Existence of connections

Example 1: If $L \to X$ is a line bundle then L has an alg. connection iff its Atiyah class $a(L) = c_1(L) = 0$.

Example 2: $E \to X$ v.b. on a curve (smooth projective)

E has connection iff $a(E) = 0$. By construction a is additive under direct sum. (split the Čech cocycles...)

If $E = \mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, $c_1(E) = 0$

$a(E) \in H^1(\mathrm{End}(\mathcal{O}(1) \oplus \mathcal{O}(-1)) \otimes \Omega_{\mathbb{P}^1}) = $ (Serre duality)

$H^0(\mathrm{End}(\mathcal{O}(1) \oplus \mathcal{O}(-1)))^\vee$

$H^0(\mathrm{End}\, \mathcal{O}(1) \oplus \mathcal{O}(-1)) = H^0(\mathcal{O}) \oplus H^0(\mathcal{O}(2)) \oplus H^0(\mathcal{O})$

$\Rightarrow a(E) \in H^0(\mathcal{O})^\vee \oplus H^0(\mathcal{O}(2))^\vee \oplus H^0(\mathcal{O})^\vee$

$a(E) = \overset{\Psi}{a(\mathcal{O}(1))} \oplus \overset{\Psi}{0} \oplus \overset{\Psi}{a(\mathcal{O}(-1))} \neq 0.$

Def $E \to X$ vector bundle is indecomposable (simple) if it doesn't split into a direct sum of vector subbundles

If X is compact complex, every $E \to X$ has decomposition $E = E_1 \oplus \dots \oplus E_k$ into indecomposables, unique up to iso. (Remak decomposition)

Thm (Weil) $E \to X$ v.b. on smooth projective curve, & $E = E_1 \oplus \dots \oplus E_k$ Remak decomposition $\Rightarrow$ E has an algebraic connection iff $\deg E_i = 0$ $\forall i$.

Proof (Atiyah) Since Atiyah class is additive, enough to check that if E is indecomp, then $a(E) = 0$ iff $\deg E = 0$.

Proposition X compact complex manifold, $E \to X$ ~~indecomp~~ vb $\Rightarrow$ E is indecomp iff $A = H^0(X, \mathrm{End}\, E)$ satisfies

(1) The nilpotent elements form a subalgebra $N \subset A$;

(2) As a vector space $A = \mathbb{C}\, id_E \oplus N$

Proof ($\Rightarrow$) E indecom $\Rightarrow$ $\forall \varphi \in A \longrightarrow \det(t\, id_E - \varphi) = t^r + a_1 t^{r-1} + \dots + a_r$ char poly, $a_i \in \Gamma(X, \mathcal{O}_X) \Rightarrow$ constants.

$\Rightarrow$ eigenvalues constant. $\Rightarrow$ take generalized eigenspaces for any λ, $E_\lambda = \{ s \mid (\varphi - \lambda\, id)^k s = 0 \text{ some } k \}$ subbundle $\longrightarrow$ so φ can have only one eigenvalue

$\to \det(t\, id - \varphi) = (t-\lambda)^r$, $\varphi - \lambda\, id$ is nilpotent

(any endo is either nilpotent or an isomorphism!)

($\Leftarrow$) If (1), (2) hold and $E = E_1 \oplus E_2$ can look at projections $f_1 = id_{E_1} \oplus 0$, $f_2 = 0 \oplus id_{E_2}$ . . . ∎

Now if $E \to X$ v.b. on curve, $H^1(X, \mathrm{End}\, E \otimes \Omega^1_X) = H^0(X, \mathrm{End}(E))^\vee$

$\to$ calculate $a(E)$ as functional on $H^0(\mathrm{End}\, E)$

Proposition Let $\varphi \in H^0(\mathrm{End}\, E)$ be nilpotent.
then $\langle a(E), \varphi \rangle = 0$, $\langle a(E), id_E \rangle = -(2\pi i) \deg E$

Proof For any $\psi \in H^0(\mathrm{End}\, E)$, $\langle a(E), \psi \rangle \cdot \omega = tr(a(E)\psi)$

ω Kahler class, but $tr(a(E)) = -2\pi i\, c_1(E)$

$\Rightarrow$ identity part. Nilpotent part: Look at
$E, \varphi(E), \varphi^2(E), \dots, \varphi^n(E) = 0, \varphi^{n-1}(E) \neq 0$.

$\varphi^k(E) \subset E$ subsheaf loc. free since we're on a curve this generates a ~~loc. free subsheaf~~ subbundle (saturation) (eg $\mathcal{O}(-1) \subset \mathcal{O}$ saturation is all $\mathcal{O}(0)$)

$\Rightarrow$ get a flag of subbundles preserved by φ

$\varphi_k : E_k / E_{k+1} \circlearrowleft$ is 0. (lowers degree.)

$\boxed{\text{Fact}}$ Atiyah class compatible with filtrations:
$a(E) \in H^1(X, \Omega^1_X \otimes \mathrm{End}_W(E))$ End_W endomorphisms preserving filtration. (later).

By the Fact, $a(E) \in H^0(\mathrm{End}_{(W)} E)^\vee$
$\langle a(E), \varphi \rangle \omega = tr(a(E)\varphi) = \sum tr(a(E)\varphi)_k$ assoc graded
$= 0$

$\Rightarrow$ Prop'n & Weil's theorem $\blacksquare$

Why is the Atiyah class compatible with filtrations?

Differential operators revisited

$a(E)$ will be compatible with filtrations if $a(E)$ is an exact functor.

$$0 \to \mathrm{End}\, E \to \mathcal{A}_X(E) \to T_X \to 0$$

not exact — reinterpret $a(E)$.

X smooth $\Delta : X \hookrightarrow X \times X$

$X^{(n)}_\Delta = (X, \mathcal{O}_{X\times X} / I^{n+1})$ n^{th} inf neighborhood.

n^{th} jets of functions on X is $J^n_X = \mathcal{O}_{X\times X}/I^{n+1}$ as $\mathcal{O}_X$-module in the following sense:
J^n_X is amalgamated $J^n_X \twoheadrightarrow \mathcal{O}_X$
$+$ $p_i^{(n)} : X^{(n)}_\Delta \hookrightarrow X \times X \xrightarrow{p_i} X$ induces via pullback

two maps $\mathcal{O}_X \to J^n_X$ inverse to amalgamation,
choose the ~~right~~ left one

$d^n_X : \mathcal{O}_X \to J^n_X$, induced by $p_2^{(n)}$.

J^n_X qcoh sheaf of amalgamated $\mathcal{O}_X$-algebras

$J_X = \varprojlim J^n_X$; $\Omega^1_X \nearrow I/I^2 = J^1_X \to J^0_X = \mathcal{O}_X$

amalgamation ideal

There is always $S^\bullet \Omega^1_X \twoheadrightarrow gr_\bullet J_X$, iso for X smooth.

We have short exacts $0 \to S^n\Omega^1_X \to J^n_X \to J^{n-1}_X \to 0$

<u>Note</u> $f \in \mathcal{O}_X$: $d^1_X f - f = df$ usual deRham d.

A, B quasicoh sheaves, $D^k_X(A,B) = Hom_{\mathcal{O}_X}(J^k(A), B)$

In particular symbol sequences come from jet sequences :

F loc free $\quad 0 \to D^{k-1}_X(F) \to D^k_X(F) \to S^k T_X \otimes End\, F \to 0$

$0 \to J^{k-1}_X(F)^\vee \otimes F \to J^k_X(F)^\vee \otimes F \to (S^k\Omega^1_X \otimes F)^\vee \otimes F \to 0$

Note if F sheaf, $J^k_X(F) = J^k_X \otimes_{\mathcal{O}_X, \text{right}} F = p_{2*}^{(k)} p_1^{(k)*} F$

Claim : $a(F) = -j^1(F)$

where $j^1(F)$ is class of $0 \to \Omega^1_X \otimes F \to J^1(F) \to F \to 0$.

$\Rightarrow j^1(F)$ is exact.

<u>Higher dimensions</u>

<u>Example 3</u> (Atiyah) Weil's theorem doesn't generalize to dim > 1

$X = Y \times Z$, $Y = \mathbb{P}^1$, $Z =$ elliptic curve

Choose base points $y_0 \in Y$, $z_0 \in Z$: $Y = Y \times \{z_0\} \subset X$,

$Z = \{y_0\} \times Z \subset X$,

Look at the exact seq on X

$0 \to \mathcal{O}_X(Z) \to \mathcal{O}_X(2Z) \to \mathcal{O}_Z(2Z) \to 0$ $\quad (\mathcal{O}_Z(2Z) = \mathcal{O}_Z)$

$\Rightarrow$ in cohomology

$H^1(\mathcal{O}_X(2Z)) \to H^1(\mathcal{O}_Z) \to H^2(\mathcal{O}_X(Z))$

$= H^0(\mathcal{O}_X(-Z) \otimes K_X)^\vee$

$= H^0(\mathcal{O}_X(-3Z))^\vee = 0$

$\Rightarrow \exists \xi \in H^1(\mathcal{O}_X(2Z))$ mapping to the generator of $H^1(\mathcal{O}_Z) \cong \mathbb{C}$ $\Rightarrow$ rank two vector bundle

$(\xi) \quad 0 \to \mathcal{O}_X(Z) \to E \to \mathcal{O}_X(-Z) \to 0$

<u>Claim</u> (1) $c_1(E) = 0$ (2) E is indecomposable

(3) $a(E) \neq 0$.

Proof (1) $\det E = \mathcal{O}_X$, $c_2(E) \sim Z \cdot (-Z) = 0$.

(2) $E|_Z$: $0 \to \mathcal{O}_Z \to E|_Z \to \mathcal{O}_Z \to 0$ nontrivial extension. If $E|_Z = L_1 \oplus L_2$, eg $\deg L_1 \geq 0$

$$L_1 \to E|_Z \to \mathcal{O}_Z \quad \ldots \Rightarrow L_1 = \mathcal{O}_Z$$
$$\underbrace{L_1 \to \mathcal{O}_Z}_{0 \text{ or isom}}$$

(3) restrict on Y, get $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

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Example 4 Give a procedure modifying every variety X so that all vector bundles have connections.

Def. S scheme, $\mathcal{G} \to S$ group-scheme. A $\mathcal{G}$-torsor is a scheme $P \to S$ equipped with a right $\mathcal{G}$-action so that locally (étale) $P \simeq \mathcal{G}$ as right $\mathcal{G}$-space.

Examples: 1. $\mathcal{G}$ acts on itself by right translations.

2. If G an alg. group $\underline{G} \to S$ the trivial G-bundle $/S$

A $\underline{G}$-torsor is a principal G-bundle.

Such torsors can be constructed out of vector bundles: $E \downarrow S$ v.b.
$\Rightarrow$ look at its frame bundle $\mathrm{Isom}(E, \mathbb{C}^r \otimes \mathcal{O}_S) \subset \mathrm{Hom}_{\mathcal{O}}(E, \mathbb{C}^r \otimes \mathcal{O}_S)$

Natural action of $GL_r\mathbb{C}$ on $\mathbb{C}^r$ induces right action on the frame bundle $\Rightarrow$ principal $GL_r\mathbb{C}$ bundle

3. If E vector bundle a torsor over E (as sheaf of abelian groups) is called an affine bundle modeled on E.

If $\mathcal{G} \to S$ is a group scheme char for local triviality is given by cover $\{U\}$ of S C $g_{UV} \in \Gamma(U \cap V, \mathcal{G})$
$\rightsquigarrow H^1(S, \mathcal{G})$ ($g \sim \tilde{g}$ if $\exists f$ $f_U g_{UV} f_V^{-1} = \tilde{g}_{UV}$)

In particular affine bundles modeled on $E \to S$ are classified by $H^1(S, E)$.

Example An affine bundle modeled on Ω^1_S is given by an element in $H^1(S, \Omega^1_S)$

e.g. S = elliptic curve. $T \to S$ the affine bundle corresp to the polarization $\omega \in H^1(S, \Omega^1_S)$

T quasi-projective, doesn't have non constant algebraic functions but the analytic function separate points. $\boxed{\text{Stein}}$

$\longleftrightarrow$ tdo's: quantization of functions on twisted cotangent bundle.

