $H^1(N_{C_n/X_n}) = 0 \ \forall n \Rightarrow \beta$ will be surjective (diagram chase...)

By Leray s.s. suffices to check that $H^0(S_n, R^1\pi_* N_{C_n/X_n})$ and $H^1(S_n, \pi_* N_{C_n/X_n})$ are $0$ ($\Rightarrow E_\infty$ term $0$)
— second is automatic since $S_n$ affine, and first follows from infinitesimal version of semi-continuity + $H^1(N_{C/X}) = 0$ ■

<u>Corollary</u> $X$ smooth 3d C-Y, $C \subset X$ rigid l.c.i. curve $\Rightarrow$ $D(C \subset X)$ unobstructed.
<u>Proof</u> $N_{C/X}^\vee = N_{C/X} \otimes \omega_C^{-1} \Rightarrow h^0(N_{C/X}) = h^1(N_{C/X})$ so rigidity gives vanishing of the obstruction space ■

<u>Mumford's example</u> Gives a smooth space curve of degree 14, genus 24 s.t. for any nearby pt $C'$ in the Hilbert scheme $D(C' \subset \mathbb{P}^3)$ is obstructed
- Mumford: A.J.M. 84 1962 p.642 – 648
"Further pathologies in algebraic geometry"
$C \subset \mathbb{P}^3$ curve, $\mathcal{H} = \mathcal{H}_{14,24,\mathbb{P}^3}$.
$F \subset \mathbb{P}^3$ surface. $H = \mathcal{O}(1)|_F$, $h = \mathcal{O}(1)|_C$
<u>Fact</u> Any nonsingular space curve $C \in \mathcal{H}$ is contained in a pencil of quartics
<u>Proof</u>: $H^0(\mathbb{P}^3, \mathcal{O}(4)) \to H^0(C, h^{\otimes 4})$, want to estimate dim of kernel:
$h^0(\mathbb{P}^3, \mathcal{O}(4)) = \binom{4+4-1}{4} = 35$
$\deg h^{\otimes 4} = 56 > 46$
$\Rightarrow$ (R-R) $h^0(h^{\otimes 4}) = 56 - 23 = 33$, dim ker $\geq 2$ ■
Let now $C \in \mathcal{H}$ be smooth, $P$ pencil of quartics through $C$. Assume $P$ has no fixed components $\Rightarrow$ $F', F''$ span $P \Rightarrow F' \cdot F'' = C + q$, $q$ conic
$\Rightarrow C + q$ has at most triple points ($C$ smooth, $q$ at most double line)
$\Rightarrow F', F''$ share no double points, $\forall x \in F' \cap F''$ will be smooth either in $F'$ or in $F''$.
Also $C \cup q$ is smooth transverse intersection of $F', F''$, both are smooth along it. $\Longrightarrow$ generic $F \in P$ is smooth along all of $C$.
<u>Fact 2</u> Any algebraic family of smooth curves in $\mathcal{H}$ that are contained in pencils of quartics w/out fixed components has dimension $\leq 56$

Pf Suffices to check that any family of pairs $C \subset F$
$C$ smooth & in $\mathcal{H}$, $F$ quartic smooth along $C$, is of dim
$\leq 57$. The quartics $F$ in such a family
all contain conics, but the general quartic
doesn't $\Rightarrow$ nongeneric.

So dim of space of such $F$'s $\leq 34-1=33$.

Start with pair $C \subset F$. $F$ is Cohen-Macaulay (divisor)!
$\Rightarrow \omega_F = \mathcal{O}_F$ (it's a $K_3$!)

Use R.R for $C \subset F$. $0 \to \mathcal{O}_F \to \mathcal{O}_F(C) \to \mathcal{O}_C(C) \to 0$.

$\Rightarrow \dim |C|_F = \frac{(C.C)_F}{2} + 1 + h^1(\mathcal{O}_F(C)) - h^2(\mathcal{O}_F(C))$

Adjunction $\Rightarrow \mathcal{O}_C(C) = \omega_C$, $\deg (C^2)_F = 4r$

$h^i(\mathcal{O}_F(C)) = h^{2-i}(\mathcal{O}_F(-C))$ by duality

$0 \to \mathcal{O}_F(-C) \to \mathcal{O}_F \to \mathcal{O}_C \to 0 \Rightarrow$

$h^0(\mathcal{O}_F(-C)) = 0$, $h^1(\mathcal{O}_F(-C)) = 0$

$\Rightarrow 24 + 33 = 57$ ■

If $P$ has a fixed component, must be a cubic surface
($\deg = 1, 2$ or $\underline{3}$). $\Rightarrow C \subset F$ cubic & $F$ has to
be unique ($\deg C = 14 > \deg (F', F'')$ for any two cubics).

Fact 3 Any maximal algebraic family of smooth $C$'s
contained in cubics is of dimension exactly 56.

Proof If $C \subset F$ such a curve $\Rightarrow \omega_F = -H$.

By RR $\dim |C|_F = \frac{(C.(C+H))_F}{2} + h^1(\mathcal{O}_F(C)) - h^2(\mathcal{O}_F(C))$

$\omega_C = C.(C+\omega_F)$

$(C.C)_F - (C.H)_F = 46$. $\deg C = 14 \Rightarrow (C.C)_F = 60$

$h^i(\mathcal{O}_F(C)) = h^{2-i}(\mathcal{O}_F(-H-C))$ duality

$0 \to \mathcal{O}_F(-H-C) \to \mathcal{O}_F \to \mathcal{O}_{H+C} \to 0$

$H+C$ reduced, connected $\Rightarrow h^0(\mathcal{O}_{H+C}) = 1$

$\Rightarrow h^i(\mathcal{O}_F(C)) = 0 \quad i = 1, 2. \quad \Rightarrow \dim |C|_F = 37$

If $C \subset F$ is generic in a maximal family, define $C^* \subset F^*$
(generic fiber) specializing to $C$

– Picard doesn't move for cubic, by upper semicontinuity
get the count we want $= 19 + 37$?