Stein - no nontrivial (proj) subvarieties

If $E \to S$ a vector bundle, replace S with an affine bundle $\pi: T_E \to S$ so that $\pi^* E$ has a canonical connection: $a(E) \in H^1(S, End\, E \otimes \Omega^1_S)$ $\quad 0 \to End\, E \to A(E) \to T_S \to 0$
— take T_E to be affine bundle modeled on $End\, E \otimes \Omega^1_S$ corresp to $a(E)$.

Check - there is a canonical connection on $\pi^* E$

Q: Can we do that for all vector bundles on S simultaneously?

Theorem (Jouanolou's trick) If X is quasi-projective $\Rightarrow$ $\exists$ affine bundle $Y \to X$, Zariski loc. trivial so that Y is an affine variety

Proof 1. May assume X projective: indeed $X \subset \bar{X}$ projective Zar open
$\Rightarrow$ can blow up $\bar{X} \setminus X$ to get $\tilde{X} \supset X$ s.t. $\tilde{X}$ is projective, complement is Cartier divisor.

If now $\tilde{Y} \xrightarrow{\pi} \tilde{X}$ an affine bundle with $\tilde{Y}$ affine
$\Rightarrow \tilde{Y}|_X = \pi^{-1}(X)$ is an affine bundle$/X$,
$\tilde{Y} \setminus \tilde{Y}|_X = \pi^{-1}(\tilde{X} \setminus X)$ is Cartier

But the complement of a Cartier divisor in an affine variety is affine.

2. May assume X is a $\mathbb{P}^N$: indeed $X \subset \mathbb{P}^N$ closed subvariety. If $Y \to \mathbb{P}^N$ affine affine bundle, $Y|_X$ is aff bundle & closed subvariety in $Y \Rightarrow$ affine

3. For $\mathbb{P}^N$ take $GL(N+1)/GL(1) \times GL(N) \to \mathbb{P}^N = GL(N+1)/P$
Levi form. Fibers are unipotent radical $R^u P$

This is the affine bundle modeled on $\Omega_{\mathbb{P}^N}$ corresp to hyperplane $h \in H^1(\mathbb{P}^N, \Omega_{\mathbb{P}^N})$

Corollary $\forall X$ quasi-proj $\Rightarrow \exists Y \xrightarrow{\pi} X$ affine bundle s.t. $\pi^* E$ admits a connection $\forall E \to X$.

iii) Connections as 1forms on principal bundles
(i = endo with Leibniz, ii = infin. action of $\mathcal{I}$).

We saw that a vector bundle E gives us a principal $GL(r, \mathbb{C})$ bundle for r = rank E. Conversely a rep $\rho: G \to GL(V)$ turns principal bundles into v.b. $\quad P \times V/G$

Look for a connection on P, inducing connections on all associated vector bundles

A connection on P is a splitting of

$$0 \to T_{P/S} \overset{\omega}{\dashleftarrow\!\!\to} T_P \xrightarrow{d\pi} \pi^* T_S \to 0 \qquad (*)$$

Has to be compatible with G action — ask for the splitting to be equivariant (all terms have G actions).

It is easy to check that $T_{P/S} \simeq \mathfrak{g} \otimes \mathcal{O}_P$.

lift of G-action on $T_{P/S}$ is adjoint action of G on $\mathfrak{g}$.

$$\begin{array}{ccc} (T_{P/S})_x & \longrightarrow & (T_{P/S})_{xg} \\ \| & & \wr\| \\ \mathfrak{g} & \xrightarrow{Ad(g^{-1})} & \mathfrak{g} \end{array}$$

<u>Def</u> a connection on P is a G-equivariant 1-form $\omega \in H^0(P, \mathfrak{g} \otimes \Omega^1_P)^G$ s.t. it is horizontal, i.e. splits $\mathfrak{g} \otimes \mathcal{O}_P \hookrightarrow T_P$

For existence, notice that if we push $*$ forward to S and take G-invariants again get short exact

Check exact:

$$\begin{array}{ccccccccc} 0 \to & (\pi_* T_{P/S})^G & \longrightarrow & (\pi_* T_P)^G & \longrightarrow & (\pi_* \pi^* T_S)^G & \to 0 & & * \\ & \| & & \wr\| & & \| & & & \\ 0 \to & P \times_{Ad} \mathfrak{g} & \longrightarrow & \mathcal{A}(P) & \longrightarrow & T_S & \longrightarrow 0 & & ** \end{array}$$

by definition

A G-equiv splitting of $*$ is same as splitting of $**$, so obstruction to existence of a connection on P is $a(P) \in H^1(S, \Omega^1_S \otimes \mathfrak{g}_P)$, extension class of $**$.

If P was the frame bundle of $E \Rightarrow a(P) = a(E)$.

For general P and a representation $\rho : G \to GL(V)$ take $E = P \times_\rho V$, Atiyah sequence for E is pushforward of $a(P)$ via map $\mathfrak{g}_P \to End\, E$.

— a splitting of $a(P)$ gives a splitting

A connection d^∇ on a q-coh sheaf is integrable if $d^\nabla \circ d^\nabla = 0$ i.e. $F \xrightarrow{d^\nabla} F \otimes \Omega^1_S \to F \otimes \Omega^2_S \to \ldots$ is a complex

In version (i) a connection was a splitting

$$0 \to End\, E \to \mathcal{A}(E) \underset{\nabla}{\leftrightarrows} T_S \to 0$$

But $\mathcal{A}_E$ is a sheaf of $\mathbb{C}$-linear Lie algebras (like T_S), $End\, E$ $\mathcal{O}$-linear Lie alg.

$\Rightarrow$ the Atiyah sequence is an exact seq of sheaves of Lie algebras.
Integrability $\Leftrightarrow$ $\nabla : T_S \to A_E$ a Lie algebra homomorphism
In interpretation iii, ω is integrable iff it's closed.

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Integrability & the fundamental group

Can impose additional restriction on a connection ... in the 3 incarnations of a connection, integrability is:

1) As a twisted endomorphism $d^\nabla : F \to F \otimes \Omega^1_S$
integrable if $F \to F \otimes \Omega^1_S \to F \otimes \Omega^2_S \to \dots$ is a complex
$\Leftrightarrow$ $d^\nabla \circ d^\nabla = 0$ on first level. Obstruction: curvature
$curv(\nabla) = (d^\nabla)^2 : F \to F \otimes \Omega^2_S$, always an $\mathcal{O}_S$-linear operator
($a \in F$ section, $f \in \mathcal{O}_S$ $\Rightarrow$ $curv\nabla(fa) = d^\nabla(a \otimes df + f d^\nabla a)$
$= -d^\nabla a \otimes df + a \otimes d^2 f + f (d^\nabla)^2 a + d^\nabla a \otimes df = f\,curv(\nabla a)$)

Remarks 1. On a curve any connection is integrable ($\Omega^2_S = 0$)

2. S smooth, connected, projective, $L \to S$ line bundle,
d^∇ connection $\Rightarrow$ d^∇ integrable ...

Indeed let $\bar\partial_F$ be the complex structure operator on F.
$D = \bar\partial_F + d^\nabla : F \to F \otimes A^1_S$ (all 1-forms)
is a C^∞ connection on F
$curv(D) \in H^2(S, \mathbb{C})$ - closed by Bianchi identity
$curv(\nabla) \in H^0(S, \Omega^2_S)$
since $D \circ D = curv(D) = \underbrace{(\bar\partial_F)^2}_{=0} + (d^\nabla)^2 + \{\bar\partial_F, d^\nabla\}$

Let $U \subset S$ be open set where F is trivialized, and
let $e_U \in \Gamma(U, F)$ nonvanishing holomorphic section.
$\Leftrightarrow$ $d^\nabla = \partial_F + \theta_U$, $\theta_U \in \Gamma(U, \Omega^1_S)$ uniquely
determined by $d^\nabla e_U = e_U \otimes \theta_U$.
$\Rightarrow \{\bar\partial_F, d^\nabla\} = \bar\partial_F \circ (\partial_F + \theta_U) + (\partial_F + \theta_U) \circ \bar\partial_F$
$= \underbrace{\bar\partial_F \circ \partial_F + \partial_F \circ \bar\partial_F}_{= 0} + \underbrace{\bar\partial_F \circ \theta_U + \theta_U \circ \bar\partial_F}_{= \bar\partial\theta_U = 0}$

So the curvatures $curv(D) = curv(\nabla)$ .. by
Chern-Weil $c_1(F) = 2\pi i\, curv(D)$
but $c_1(F) = -2\pi i\, tr(\text{Atiyah class}) = 0$ since F has a connection

3. S smooth, projective $\Rightarrow$ if $F \to S$ has connection d^∇ $\Rightarrow$
all invariant polynomials of $curv(\nabla)$ vanish
- by same argument.

Conjecturally, $\mathrm{curv}(\nabla) = 0$ ..

(2) connection as infinitesimal action, i.e. splitting of

$a(F) : 0 \to \mathrm{End}\, F \to \mathcal{A}_S(F) \underset{\nabla}{\overset{\sigma}{\rightleftarrows}} T_S \to 0$

Note : End F is a sheaf with $\mathcal{O}_S$-linear Lie bracket.

T_S is equipped with a $\mathbb{C}$-linear Lie bracket.

$\mathcal{A}_S$ also has a $\mathbb{C}$-linear Lie bracket) : subsheaf of $\mathcal{D}_S(F)$ which has natural multiplication..

$\to$ End $\subset \mathcal{A}_S$ is ideal.

Lemma ∇ is integrable iff ∇ is a morphism of Lie algebras.

Proof $\xi, \eta \in T_S$, $\mathrm{curv}\, \nabla \lrcorner\, \xi \wedge \eta = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$

Different interpretation : T_S is naturally a subsheaf of the Atiyah algebra $\mathcal{D}_S^1(\mathcal{O}_S) = \mathcal{D}_S^1$.

because $\mathcal{O}_S$ has canonical integrable connection d.

$0 \to \mathcal{O}_S \to \mathcal{D}_S \rightleftarrows T_S \to 0$

Lie derivative

So lemma $\iff$ the ~~action~~ lifting of infinitesimal symmetries $\nabla : T_S \to \mathcal{A}_S(F)$ extends to an action of $\mathcal{D}_S$, i.e. a hom $\mathcal{D}_S \to \mathcal{D}_S(F)$, i.e. a $\mathcal{D}$-module

(3) Interpret by one-forms on principal bundles

G-equiv $\mathfrak{g}$-valued 1-form ω on a principal G-bundle

$\Rightarrow$ integrable $\iff$ $d\omega = 0$ .. $d\omega$ is the curvature

(after pushing down & taking invariants)

Integrable connections carry topological info...

Assume S connected, Universal cover $\tilde{S} \to S$ is a principal $\pi_1(S)$ bundle $\Rightarrow$ can take associated vector bundles

Given $\rho : \pi_1(S) \to G$ rep, form $P = \tilde{S} \times_\rho G = \tilde{S} \times G / \pi_1$

$\gamma(x, g) = (x\gamma, \rho(\gamma) \cdot g) \quad \gamma \in \pi_1$.

To describe all P's coming in this way, need notation:

G complex algebraic group $\Rightarrow$ G^δ abstract group G with discrete topology

Def G complex algebraic, $P \to S$ principal G-bundle $\Rightarrow$ say that P has a discrete form if exists a principal G^δ bundle $P^\delta \to S$ and a continuous bijective $G^\delta \to G$-equivariant map $P^\delta \to P$.

Def A principal G bundle P is called a G local system if it can be expressed by constant coordinate transitions

$\Rightarrow$ transitions in $H^1(S, G) < H^1(S, G(\mathbb{Q}))$

<u>Proposition</u> $P \to S$ principal G-bundle. T.f.a.e.

a. P arises from a rep of π_1

b. P is a G-local-system

c. P has an integrable connection

d. P has a discrete form.

<u>Pf</u> $a \Rightarrow b$ discreteness of π_1 action. $b \Rightarrow c$ since coord transforms are constant $\Rightarrow$ choosing d to be our connection on every patch is consistent ($g^{-1}dg$ term vanishes).

$c \Rightarrow d$ Let $\mathcal{P}$ be the sheaf of germs of analytic sections of P. — sheaf of sets with natural topology on total space with discrete fibers. Continuous maps:

$\sigma: \mathcal{P} \to P$, germ $p_x \to p_x(x)$.

There's a continuous G action on $\mathcal{P}$, let $\sigma: \mathcal{P} \to \mathcal{P}/G$ the topological quotient. Integrable connection gives a continuous section $\alpha \in \Gamma(S, \mathcal{P}/G)$

$P^\delta = \sigma^{-1}(\alpha(S))$

$d \Rightarrow a$ G^δ is discrete $\Rightarrow P^\delta \to \mathcal{P}$ has connected components which are covering spaces $\Rightarrow$ $\rho: \pi_1(S) \to G^\delta \to G$ ....

<u>Corollary</u> S smooth, proj. $L \to S$ line bundle $\to$ L comes from a character of π_1 iff L alg. equivalent to 0.

(Weil) <u>Cor</u> If S is a proj curve $\Rightarrow$ a v.b. F comes from a rep of π_1 iff all its irreducible summands have degree zero...

$\Rightarrow$ Topological interp of integrability: F is a v.b. with connection d^∇ can look at $D = \bar\partial_F + d^\nabla$, s.t. $\operatorname{curv}(D) = \operatorname{curv}(\nabla)$.

Given a path $\gamma: [0,1] \to S$ w/endpoints x, y, piecewise smooth, can use D to lift γ to a path in F with any base $v \in F_x$.

$\Rightarrow$ attach to any path γ an iso $\tau_\gamma: F_x \to F_y$.

parallel transport wrt D along γ. $\to$ groupoid

If Π_x is the semigroup of all loops in S based at x get hom $\tau: \Pi_x \to \operatorname{Aut} F_x$, holonomy action of D, group generated by image is holonomy group

$\to$ D integrable $\Leftrightarrow$ τ factors through $\pi_1(S, x)$ $\sim\to$ monodromy.

<u>Crystalline interpretation of an integrable connection</u>:

S smooth proj, $(S\times S)^\wedge$, complete formal neighborhood of the small diagonal. $(S\times S\times S)^\wedge$ $\quad (S\times S)^\wedge \underset{p_i}{\downarrow\downarrow} S \quad (S\times S\times S)^\wedge \underset{p_{ij}}{\downarrow\downarrow\downarrow} (S\times S)^\wedge$

- morphisms, + canonical retractions

<u>Lemma</u> Let $F \to S$ be a vector bundle. Then an integrable connection ∇ on F is the same as an isomorphism $\varphi: p_1^* F \to p_2^* F$ on $(S\times S)^\wedge$ s.t. φ satisfies cocycle condition $(p_{12}^*\varphi)(p_{23}^*\varphi) = (p_{13}^*\varphi)$ on $(S\times S\times S)^\wedge$. + $\varphi = id$ when restricted on S.

<u>Proof</u> Given φ can construct $d^\nabla: F \to F\otimes\Omega^1_S$ as follows:

Let $\Delta^{(2)}: S \hookrightarrow S\times S$, $\Delta^{(3)}: S \hookrightarrow S\times S\times S$

For any $a\in F$, can look at $p_2^* a - \varphi(p_1^* a) \mod I^2$

I = ideal of $\Delta^{(2)}(S)$. This belongs to $p_2^*F \otimes \mathcal{O}_{S\times S}/I^2$ but is actually in $p_2^* F \otimes I/I^2$

$\Rightarrow d^\nabla(a) := \Delta^{(2)*}(p_2^* a - \varphi(p_1^* a) \mod I^2) \in F\otimes\Omega^1_S$

So we've constructed a functor from pairs (F,φ) to pairs (F, d^∇), $d^\nabla \in Hom_{\mathbb{C}}(F, F\otimes\Omega^1_S)$.

We want to show that ~~when~~ this gives an equivalence with the full subcategory $(F, d^\nabla$ integrable$)$. — <u>local in S</u>.

$\Rightarrow$ reduce to case S affine: cover S by opens V - étale covers of affine, + take direct images..

neither site localizes well in Zariski, only étale (or formally..)

So assume $S=U$ affine & $F|_U$ is trivial

- want U affine open in $\mathbb{A}^d$ $d=\dim S$ - can be done in étale. $F \simeq \mathcal{O}_U^{\oplus n}$, φ is given by a function

$g: (U\times U)^\wedge \to GL_n(\mathbb{C})$ - $GL_n\mathbb{C}$ valued function on pairs of infinitesimally close pts in U

$$\begin{cases} \cancel{g(x,y)\,g(y,z)}\; g(x,z) = g(y,z)\, g(x,y) \\ g(x,x) = 1 \end{cases}$$

write $g(x,y) = \mathbb{1} + A(x)(x-y) + O(x-y)^2$

where is a $M_n(\mathbb{C})$ valued 1-form on U.

$\to d^\nabla a\,(x-y) = a(y) - g(x,y)\,a(x)$

$= a(y) - a(x) - A(x)\,a(x)\,(x-y) \to d^\nabla a = \dfrac{a(y)-a(x)}{y-x} = A(x)a(x)$

i.e. $d^\nabla = d - A$, so d^∇ is a connection

The cocycle condition on g gives a formal DE on A:

take $y-z$ to be a first-order infinitesimal $\Rightarrow$ cocycle condition

$$g(x,z) = g(y,z)\,g(x,y) = (1 + A(y)(y-z))\,g(x,y)$$

$$\Rightarrow \frac{g(x,z) - g(x,y)}{y-z} = A(y)\,g(x,y) \Rightarrow \partial_y g(x,y) = A(y)\,g(x,y)$$

projection on dy part of the differential

Get initial value problem $\boxed{\partial_y g(x,y) = A(y) g(x,y), \quad g(x,x) = \mathbb{1}}$ $(*)$

– g is unique if it exists, for given A.

Remark: If g is a soln then cocycle condition is automatically satisfied since $g(x,z)$, $g(y,z)\,g(x,y)$ are solutions & they coincide for $y=z$.

Need to show $*$ has solution iff d^∇ is integrable:

Change variables $t = (y-x) = (t_1, \dots, t_d)$

$A(x+t) = \sum_1^d A_i(x,t)\,dt_i$. The system $(*)$ in these coords $\rightarrow$

$$\boxed{\frac{\partial g(x,t)}{\partial t_i} = A_i(x,t)\,g(x,t), \quad g(x,0) = \mathbb{1}}$$

Standard form, Integrability $\Leftrightarrow$ commuting of characteristics (second partials) $\frac{\partial A_i}{\partial t_j} + A_i A_j = \frac{\partial A_j}{\partial t_i} + A_j A_i$

$\Longleftrightarrow \frac{\partial A_i}{\partial t_j} - \frac{\partial A_j}{\partial t_i} = [A_i, A_j]$ i.e. $(d^\nabla)^2 = 0$ ∎

Def A stratification of schemes $/S$ is a scheme $X \to S$ together with $\varphi : p_1^* X \xrightarrow{\sim} p_2^* X$ on $(S \times S)^\wedge$ s.t. φ/S is identity & satisfies the cocycle condition.

Replace $(S\times S)^\wedge$ by <u>any</u> thickening & <u>any</u> two retractions $\rightsquigarrow$ crystal .. two notions agree for smooth bases.

$\boxed{\text{Gauss – Manin}}$ $f : X \to S$ smooth, proper between smooth varieties. Relative de Rham: $H^k_{DR}(X/S, \mathbb{C}) = \mathbb{R}^k f_* \Omega^\bullet_{X/S}$ $\downarrow S$

Want to exploit topological nature of $H^k_{DR}(X/S)$, ... $\dashrightarrow$ connection on $H^k_{DR}(X/S, \mathbb{C})$.

1. Topological construction: By Grothendieck comparison thm, the coherent analytic sheaf corresp to $H^k_{DR}(X/S)$ is

$$R^k f_*^{an} \Omega^\bullet_{X/S} = (R^k f_*^{an} \mathbb{C}_{X^{an}}) \otimes_{\mathbb{C}_{S^{an}}} \mathcal{O}_{S^{an}}$$
$$= (R^k f_*^{an} \mathbb{Z}_{X^{an}}) \otimes_{\mathbb{Z}_{S^{an}}} \mathcal{O}_{S^{an}}$$

by universal coefficient thm valid in analytic context...
in particular $H^k_{DR}(X/S)$ admits a discrete form, i.e. it is associated with the frame bundle of $R^k f_* \mathbb{Z}_{X^{an}}$, so has an integrable connection.

Explicitly G-M is given by declaring the horizontal sections to be the sections of $R^k f_*^{an} \mathbb{C}_{X^{an}}$

Topologically $R^k f_* \mathbb{Z}$ is a covering space of S $\rightsquigarrow$ canonical connection, & this spans.

2. Algebraic construction Given any smooth morphism of schemes $f: X \to S$ Grothendieck constructs a connection on $\mathbb{R} f_* \Omega^\bullet_{X/S} \in D^b(S)$. Moreover if $\mathbb{R}^k f_* \Omega^\bullet_{X/S}$ are locally free (in particular, have the base change property) he shows the connection induces connections on every $\mathbb{R}^k f_*$.

$\boxed{\text{Def}}$ S scheme, $F \in D^b(S)$. An integrable connection on F is a consistent way of assigning to every diagram $S \overset{h}{\hookrightarrow} S' \underset{q_2}{\overset{q_1}{\rightrightarrows}} S$
$h: S \hookrightarrow S'$ is a square 0 extension, q_1 & q_2 retractions of h
an isomorphism $\varphi_{q_1 q_2}: Lq_1^* F \xrightarrow{\sim} Lq_2^* F$
that restricts to Id & is associative for any given third retraction, ie cocycle condition.

Special cases:
(a.) $f: X \to S$ smooth & affine
Preliminaries: If ∂ is a $f^{-1}(\mathcal{O}_S)$-linear derivation of $\mathcal{O}_X$ (vertical vector field) i.e. $\partial \in \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) = T_{X/S}$
$\Rightarrow$ ∂ induces a $f^{-1}(\mathcal{O}_S)$-linear homomorphism
$\Theta(\partial): \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$ (Lie derivative along fibers)
Characterized by: Acts on $\Omega^0_{X/S} = \mathcal{O}_X$ as ∂, commutes with d, Leibniz (Hom of DGA's!)
wrt interior, exterior, scalar

Special cases of the algebraic definition of Gauss-Manin

a. $f: X \to S$ smooth, affine morphism.

Last time: to any $\partial \in Der_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X) = Hom_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) = T_{X/S}$ associated a Lie derivative along the fibers $\theta(\partial)$ taking $\Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S}$, endomorphism as differential graded algebra.

Main Observation: Cartan homotopy property:

$\theta(\partial) = d \circ i_\partial + i_\partial \circ d$ $\quad i_\theta$ contraction

– so $\theta(\partial)$ homotopic to the identity.

[LHS & RHS agree on $\mathcal{O}$ & commute with d, Leibniz]

$\longrightarrow$ $\theta(\partial)$ acts as 0 on hypercohomology sheaves.

$\boxed{\text{Want}}$: map $G\text{-}M : Der_{\mathbb{C}}(\mathcal{O}_S) \longrightarrow End_{\mathbb{C}}(H^k_{DR}(X/S))$ connection.

First given $v \in Der\, \mathcal{O}_S$ try to lift to an infinitesimal automorphism of X, i.e. a derivation of $\mathcal{O}_X$.

- Local in S, so may assume S, X affine

$S = Spec\, A$, $X = Spec\, B$, $f: A \to B$.

Every element of $Der\, \mathcal{O}_S$ will lift to an element in $\mathcal{O}_X$ iff the natural map $Der_{\mathbb{C}}(B, B) \to Der_{\mathbb{C}}(A, B)$ has image containing $Der_{\mathbb{C}}(A, A)$.

Recall for any morphism of (Nötherian) rings $f: A \to B$ and any B-module F there is an exact sequence

$0 \to Der_A(B, F) \to Der_{\mathbb{C}}(B, F) \to Der_{\mathbb{C}}(A, F)$

$\xrightarrow{\delta} Ex^1_A(B, F) \to \ldots$

$Ex^1_A(B, F)$ = equivalence classes of A algebras that are square 0 extensions of B by F, i.e. eq. classes of

$0 \to I \to E \to B \to 0$, $\quad I^2 = 0$, $I \cong F$ as an A module.

(this sequence terminates for rings – but not in general for operads).

The map δ can be described explicitly:

if $\xi \in Der_{\mathbb{C}}(A, F)$, $\delta(\xi)$ has underlying vector space $B \oplus F$ as central term, multiplication is:

$(b,e)\cdot(b',e') = (bb', be'+b'e)$ and A-module structure is given by $a\cdot(b,e) = (f(a)\cdot b, \xi(a) + f(a)e)$

Since our map f is smooth $\Rightarrow$ f satisfies the inf. lifting property, i.e.

$\operatorname{Ex}'_A(B,F) = 0$ $\quad$ in nonrelative case $\leftarrow$ $\quad 0 \to I \to E \to B \to 0$, $A \to B$, $A \dashrightarrow E$

Take $F \simeq B$ as an A module $\Rightarrow$

$0 \to \operatorname{Der}_A(B,B) \to \operatorname{Der}_{\mathbb{C}}(B,B) \to \operatorname{Der}_{\mathbb{C}}(A,B) \to 0$

so any derivation from $\operatorname{Der}_{\mathbb{C}} A,A \subset \operatorname{Der}_{\mathbb{C}} A,B$ can be lifted to $\operatorname{Der}_{\mathbb{C}}(B,B)$ with ambiguity exactly $\operatorname{Der}_A(B,B)$ <u>namely</u> vertical vector fields, which act homotopic to 0

$\Longrightarrow$ <u>G-M connection</u>.

Integrability — above are morphisms of $\mathbb{C}$-Lie algebras. ($\operatorname{Der}_{\mathbb{C}}(A,B)$ is Lie module, $\operatorname{Der}_{\mathbb{C}} A,A$ Lie algebra in it.)

<u>b.</u> $f: X \to S$ smooth morphism of smooth varieties.

Again we want to construct $GM: T_S \to \mathcal{A}_S(H^k_{DR}(X/S))$ integrable.

By hypothesis, we have a tangent short exact sequence of vector bundles

(1) $\quad 0 \to f^*\Omega^1_S \xrightarrow{df^\vee} \Omega^1_X \to \Omega^1_{X/S} \to 0$

$\Omega^\bullet_X$ has a filtration $\Omega^\bullet_X = I^0 \supset I^1 \supset I^2 \supset \dots$ $\quad$ (2)

where $I^k = \operatorname{Im}\,[f^*\Omega^\bullet_S \otimes_{\mathcal{O}_X} \Omega^{\bullet-k}_X \to \Omega^\bullet_X]$

Since Ω^i_X, Ω^i_S are locally free $\Rightarrow$ the assoc. grad'd of I are $\quad gr^k := gr^k_I \Omega^\bullet_X = f^*\Omega^k_S \otimes \Omega^{\bullet-k}_{X/S}$ $\quad$ (3)

by (1).

$$0 \to I^1/I^2 \to I^0/I^2 \to I^0/I^1 \to 0$$

$$\qquad \| \qquad\qquad \| \qquad\qquad \|$$

(4) $\quad 0 \to f^*\Omega^1_S \otimes \Omega^{\bullet-1}_{X/S} \to \Omega^\bullet_X/I^2 \to \Omega^\bullet_{X/S} \to 0$

Take the long exact sequence of hyperderived images.

The k^{th} edge homomorphism of this sequence is

$$\begin{array}{l} \mathbb{R}^k f_* \Omega^\bullet_{X/S} \longrightarrow \mathbb{R}^{k+1} f_*(f^*\Omega^1_S \otimes \Omega^{\bullet-1}_{X/S}) \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^{k+1} f_* \Omega^{\bullet-1}_{X/S} \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^k f_* \Omega^\bullet_{X/S} \end{array}$$

(projection formula, + fact that differential) in $\Omega^\bullet_{X/S}$ is $f^{-1}\mathcal{O}_S$ linear

$\Rightarrow$ map $d^{GM} : H^k_{DR}(X/S) \to H^k_{DR}(X/S) \otimes \Omega^1_S$

— Gauss–Manin as differential in (Leray?) spectral sequence: — s.s. of pushforward of filtered object.

$$E_1^{pq} \Longrightarrow gr(\mathbb{R}^{p+q} f_* \Omega^\bullet_X)$$

$\| \; \mathbb{R}^{p+q} f_*(gr^p) = \Omega^p_S \otimes_{\mathcal{O}_S} \cdot H^q_{DR}(X/S)$.

This spectral sequence is multiplicative: $I^\bullet$ is compatible with $\wedge$ (D.G.A. structure) i.e. $I^k \wedge I^{k'} \subset I^{k+k'}$

get multiplicative structure on the s.seq.

$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$ supercommutative

$e \cdot e' = (-1)^{(p+p')(q+q')} e' \cdot e$.

$d_r(e \cdot e') = (d_r e) \cdot e' + (-1)^{p+q} e \cdot d_r e'$.

The differential on E_1 is given as follows: set $k = p+q$

$d_1^{pq} : E_1^{p,q} \to E_1^{p+1,q}$ is a homomorphism

$\mathbb{R}^k f_*(gr^p) \longrightarrow \mathbb{R}^{k+1} f_*(gr^{p+1})$ that is the k^{th} edge homomorphism for $0 \to gr^{p+1} \to I^p/I^{p+2} \to gr^p \to 0$.

In particular $\forall k$ the Gauss-Manin operator d^{GM} is just $d_1^{0,k}$

$\forall k$ we have a complex

$E_1^{\bullet,k}$ $\quad 0 \to H^k_{DR} \xrightarrow{d_1^{0,k}} H^k_{DR} \otimes \Omega^1_S \xrightarrow{d_1^{1,k}} H^k_{DR} \otimes \Omega^2_S \to \ldots$

diagonal complex of our spectral sequence.

If $k=0$ this complex is $H^0_{DR}(X/S) \otimes \Omega_{X/S}$,

$d_1^{1,0}$ is just usual exterior differentiation

— $H^0_{DR}(X/S)$ is locally trivial bundle - smooth map is (very) locally a product ... push forward of constant sheaf (étale)

— locally constant — know how to differentiate functions upstairs.

If $\alpha \in \Omega^i_S \subset E_1^{i,0}$ local section, $e \in E_1^{0,k} = H^k_{DR}(X/S)$

$\Rightarrow d_1^{i,k}(\alpha \cdot e) = d\alpha \cdot e + (-1)^i \alpha \cdot e$ $\to$ Leibniz holds.

$d_1^{0,k}$ is a connection + $d_1^{i,k}$ are recovered from $d_1^{0,k}$ by imposing the Leibniz rule.

But $E_1^{\cdot,k}$ is a <u>complex</u> $\Rightarrow$ <u>curvature vanishes</u>

$\blacksquare$

<u>Remark</u> • $H^\cdot_{DR}(X/S)$ is a sheaf of algebras, + by multiplicativity of $E \to$ $e \in H^k_{DR}$, $e' \in H^{k'}_{DR}$

$d_1^{0,k+k'}(e \cdot e') = (d^{0,k+k'}e) e' + (-1)^k e \cdot (d_1^{0,k'} e')$

which rewritten for G-M gives

$\forall v \in T_S \quad GM_v(e \cdot e') = GM_v(e) \cdot e' + e \cdot GM_v(e')$

multiplicative

• <u>Čech calculation of d^{GM}</u> Assume S affine (everything local in S). Choose a finite cover $\mathcal{U}$ of X by affine opens that are étale covers of open sets in $\mathbb{A}^d_S$, $d = \dim(X/S)$, and such that $\Omega^1_{X/S}$ is a free $\mathcal{O}_U$-module for every $U \in \mathcal{U}$ with basis $dx_1^U \dots dx_d^U$.

$H^k_{DR}(X/S)$ is an $\mathcal{O}_S$-module that is the k^{th} cohomology of the total complex of the double complex $C^{p,q} = C^p(\mathcal{U}, \Omega^q_{X/S})$

We will describe a noncanonical, non integrable connection on $tot(C^{\cdot\cdot})$ which gives Gauss-Manin on passing to cohomology.

For a derivation $\xi \in Der\, \mathcal{O}_S$, $U \in \mathcal{U}$ denote by ξ^U the unique lifting of ξ to a derivation of $\mathcal{O}_U$ that kills $dx_1^U \dots dx_d^U$. Choose a complete order "<" of the finite set $\mathcal{U}$. Define $\tilde{\xi} \in End_S(C^{\cdot\cdot})$ via

$\tilde{\xi}|_{\Gamma(U_0 \cap \dots \cap U_p, \Omega^q_{X/S})} = \xi^{U_{min}}$, U_{min} = minimal element in $\{U_0, \dots, U_p\}$.

$\tilde{\xi}$ is an endo. of bidegree $(0,0)$

Čech calculation of Gauss-Manin

$f: X \to S$ smooth between smooth varieties. May assume S affine.
$\mathcal{U}$ - finite cover of X by affine open sets, & take over affine open sets in $\mathbb{A}^d_S$, $d = \dim X$.
Choose a trivialization of $\Omega^1_{X/S}$ over every $V \in \mathcal{U}$: $dx_1^V, \ldots, dx_d^V$.

We have $C^{p,q} = C^q(\mathcal{U}, \Omega^q_{X/S})$ - realize G-M on total complex.

Want connection on $\mathrm{tot}(C^{\bullet\bullet})$ inducing G-M on cohomology.
Problem of ambiguity - can lift derivations on open sets, but which lift to choose over intersections? $\Rightarrow$ order sets...

For $\xi \in \mathrm{Der}(\mathcal{O}_S)$ denote by $\xi_U \in \mathrm{Der}(\mathcal{O}_U)$ the unique lifting of ξ that kills $dx_1^U, \ldots, dx_d^U$.

Choose a total order $<$ on $\mathcal{U}$. For $\xi \in \mathrm{Der}(\mathcal{O}_S)$ choose $\tilde{\xi} \in \mathrm{End}_{\mathbb{C}}(C^{\bullet\bullet})$ by setting
$$\tilde{\xi}|_{\Gamma(U_0 \cap \ldots \cap U_p, \Omega^q_{X/S})} = \xi_{U_{min}},$$
where U_{min} is the minimal among $U_0, U_1, \ldots U_p$.

$\tilde{\xi}$ has bidegree $(0,0)$ - doesn't change order of diff form or order of intersections.

For each pair $U, V \in \mathcal{U}$, have an $\mathcal{O}_X$-linear morphism $\lambda(\xi)_{UV}: \Omega^q_{X/S}|_{U \cap V} \to \Omega^{q-1}_{X/S}|_{U \cap V}$, contraction by $\xi_U - \xi_V$.

This gives an $\mathcal{O}_X$-linear endomorphism $\lambda(\xi)$ of $C^{\bullet\bullet}$ of bidegree $(1,-1)$ by $\lambda(\xi)(\sigma)_{U_0 \ldots U_{p+1}} :=$
$= (-1)^q \lambda(\xi)_{U_0 U_1}(\sigma_{U_1 \ldots U_{p+1}})$, $\sigma \in C^p(\mathcal{U}, \Omega^q_{X/S})$
where $U_0 < U_1 < \ldots < U_{p+1}$.

Define $\check{GM}: \xi \to \mathrm{End}_{\mathbb{C}}(\mathrm{tot}\, C^{\bullet,\bullet})$, $\xi \mapsto \tilde{\xi} + \lambda(\xi)$

$\lambda(\xi)$ also has total degree 0 & is linear part:
diff operator part + deg 0 linear part (endomorphism)

Depends on all the choices, not integrable in general.

Everything here comes from $C^{\bullet\bullet}$, & we have the two s. seqs. $\Rightarrow H^*(\mathrm{tot}\, C^{\bullet\bullet})$ from the two filtrations:

The horizontal filtration of $\mathrm{tot}(C^{\bullet\bullet})$ is called the Zariski filtration

$F^i_{Zar} = \bigoplus_{p\geq i} C^{p,q}$, vertical filtration is the Hodge filtration $F^i_{Hodge} = \bigoplus_{q\geq i} C^{p,q}$.

By examining how $\check{GM}$ shifts the filtrations $\Rightarrow$ consequences:

i) $\check{GM}$ is compatible with the Zariski filtration of $\mathrm{tot}\, C^{\bullet\bullet}$

$\Rightarrow$ acts on the assoc. s. s.

Zar $E_1^{p,q} = C^p(\mathcal{U}, \mathcal{H}^q_{DR}(X/S)) \Rightarrow H^{p+q}_{DR}(X/S)$

$\Rightarrow$ $\check{GM}$ induces a connection on $H^k_{DR}(X/S)$

- filtered by horizontal degrees. . check that this is Gauss-Manin as before.

ii) $\check{GM}$ is not compatible with Hodge $\Rightarrow$ doesn't act on HdR ss

$E_1^{pq} = R^p f_* \Omega^q_{X/S} \Rightarrow H^{p+q}_{DR}(X/S)$

but, it only shifts F_{Hodge} at most by one $\Rightarrow$

$GM: F^i_{Hodge} H^k_{DR}(X/S) \to F^{i-1}_{Hodge} H^k_{DR}(X/S)$

$\Rightarrow$ <u>Griffiths Transversality</u>

(<u>doesn't depend</u> on degeneration of Hodge-deRham! filtration is always there, as is GM, as is transversality!)

Consider the Kodaira-Spencer map for the family $f: X\to S$:

By definition this is the map $\rho_{X/S}: T_S \to R^1 f_* T_{X/S}$ first edge homomorphism of the direct image of the tangent sequence $0\to T_{X/S} \to T_X \xrightarrow{df} f^* T_S \to 0$

$\rho_{X/S}$ is the composition $T_S \to f_* f^* T_S \xrightarrow{\partial} R^1 f_* T_{X/S}$.

- Over dual numbers this is our usual deformation map.

Let $\alpha_{X/S} \in H^0(S, \Omega^1_S \otimes R^1 f_* T_{X/S})$ correspond to $\rho_{X/S}$.

Note that for any $\xi \in T_S$, the image $\rho_{X/S}(\xi) \in H^0(R^1 f_* T_{X/S})$ into $H^1(X, T_{X/S})$ by Leray is represented by the Čech cocycle $\xi_v - \xi_u$.

$\Rightarrow$ So degree 0 part of $\check{GM}$ is just contraction with the Kodaira–Spencer class (cup product)

iii) Griffiths infinitesimal period relations:

Assume f is such that H-dR s.s. degenerates at E_1.

The assoc. graded of GM $d^{GM}: H^k_{DR} \to H^k_{DR} \otimes \Omega^1_S$ is an $\mathcal{O}_X$-linear homomorphism

KS: $H^k_{Dol}(X/S) \to H^k_{Dol}(X/S) \otimes \Omega^1_S$

given by a cup product with $\alpha(X/S)$, i.e.

$KS: R^p f_* \Omega^q_{X/S} \to R^{p+1} f_* \Omega^{q-1}_{X/S} \otimes \Omega^1_S$

$e \mapsto e \cup \alpha(X/S)$.

KS map comes the same way as GM: take relative dR with 0 differential, inflate to full dR with 0 differential, push downstairs, (filter first) $\Rightarrow$ s.s., degenerates $\Rightarrow$ Dolbeault, corresponding $\mathcal{O}_X$-linear edge homomorphism is KS class

Remark A pair (E, θ) of a vector bundle $E \to S$ and an $\mathcal{O}_S$ linear homomorphism $\theta: E \to E \otimes \Omega^1_S$, is called a Higgs bundle.

iii) says the loc sys (H^k_{DR}, d^{GM}) has a canonically associated Higgs bundle (H^k_{Dol}, KS)

Equivalently, the pushforwards of the trivial local system $(\mathcal{O}_X, d)$ & trivial Higgs bundle $(\mathcal{O}_X, 0)$ correspond to each other as above ... assoc graded!

C Let $f: X \to S$ be a smooth map between arbitrary schemes.

$\rightsquigarrow$ define a connection in the derived category on $R f_* \Omega^\bullet_{X/S} \in D^b(S)$.

This means that given a diagram

$S \xhookrightarrow{h} S' \overset{q_1}{\underset{q_2}{\rightrightarrows}} S$

h an inclusion given by a square zero ideal, q_i retractions of h

$\Longrightarrow$ functorial isomorphism

$$\mathbb{L}q_1^*(\mathbb{R}f_*\Omega^\bullet_{X/S}) \simeq \mathbb{L}q_2^*(\mathbb{R}f_*\Omega^\bullet_{X/S})$$

identity on S, with natural cocycle condition..

Since f is smooth, it commutes with base change in the derived category $\Rightarrow$ suffices to construct a functorial isomorphism (*) $\mathbb{R}f_{1*}(\Omega^\bullet_{X_1/S'}) \to \mathbb{R}f_{2*}(\Omega^\bullet_{X_2/S'})$ where $X_i = X \times_{q_i} S' \xrightarrow{f_i} S'$. i.e. X_i smooth liftings of X to S'.

$$\begin{tikzcd} X \arrow[d] & X_1 \arrow[d] & X_2 \arrow[dl] \\ S \arrow[r, hook] & S' & \end{tikzcd}$$

We'll find (*) in general for any two smooth liftings X_1, X_2

– difference between crystal & stratification: crystal need isos for any lifting, stratification for liftings given by retractions .— smooth case any lifting is (locally) given by retractions ...

$\Rightarrow$ in fact we'll be constructing a crystal: build (*) for any two liftings X_1, X_2.

Consider $G = \{ g \in Aut_{S'}(X_1) \,/\, g|_X = id \}$

$P = \{ p \in Isom_{S'}(X_1, X_2) \,/\, p|_X = id \}$

G is a commutative group scheme, P a right torsor over G,

<u>SGA I III Prop 5.3</u> : $G \simeq Hom_{\mathcal{O}_X}(\Omega^1_{X/S}, I\mathcal{O}_{X_1}) = Der_S(\mathcal{O}_X) \otimes I\mathcal{O}_{X_1}$

$I\mathcal{O}_{X_1}$ – ideal sheaf of the pullback of X.

Geometrically this says given $g : X_1 \to X_1$ inducing id on X $\Rightarrow$ can lift g uniquely to $\tilde{g} : X_1 \to X_1 \times_S X_1$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X_1 \times_S X_1 \arrow[d, "pr_1"] \\ X_1 \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & X_1 \end{tikzcd}$$

I square zero $\Rightarrow G \simeq Der_S(\mathcal{O}_X) \otimes_{\mathcal{O}_X} I\mathcal{O}_{X_1}$

Also get natural iso's of $\mathcal{O}_X$ modules

$$\Omega^p_{X/S} \simeq \Omega^p_{X_1/S'} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_X \quad , \quad I\mathcal{O}_{X_1} \simeq I\mathcal{O}_{X_2}$$

+ there are natural maps :

- <u>transport of structure</u> $u : P \longrightarrow Hom^{(0)}_{f^{-1}\mathcal{O}_{S'}}(\Omega^\bullet_{X_1/S'}, \Omega^\bullet_{X_2/S})$
  $f^{-1}\mathcal{O}_{S'}$ sheaf of algebras, these are deg 0 homomorphism between $f^{-1}\mathcal{O}_{S'}$ modules.
- <u>interior product</u> $i : G \longrightarrow Hom^{(-1)}_{f^{-1}\mathcal{O}_{S'}}(\Omega^\bullet_{X_1/S'}, \Omega^\bullet_{X_2/S})$

$i_\partial(f d_1 x_1' \wedge \dots \wedge d x_d') =$
$= p\, f\, \Sigma(-1)^{j+1}\langle \partial, d_1 x_j' \otimes 1\rangle\, p_*(dx_1' \wedge \dots \wedge \widehat{d_1 x_j'} \wedge \dots \wedge dx_d')$

where $\partial \in \Gamma(U, G)$,
$f \in \Gamma(U, \mathcal{O}_{X_1})$
$p \in \Gamma(U, P)$ — operation independent of choice of p . .
(we're differentiating and S' is square zero . .)

- <u>Lie derivative along fibers</u>
  $\theta : G \longrightarrow Hom^0_{f^{-1}\mathcal{O}_{S'}}(\Omega^\bullet_{X_1/S'}, \Omega^\bullet_{X_2/S})$
  defined on $\Omega^0_{X_1/S'} = \mathcal{O}_{X_1}$ by $\theta(\partial)(f) = \langle \partial, d_1 f \otimes 1\rangle$
  + extended by the usual rules (Leibniz)

$\Omega^p_{X/S} = \Omega^p_{X_1'/S'} \underset{\mathcal{O}_{X_1'}}{\otimes} \mathcal{O}_X$ — everything sheaves of $\mathcal{O}_X$ modules . . so 5/2

$\theta(\partial)(f) = \langle \partial, df \otimes 1\rangle$ defines form on X which we consider as living on X_2 . . .

<u>Cartan homotopy formula</u> $\theta(\partial) = i_\partial \circ d_1 + d_2 \circ i_\partial$

<u>Relations</u> $\theta(\partial + \partial') = \theta(\partial) + \theta(\partial')$
$u(p \circ \partial) = u(p) + \theta(\partial)$: u is G-equivariant w.r.t. θ. .

Since G is commutative, P a right G-torsor : $P \in H^1(S', G)$. (flat topology)

This generates a cyclic ~~subgroup~~ extension: $0 \to G \xrightarrow{e} \tilde{G} \xrightarrow{j} \mathbb{Z} \to 0$
i.e. $\tilde{G}$ is the unique extension s.t. $j^{-1}(1) = P$ as G-torsor :
we can add G-torsors (G commutative), take cyclic group generated by it, gives extension & pull this back to $\mathbb{Z}$.

Let $H^\bullet = Hom^\bullet_{f^{-1}(\mathcal{O}_{S'})}(\Omega^\bullet_{X/S'}, \Omega^\bullet_{X_2/S'})$

and define a complex $\dots \to 0 \to L^{-1} \xrightarrow{d} L^0 \to 0 \dots$

$$\begin{array}{ccc} L^{-1} & \xrightarrow{d} & L^0 \\ \| & & \| \\ G & \xrightarrow{e} & \tilde{G} \end{array}$$

This complex is quasi isomorphic to $\mathbb{Z}$..
There's a morphism $\varphi : L^\bullet \to H^\bullet$:
$\varphi^{(-1)} = i \qquad \varphi^{(0)}|_P = u \qquad \varphi^{(0)} \circ d = 0 \qquad \Rightarrow$ defines it uniquely.

In the derived category of abelian sheaves on the topological space underlying X we have a morphism $\mathbb{Z} \xrightarrow{\sim} L^\bullet$
$\Rightarrow$ get $\quad \mathbb{Z} \xrightarrow{\sim} L^\bullet \xrightarrow{\varphi} H^\bullet$ or equivalently
an element $\psi \in \mathbb{R}^0 \Gamma_X (H^\bullet)$

Composing with the canonical morphism $H^\bullet \to R\,\mathcal{H}om_{f^{-1}\mathcal{O}_{S'}}(\Omega^\bullet_{X_1/S'}, \Omega^\bullet_{X_2/S'})$
$\Rightarrow$ get element in $\mathrm{Hom}^{(0)}_{D(f^{-1}\mathcal{O}_{S'})}(\Omega^\bullet_{X_1/S'}, \Omega^\bullet_{X_2/S'}) \ni \Phi$.

$D(f^{-1}\mathcal{O}_{S'})$ derived category of sheaves of $f^{-1}\mathcal{O}_{S'}$ algebras

The element Φ is an isomorphism because it gives us a transitive system of morphisms between different liftings $\Omega^\bullet_{X_1/S'}$ of $\Omega^\bullet_{X/S}$ & also specializes to the identity if $X_1 = X_2$ (descent datum..)
$\Rightarrow$ applying $\mathbb{R}f_*$ get an isomorphism $\mathbb{R}f_*\Phi : \mathbb{R}f_{1*}(\Omega^\bullet_{X_1/S'}) \xrightarrow{\sim} \mathbb{R}f_{2*}(\Omega^\bullet_{X_2/S'})$. $\blacksquare$

Application! __Theorem__ (Deligne) Let $f: X \to S$ be a smooth projective morphism between smooth varieties.
Then the topological Leray spectral sequence
$$E_2^{pq} = H^p(S, R^q f_* \mathbb{C}) \Rightarrow H^{p+q}(X, \mathbb{C})$$
degenerates at E_2.

__Proof__ (Sketch) f is projective $\Rightarrow \exists$ a global line bundle $H \to X$ that is relatively ample (f-ample)
Consider the first relative Chern class of H :
$\gamma \in \Gamma(S, \mathcal{H}^2_{DR}(X/S, \mathbb{C}))$, $\gamma(s) = c_1(H/_{f^{-1}(s)})$
Notice that since $\gamma(s)$ is an integral class for every s
$\Rightarrow \quad \gamma \in \Gamma(S, R^2 f_* \mathbb{Z}) \subset \Gamma(S, \mathcal{H}^2_{DR}(X/S, \mathbb{C}))$

So γ is flat wrt G-M, $\nabla^{GM}(\gamma) = 0$

Algebraically this can be seen by looking at $c_1(H) \in H^2(X, \mathbb{C})$
Since $\Omega_X^\bullet$ is a resolution of $\mathbb{C}$, $H^2(X,\mathbb{C}) \simeq \mathbb{H}^2(\Omega_X^\bullet)$
There's a morphism $\Omega_X^\bullet \to \Omega_{X/S}^\bullet$ induced by the canonical quotient map $\Omega_X^1 \twoheadrightarrow \Omega_{X/S}^1$.
The Leray s.s. gives a map $\mathbb{H}^k(X, \Omega_{X/S}^\bullet) \to H^0(\mathbb{R}^k f_* \Omega_{X/S}^\bullet)$
$\Rightarrow$ γ is identified with the image of $c_1(H)$ under
$H^2(X,\mathbb{C}) \simeq \mathbb{H}^2(\Omega_X^\bullet) \to \mathbb{H}^2(\Omega^2_{X/S}) \to H^0(\mathbb{R}^2 f_* \Omega_{X/S}^\bullet)$
which is the relative de Rham

From the definition of d^{GM} as edge homomorphism for the pushforward of $0 \to \Omega_{X/S}^{\bullet-1} \otimes f^*\Omega_S^1 \to \Omega_X^\bullet/I^2 \to \Omega_{X/S}^\bullet \to 0$
$\Rightarrow \operatorname{Ker}(d^{GM}: \mathbb{R}^2 f_* \Omega_{X/S}^\bullet \to R^2 f_* \Omega_{X/S}^\bullet \otimes \Omega_S^1)$
$= \operatorname{Im}(\mathbb{R}^2 f_* \Omega^\bullet_X/I^2 \longrightarrow \mathbb{R}^2 f_* \Omega^\bullet_{X/S})$

But $\Omega_X^\bullet \to \Omega_{X/S}^\bullet$ factors through $\Omega_X^\bullet/I^2$ & we have a commutative diagram

$$\begin{array}{ccc}
c_1(H) \in \mathbb{H}^2(\Omega_X^\bullet) & \longrightarrow & H^2(\Omega_{X/S}^\bullet) \\
\downarrow & & \downarrow \\
\mathbb{H}^0(\mathbb{R}^2 f_* \Omega_X^\bullet) & \longrightarrow & H^0(\mathbb{R}^2 f_* \Omega_{X/S}^\bullet) \ni \gamma
\end{array}$$

So the variation of Hodge structures is polarized: we have flat Kähler form γ moving with the variety . .
$\Rightarrow$ do Lefschetz: get sl_2 triple,
Lefschetz decomposition of the fibers of $H^2_{DR}(X/S, \mathbb{C})$ which is horizontal wrt d^{GM}.

$\Rightarrow$ the Lefschetz operators $L^k = \gamma^k \cup \cdot : R^q f_* \mathbb{C} \to R^{q+2k} f_* \mathbb{C}$ are compatible with the differentials of the Leray spectral sequence - (come from topological data: ~~Hod~~ de Rham $\leftrightarrow$ Betti)

$\Rightarrow$
1) Suffices to check the vanishing of the differentials of Leray on the <u>primitive</u> cohomology along the fibers.

2) If $d = \dim X/S$ $\Rightarrow$ we have $H^p((R^q f_* \mathbb{C})_{prim})$:

$$\begin{tikzcd}
H^p((R^q f_*\mathbb{C})_{prim}) \arrow[r, "L^{d-q+1}"] \arrow[d, "d_2"] & H^p(R^{2d-q+2} f_* \mathbb{C}) \arrow[d, "d_2"] \\
H^{p+2}(R^{q-1} f_* \mathbb{C}) \arrow[r, "L^{d-q+1}"] & H^{p+2}(R^{2d-q+1} f_* \mathbb{C})
\end{tikzcd}$$

By Hard Lefschetz, top $L^{d-q+1}=0$ and the bottom is an isomorphism $\Rightarrow$ d_2 on the left must be zero.
Combined with (1) $\Rightarrow$ $d_2 = 0$
+ induction for higher differentials, same argument. ∎

5/7

$\boxed{\text{Cycle maps}}$

X smooth $/\mathbb{C}$, (usually projective), irreducible, $\dim = d$.

$Z^p(X)$ – group of cycles of codimension p = free abelian gen. by all irred. codim p subvarieties.

(i) The classical/topological cycle map γ:
There's a natural map $\gamma_{top}: Z^p(X) \to H^{2p}_{Betti}(X, \mathbb{C})$ by Poincaré duality – consider $i: Z \hookrightarrow X$ smooth codim p subvariety, defines functional on $H^{2d-2p}_B(X, \mathbb{C}) \xrightarrow{\gamma_{top}(Z)} \mathbb{C}$

$H^{2d-2p}_{DR}(X, \mathbb{C}) \xrightarrow{\int_Z i^*(\cdot)}$

~~H^{2p}~~ Poincaré duality:
$(H^{2d-2p}_B(X, \mathbb{C}))^\vee \simeq H^{2p}_B(X, \mathbb{C})$.

For general irreducible Z, take a resolution of singularities $\tilde{Z} \to Z \xrightarrow{i} X$, and integrate the pull back of forms to $\tilde{Z}$.. any two resolutions will differ by a set of measure zero ...
extend by linearity.

– Betti natural place to compare topological & analytic properties. –

Consider $H_Z^{p,p}(X, \mathbb{C}) \subset H^{2p}_B(X, \mathbb{C})$

$\forall Z$ codim p $\Rightarrow$ $\tilde{Z}$ is of dimension $d-p$
$\Rightarrow$ a C^∞ form ω of deg $2d-2p$ will be non-zero restricted on $\tilde{Z}$ if it has type $(d-p, d-p)$ after restriction

$\Rightarrow \quad \gamma_{top}(Z) \in H^{p,p}_{\mathbb{Z}}(X)$.

γ_{top} can be generalized to a map with values in a Tate twist of H_{Betti} ... Tate twist is a device to keep track of the way multiplicative structure on H^* interacts with possible embeddings of the abelian group of coefficients in $\mathbb{C}$.

Let $A \subset \mathbb{C}$ subring ($\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ usually) & introduce objects $A(n) \subset \mathbb{C}$, abelian groups abstractly isomorphic to the additive group ~~of~~ A, & when used as coeffs they remember weight of a cohomology class – in terms of primitive generators. – the n's add up..

Intrinsic way to define it is put $\mathbb{Z}(1) = H_2(\mathbb{P}^1)$ (noncanonically isomorphic to $H_0(\mathbb{P}^1)$) – if we know what this is in our cohomology theory. Set $A(1) = \mathbb{Z}(1) \otimes_{\mathbb{Z}} A$

$A(n) = A(1)^{\otimes n} = \mathbb{Z}(1)^{\otimes n} \otimes A$

Explicitly, define $A(n) = (2\pi i)^n A < \mathbb{C}$

The coeff $(2\pi i)$ in $\mathbb{Z}(1)$ comes from the fact that the fundamental class $[\mathbb{P}^1]$ gets identified w/ $2\pi i$ under the cycle class map $= c_1(\mathcal{O}(1))$

<u>Def</u> The nth Tate twist of the Betti cohomology with coeffs in A is $H^k_B(X,A)(n) = H^k_B(X,A) \otimes A(n) = H^k_B(X, A(n))$

$\Rightarrow$ twisted version of γ_{top}:

$\gamma : Z^p(X) \longrightarrow \cancel{H^{2p}_B(X, \mathbb{Z}}$ $\quad H^{p,p}_{\mathbb{Z}(p)}(X) \subset H^{2p}_B(X, \mathbb{C})$

$\eta \longmapsto (2\pi i)^p \gamma_{top}(\eta)$

Here $H^{p,p}_{\mathbb{Z}(p)}(X)$ is the intersection of $H^{p,p}$ w/ $H^{2p}_B(X, \mathbb{Z}(p))$.

γ takes into account how deeply a cohomology class sits wrt weights coming from products in cohomology.

$H^{p,p}_{\mathbb{Z}(p)}$ subquotient of deRham & integral cohomology – would like to lift our map γ to more refined maps to $H^\bullet_{DR}$, $H^\bullet_B(X,\mathbb{Z})$

The natural place where the fundamental class of Z lives is in local cohomology with support in Z ...

<u>Digression - Local Cohomology</u> X - complex $\overset{\text{smooth}}{\text{variety}}$, $Z \subset X$ irreducible, codim $= p$. For a complex $K^\bullet$ of sheaves of abelian groups on X, denote $\mathbb{H}^i_Z(X, K^\bullet)$ local hypercoh of X, supports in Z:

$\Gamma_Z(X, —) : \{\text{category of sheaves}\} \to Ab$

- sections supported on Z.

Compute via injective resolution or Čech complex - $K^\bullet \to I^{\bullet\bullet}$

$\mathbb{H}^i_Z(X, K^\bullet) = H^i(\Gamma_Z(X, \text{tot } I^{\bullet\bullet}))$

<u>Important properties</u>:

a. <u>Gysin sequence</u>: $U = X \setminus Z$ then a long exact

$\cdots \longrightarrow \mathbb{H}^i_Z(X, K^\bullet) \to \mathbb{H}^i(X, K^\bullet) \to \mathbb{H}^i(U, K^\bullet|_U) \to \cdots$

b. <u>Grothendieck</u> vanishing: If $F \to X$ loc. free, $j < p \Rightarrow$ $H^j_Z(X, F) = 0$

c. <u>Excision</u>: $V \subset X$ open s.t. $Z \subset V \Rightarrow \mathbb{H}^i_Z(X, K^\bullet) \xrightarrow{\sim} H^i_Z(V, K^\bullet|_V)$.

d. <u>Splitting</u>: $K^\bullet$ - complex of loc free sheaves with $K^i = 0$ for $i < p \Rightarrow \mathbb{H}^j_Z(X, K^\bullet) = 0$ for $j < 2p$, & $\mathbb{H}^{2p}_Z(X, K^\bullet) \hookrightarrow H^p_Z(X, K^p)$.

Remarks Gysin follows from observation that for a flasque sheaf F

$0 \to \Gamma_Z(X, F) \to \Gamma(X, F) \to \Gamma(U, F) \to 0$ is exact.

Vanishing - proven by Čech covering, counting the work.

Excision comes easily e.g. from Gysin.

Splitting: Filter $K^\bullet$ by good filtration $K^\bullet \supset K^{\geq 1} \supset K^{\geq 2} \supset \cdots$

Look at associated s.s. $E_1^{i,j} = H^j_Z(X, K^i) \Rightarrow \mathbb{H}^{i+j}_Z(X, K^\bullet)$

Now $H^j_Z(X, K^i) = 0$ $j < p$ by Grothendieck,

$H^j_Z(X, K^i) = 0$ $i < p$ by assumption

$\Rightarrow E_1^{i,j} = 0$ for $i + j < 2p \Rightarrow E_\infty^{ij} = 0$ $j + i < 2p$,

$E_\infty^{i,j}$ embeds into $E_1^{i,j}$ for $i + j = 2p$.

(ii) Betti cycle map γ_B:

Use local cohomology with supports for sheaves of abelian gps.

The Gysin sequence for $\mathbb{Z}(p)$ is

$\cdots \to H_B^{2p-1}(U, \mathbb{Z}(p)) \to H_Z^{2p}(X, \mathbb{Z}(p)) \to H_B^{2p}(X, \mathbb{Z}(p)) \to \cdots$

By Lefschetz duality, $H^{2p}_Z(X,\mathbb{Z}) \simeq H_{2d-2p}(Z,\mathbb{Z})$
$\Rightarrow H^{2p}_Z(X,\mathbb{Z}(p)) \simeq \mathbb{Z}(p)$, generator is fundamental class $c_B(Z) \in H^{2p}_Z(X,\mathbb{Z}(p))$, and $c_B(Z)$ maps to $\gamma(Z) \in H^{2p}_B(X,\mathbb{C})$

Define $\gamma_B(Z) \in H^{2p}_B(X,\mathbb{Z}(p))$ as the image of $c_B(Z)$
- $\gamma_B(Z)$ projects on $\gamma(Z)$.

(iii) <u>de Rham cycle map γ_{DR}</u> $Z \subset X$ irred — at generic pt Z is smooth $\Rightarrow$ find an affine open $X^0 \subset X$ & divisors $D_1, \dots D_p$ in X s.t. $D_i^0 = D_i \cap X^0$ smooth, intersect transversally and $Z^0 = \cap D_i^0$ (<u>theorem of the rank</u>.)

Look at $X^0 - Z^0$ and the affine cover $\{U_i^0 = X^0 - D_i^0\}$ Define $c(Z_0) \in H^{p-1}(X^0 - Z^0, \Omega^p_{X^0-Z^0})$ as the element given by the Čech cocycle $\frac{df_1 \wedge df_2 \wedge \dots \wedge df_p}{f_1 f_2 \dots f_p}$ on $U_1^0 \cap U_2^0 \cap \dots \cap U_p^0$ (Poincaré residue).

By Gysin, $H^{p-1}(\Omega^p_{X^0-Z^0}) \longrightarrow H^p_{Z^0}(X^0, \Omega^p_{X^0}) \to H^p(X^0, \Omega^p_{X^0})$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \; 0$, X^0 affine

Let $c(X^0, Z^0)$ be the image of $c(Z^0)$ in $H^p_{Z^0}(X^0, \Omega^p_{X^0})$.

We have a map (inclusion) $\mathbb{H}^{2p}_Z(X, F^p\Omega^\bullet) \subset H^p_Z(X, \Omega^p_X)$ by splitting. $\cap$ excision

It can be checked that $c(X^0, Z^0)$ $\quad c(X^0, Z^0) \in H^p_{Z^0}(X^0, \Omega^p_{X^0})$ comes from a class $c_{DR}(Z) \in \mathbb{H}^{2p}_Z(X, F^p\Omega^\bullet_X)$
- to show this project on quotient, get cohomology with coeffs logarithmic forms, use Poincaré summation of residues to see this vanishes.

<u>Fact</u> $c_{DR}(Z), c_B(Z)$ project to the same element in $H^{2p}_B(X,\mathbb{C})$.

This definition is motivated by splitting principle — reduce to case of dlog around a divisor $\rightsquigarrow \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$ around intersection of divisors.

$\Rightarrow \gamma_{DR} : \mathcal{Z}^p(X) \to \mathbb{H}^{2p}(X, F^p\Omega^\bullet_X)$ lifting γ

γ_{DR}, γ_B motivate introduction of Deligne cohomology:

$$\begin{tikzcd} & \mathbb{H}^{2p}(X, F^p\Omega^\bullet_X) \arrow[d] \\ \mathcal{Z}^p(X) \arrow[ur, "\gamma_{DR}"] \arrow[r, "\gamma"] \arrow[dr, "\gamma_B"'] & H^{2p}_B(X, \mathbb{C}) \\ & H^{2p}_B(X, \mathbb{Z}(p)) \arrow[u] \end{tikzcd}$$

Natural to look at fiber product of $H^{2p}_B(X, \mathbb{Z}(p))$, $\mathbb{H}^{2p}(X, F^p\Omega^\bullet_X)$ over $H^{2p}_B(X, \mathbb{C})$ as target of most refined cycle map, includes all dR, B, Dol information..

fiber products of cohomology theories not well behaved, e.g. wrt long exacts of all three $\Rightarrow$ do it on level of complexes:

<u>Def</u> The Deligne cohomology groups of X are the groups $\mathbb{H}^i(X, F^p\Omega^\bullet_X \times_{\Omega^\bullet_X} \mathbb{Z}(p))$ ($\mathbb{Z}(p)$ in deg 0) $= \mathbb{H}^i_D(X, \mathbb{Z}(p))$.

Category of complexes not abelian - initially not clear fiber product exists .. Also is there a cycle map to $\mathbb{H}^i_D$ 5/9

Denote $i_p : F^p\Omega^\bullet_X \to \Omega^\bullet_X$, $j_p : \mathbb{Z}(p) \to \Omega^\bullet_X$.

Existence of fiber product: take $F^p\Omega^\bullet_X \oplus \mathbb{Z}(p)$ and take kernel of difference map $i_p - j_p$: kernels do exist in category of complexes.

Rather let's try to find a simpler quasi-isomorphic complex..

To understand kernel $\to$ cone construction:

$f : A^\bullet \to B^\bullet$ morphism of complexes (in an abelian category) $\Rightarrow$ canonical complex, the cone of f

$C_f^\bullet = A^\bullet[1] \oplus B^\bullet$ as a graded object, with differential

$$d_{C_f} = \begin{pmatrix} d_{A[1]} & 0 \\ f[1] & d_B \end{pmatrix}, \quad d_{C_f}^2 = \begin{pmatrix} d_{A[1]}^2 & 0 \\ d_B f - f[1] d_{A[1]} & d_B^2 \end{pmatrix} = 0$$

$\Rightarrow$ short exact $0 \to B^\bullet \to C_f^\bullet \to A^\bullet[1] \to 0$

If $f: A^\cdot \to B^\cdot$ surjective (on cohomology even)
$\Rightarrow \quad K = \ker(f: A^\cdot \to B^\cdot) = C_f^\cdot[-1]$.

Now define $F^p\Omega^\cdot_X \times_{\Omega_X} \mathbb{Z}(p) := \mathrm{Cone}(F^p\Omega_X \oplus \mathbb{Z}(p) \xrightarrow{i_p - j_n} \Omega_X)\,[-1]$
$\Rightarrow$ definition of $H^k_{\mathcal{D}}(X, \mathbb{Z}(n)) := \mathbb{H}^k(C^\cdot_{i_n - j_n})$

$F^p\Omega^\cdot_X = 0 \to \dots \to 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \dots$
$\mathbb{Z}(p) \;:\; \mathbb{Z}(p) \to 0 \to 0 \to \dots$
cone $[-1]$ $:\; \mathbb{Z}(n) \to \mathcal{O}_X \to \Omega^1_X \to \dots \to \Omega^{n-2}_X \to \Omega^{n-1}_X \oplus \Omega^n_X \to \Omega^n_X \oplus \Omega^{n+1}_X \to \dots$

Consider the complex $\mathbb{Z}(n)_{\mathcal{D}} : \mathbb{Z}(p) \to \mathcal{O}_X \to \dots \to \Omega^{n-1}_X \to 0 \to 0 \dots$
$\Rightarrow$ there's an inclusion $\mathbb{Z}(n)_{\mathcal{D}} \hookrightarrow C^\cdot_{i_n - j_n}[-1]$
which is <u>quasi-isomorphism</u>. (Check)
$\Rightarrow \quad H^k_{\mathcal{D}}(X, \mathbb{Z}(n)) = \mathbb{H}^k(X, \mathbb{Z}(n)_{\mathcal{D}})$.

<u>Examples</u> 1. $n=0$: $\mathbb{Z}(0)_{\mathcal{D}} = \mathbb{Z}$, $H^k_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^k_B(X, \mathbb{Z})$.

2. $n=1$ $\mathbb{Z}(1)_{\mathcal{D}} = (\mathbb{Z}(1) \to \mathcal{O}_X)$ = kernel of exponential map :
$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$
$\Rightarrow \mathbb{Z}(1)_{\mathcal{D}} \xrightarrow{\exp} \mathcal{O}_X^*[-1]$ quasiisomorphism.
$\Rightarrow H^k_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^{k-1}(X, \mathcal{O}_X^*)$,
$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$.
So Deligne cohomology is not a vector space - not fin generated as abelian group etc etc.
<u>Exercise</u> make this identification explicit in terms of Čech cocycles.

3. $n=2$

$$\begin{array}{ccccc} \mathbb{Z}(2)_{\mathcal{D}} = [\mathbb{Z}(2) & \to & \mathcal{O}_X & \to & \Omega^1_X], \\ \downarrow & & \downarrow \exp(\frac{1}{2\pi i}-) & & \downarrow \mathrm{id}\cdot\frac{1}{(2\pi i)^2} \\ [0 & \to & \mathcal{O}_X^*(1) & \xrightarrow[d\log]{} & \Omega^1_X(1)] \end{array}$$

$\Omega^1_X(1) := \Omega^1_X \otimes_{\mathbb{Z}} \mathbb{Z}(1)$

Replace $\mathcal{O}^*$ by $\mathcal{O}^*(1)$ since exp kills $\mathbb{Z}(1)$ not $\mathbb{Z}(2)$..

$$\begin{array}{ccc} \mathbb{Z}(2) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow{\exp(\frac{1}{2\pi i}\cdot)} & \mathcal{O}_X^* \\ \downarrow & & \downarrow d\log \\ \Omega^1_X & \xrightarrow{\frac{1}{2\pi i}} & \Omega^1_X \end{array}$$

So $\mathbb{Z}(2)_{\mathcal{D}} \xrightarrow{q.i.} [\mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X][-1]$

$H^k_{\mathcal{D}}(X, \mathbb{Z}(2)) = \mathbb{H}^{k-1}(X, \mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X)$

$H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$: represented by Cech cocycles

$(\{g_{uv}\}, \{\alpha_u\}) \in$

$Z^1(\mathcal{U}, \mathcal{O}_X^*) \oplus C^0(\mathcal{U}, \Omega^1_X)$ - 0 chains

satisfying $d\log g_{uv} = \alpha_v - \alpha_u$.

$\Rightarrow$ $\{g_{uv}\}$ is a hol. line bundle L, $\{\alpha_u\}$ holomorphic connection on L : $d + \alpha_u$.

i.e. $\mathbb{H}^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ line bundles with hol. connections

X smooth projective, $i_p : F^p\Omega^\bullet_X \to \Omega^\bullet_X$ induces

$i_p : \mathbb{H}^{2p}(X, F^p\Omega^\bullet_X) \to H^{2p}_{DR}(X)$ with image $F^p H^{2p}_{DR}(X)$

(by degeneration of Hodge-de Rham).

The short exact sequence

$0 \to \Omega^\bullet_X[-1] \to C^\bullet_{i_p - j_p}[-1] \to F^p\Omega^\bullet_X \oplus \mathbb{Z}(p) \to 0$

induces long exact in cohomology

$\ldots \to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^k_B(X, \mathbb{Z}(p)) \oplus \mathbb{H}^k(F^p\Omega^\bullet_X) \to H^k_{DR}(X) \to \ldots$

$k = 2p$: $0 \to \dfrac{H^{2p-1}_{DR}}{F^p H^{2p-1}_{DR} + H^{2p-1}_B(\mathbb{Z}(p))} \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^{2p}_B(\mathbb{Z}(p)) \cap j_p^{-1} F^p H^{2p}_{DR} \to 0$

$\Rightarrow$ Def $J^p(X)$ = first term above is the p^{th} Griffiths intermediate Jacobian.

The cokernel : $H^{2p}(\mathbb{Z}(p)) \cap j_p^{-1} F^p H^{2p}_{DR}$ $\simeq$ (remove twist)

$H^{2p}_B(\mathbb{Z}) \cap j^{-1} F^p H^{2p}_{DR}$, $j : \mathbb{Z} \to \mathbb{C}$

$\simeq H^{2p}_B(\mathbb{Z}) \cap j^{-1} F^p H^{2p}_{DR} \cap j^{-1} \overline{F^p H^{2p}_{DR}}$

since $H^{2p}_B(\mathbb{Z})$ is real

$\simeq H^{2p}_B(\mathbb{Z}) \cap j^{-1} H^{p,p}(X)$ integral (p,p) classes $H^{p,p}_{\mathbb{Z}}$.

So $0 \to J^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(\mathbb{Z}(p)) \to H^{p,p}_{\mathbb{Z}} \to 0$

Disconnected group, connected component is torus $J^p(X)$

and group of components is $H^{p,p}_{\mathbb{Z}}$.

Ex. $p=1$: $\quad 0 \to Jac(X) \to Pic(X) \to NS(X) \to 0$
($Jac = Pic^0$)

Why is J^p a torus: Riemann-Hodge bilinear relations to check our lattice is full lattice: $\langle , \rangle$ nondegen on assoc graded of Hodge, & $\langle \ \rangle$ integral...

$NS(X)$ = divisors / numerical equivalence

iv) <u>The Deligne cycle map $\gamma_{\mathcal{D}}$</u>

X smooth irred (proj) of dim d,
$Z \subset X$ smooth irred subvariety, codim $= p$.

The short exact sequence of the cone $C_{i_p - j_p}[-1]$ induces long exact of <u>local</u> hypercohomology with supports on Z. $\quad H^{2p-1}_Z(X, \mathbb{C}) \to H^{2p}_{\mathcal{D},Z}(X, \mathbb{Z}(p)) \longrightarrow$

$\longrightarrow H^{2p}_Z(\mathbb{Z}(p)) \oplus H^{2p}_Z(F^p\Omega^\cdot_X) \xrightarrow{i_p - j_p} H^{2p}_{DR, Z}(X)$

We have $c_B(Z) \in H^{2p}_Z(\mathbb{Z}(p))$, $c_{DR}(Z) \in H^{2p}_Z(F^p\Omega^\cdot_X)$
→ want to find element in $H^{2p}_{\mathcal{D}}$ mapping to it..

By Grothendieck vanishing $\quad H^{2p-1}_Z(X, \mathbb{C}) = 0$:
$2p-1 < 2p$ = real codim of Z. – can also use Lefschetz duality.

But we know that $c_B(Z)$, $c_{DR}(Z)$ both map (under j_p, i_p) to same topological class
$\Rightarrow \quad \exists!\ c_{\mathcal{D}}(Z) \in H^{2p}_{\mathcal{D},Z}(X, \mathbb{Z}(p))$ mapping to $c_B(Z) \oplus c_{DR}(Z)$.

– Set $\gamma_{\mathcal{D}}(Z)$ = image of $c_{\mathcal{D}}(Z)$ in $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ via the natural map in the Gysin sequence.

<u>Multiplicative structure on Deligne cohomology</u>

– generalized cohomology with two indices, like Bloch Chow groups, motivic cohomology, Voevodsky Morava K-theories...

There's a multiplication of Deligne complexes $\cup : \mathbb{Z}(n)_D \times \mathbb{Z}(m)_D \to \mathbb{Z}(n+m)_D$ given by

$$x \cup y = \begin{cases} x \cdot y & \deg x = 0 \\ x \wedge dy (-1)^? & \deg x > 0, \deg y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Remark : Beilinson gave construction (Regulators paper) of $\cup$ by looking at graded complex, direct sum of all Deligne complexes: dR complex, $\mathbb{Z}(n)$ both have natural products $(\wedge, \cdot)$ try to construct a product on $\oplus \mathbb{Z}(n)_D$ compatible with these — one number of ambiguity : fiber product ~~of~~ two bilinear forms has rescaling-of-each ambiguity $\Rightarrow$ $\cup_\alpha$ for $\alpha \in \mathbb{R}$.
He proved they're all homotopic, & each is homotopy associative & commutative $\to$ shows it agrees with Deligne $\Rightarrow$
$\cup$ induces $H^k_D(\mathbb{Z}(n)) \otimes H^{k'}_D(\mathbb{Z}(n')) \to H^{k+k'}(\mathbb{Z}(n+n'))$

— The cycle map is functorial wrt morphism of varieties, it's multiplicative on intersection, & descends to chow ring (vanishes on differences of rationally equivalent), & restricts to A-J map on things homologically equiv to zero: integrate over chain which this bounds.

Characteristic Classes (Beilinson) (unpublished) 5/14

Weil algebras X smooth (algebraic or analytic) variety
Ω^1_X - Kähler differentials.
$\mathcal{P}(X)$ - category of all Ω^1_X-extensions :
Ob $\mathcal{P}(X)$ are short exacts of vector bundles on X : object P is

$$0 \to \Omega^1_X \to \tilde{\Omega}^1(P) \to M(P) \to 0 \qquad \tilde{\Omega}^1(P), M(P) \text{ bundles}$$

morphisms — morphisms of short exacts which are Id on Ω^1_X.

Observe that if $\pi : X \to Y$ is a morphism of smooth varieties $\to$ can pull back Ω^1_Y extensions $\ni P$:

$$\pi^* P : \quad 0 \to \Omega^1_X \to \tilde{\Omega}^1(\pi^* P) \to \pi^* M(P) \to 0$$

where $\tilde{\Omega}^1(\pi^* P)$ is push-out of

$$0 \to \pi^* \Omega^1_Y \to \pi^* \tilde{\Omega}^1(P) \to \pi^* M(P) \to 0$$

via the codifferential map $d\pi^\vee : \pi^*\Omega^1_Y \to \Omega^1_X$.

Equivalently we can say $\mathcal{P} \to$ (Sm. Varieties) is a fibered category.

Given $P \in Ob(\mathcal{P}(X))$, can construct $\tilde{\Omega}^\bullet(P)$ - the sheaf of commutative graded dg a generated by • subalgebra $\mathcal{O}_X$ in degree 0

- the $\mathcal{O}_X$ module $\tilde{\Omega}^1(P)$ in degree 1
- the $\mathcal{O}_X$ module $M(P)$ in degree 2

+ the only relation : $\forall f \in \mathcal{O}_X$, the differential of f in $\tilde{\Omega}^\bullet(P)$ should coincide with the usual exterior derivative $df \in \Omega^1_X \subset \tilde{\Omega}^1(P)$

— this forces the differential $\tilde{\Omega}^1(P) \to M(P)$ to be the map in the exact sequence [$M(P)$ is a quotient in deg 2, the quotient map to M is forced to be projection $\Omega^1 \to \tilde{\Omega}^1 \to M$ since $d^2(\mathcal{O}) = 0$ ... Ω^1 generated by $d(\mathcal{O})$ ...

Notice $\tilde{\Omega}^\bullet(P)$ is naturally filtered by the powers F^i of the dg ideal

$F^1\tilde{\Omega}^\bullet(P) = \tilde{\Omega}^{\geq 1}(P)$

(comes from augmentation map to $\mathcal{O}$).

<u>Examples</u> 1. Let P_0 be the trivial Ω^1_X extension

$P_0 : 0 \to \Omega^1_X \to \Omega^1_X \to 0 \to 0$

$\Rightarrow \tilde{\Omega}^\bullet(P_0) = \Omega^\bullet_X$ - the de Rham algebra (exterior algebra with single relation $d(f) = df$ ..)

$F^i\Omega^\bullet_X = \Omega^{\geq i}_X$ - the good filtration

$\tilde{\Omega}^\bullet(P_0)$ is called contractible ...

Notice that $\forall P \in Ob(\mathcal{P}(X))$ has a unique map $P_0 \to P$

$\Rightarrow P_0$ is the universal initial object in $\mathcal{P}(X)$,

$\Longrightarrow$ there is a morphism of filtered dga's $\Omega^\bullet_X \to \tilde{\Omega}^\bullet(P)$.

Equivalently $\tilde{\Omega}^\bullet(P)$ is an algebra over the de Rham algebra $\Omega^\bullet X$.

2. X is a point: $\Omega^1_X = 0$, any extension $P \in Ob(\mathcal{P}(X))$ reduces to a vector space $M = M(P)$. $0 \to 0 \to M \to M \to 0$

By definition $\widehat{\Omega}^0(M) = \mathbb{C}$, $\widehat{\Omega}^1(M) = M$,

$d : \mathbb{C} \to M$ is 0.

As a graded commutative algebra $\widehat{\Omega}^\bullet(M)$ will be freely generated by two copies of M

$M^{(1)} = M$, $M^{(2)} = M$ in degs 1, 2.

~~The differential~~

$$\widetilde{\Omega}^i(P) = \bigoplus_{a+2b=i} \Lambda^a M \otimes S^b M$$

The differential on $\widehat{\Omega}^\bullet(M)$ is determined by the rule

$$\begin{array}{ccc} \widehat{\Omega}^1(M) & \xrightarrow{d} & \widehat{\Omega}^2(M) \\ \| & & \\ M^{(1)} & \longrightarrow & \Lambda^2 M^{(1)} \oplus M^{(2)} \\ \Psi & & \Psi \\ m & \longmapsto & (0, m) \end{array}$$

Explicitly, get the usual Koszul differential

$$d(m_1 \wedge \dots \wedge m_a \otimes n_1 \dots n_b) :=$$

$$= \sum_{k=1}^{a} (-1)^k m_1 \wedge \dots \wedge \hat{m}_k \wedge \dots \wedge m_a \otimes m_k n_1 \dots n_b$$

<u>Definition</u> The Weil algebra of P is the filtered algebra $\widetilde{\Omega}^\bullet(P)$.

<u>Properties</u> (i) If $\pi : X \to Y$ morphism of smooth varieties, $\forall P \in Ob(\mathcal{P}(Y)) \Rightarrow \widetilde{\Omega}^\bullet(\pi^* P) = \Omega^\bullet_X \otimes_{\pi^{-1}\Omega^\bullet_Y} \pi^{-1}(\widetilde{\Omega}^\bullet(P))$

where π^{-1} is sheaf theoretic inverse images (differentials aren't $\mathcal{O}$-linear).

(ii) Let $P \in Ob\,\mathcal{P}(X)$, then the complex $F^1 \widetilde{\Omega}^\bullet(P) / F^2 \widetilde{\Omega}^\bullet(P)$ coincides with $\underset{1}{\widetilde{\Omega}^1(P)} \to \underset{2}{M(P)}$

— P is an exact sequence of vector bundles $\Rightarrow$ locally split $\Rightarrow$ locally P is pullback from a point so we're done by (i) and example 2.

~~[illegible]~~

Denote by $S^*(F_1/F_2)$ the graded commutative dga generated by the complex F^1/F^2, i.e.

$S^*(F_1/F_2) = \bigoplus_{i \geq 0} S^i(F_1/F_2)$, $S^i(F_1/F_2)$ is the Koszul complex $\Lambda^i \tilde{\Omega}^1(P) \to \Lambda^{i-1}\tilde{\Omega}^1(P) \otimes M(P) \to \dots \to \tilde{\Omega}^1(P) \otimes S^{i-1}M(P) \to S^i M(P)$

in degrees $i, i+1, \dots, 2i$.

Thus $S^*(F_1/F_2)$ looks like

	0	1	2	3	4
S^0	$\mathcal{O}_X$				
S^1		$\tilde{\Omega}^1(P)$	$\to M(P)$		
S^2			$\Lambda^2\tilde{\Omega}^1(P)$	$\to \tilde{\Omega}^1(P) \otimes M(P)$	$\to S^2 M(P)$

etc.

<u>Property (iii)</u> The natural map $S^*(F_1/F_2) \to gr_F^*(\tilde{\Omega}^\bullet_X)$ is an isomorphism of graded commutative dga.

(Trivialize locally and calculate for a point ...)

$S^*(F_1/F_2)$ much simpler - e.g. no differential $\Lambda^i \to \Lambda^{i+1}$ etc.

(iv) The canonical morphism $\Omega^\bullet_X \to \tilde{\Omega}^\bullet(P)$ is a quasi-isomorphism

- follows from iii and the fact that the sequence $0 \to \Omega^i_X \to \Lambda^i \tilde{\Omega}^1(P) \to \dots \to S^i M(P) \to 0$ is exact (symmetric power of short exact), so resolves $S^i M(P)$ which are grades .....

Let G be a complex algebraic group, $\mathfrak{g} = \mathrm{Lie}\, G$, $p: \mathcal{E} \to X$ principal G-bundle.

The dual of the Atiyah sequence of $\mathcal{E}$ is an Ω^1_X extension:

$$P_{\mathcal{E}}: \; 0 \to \Omega^1_X \to \tilde{\Omega}^1_{X,\mathcal{E}} \to \mathfrak{g}^\vee_{\mathcal{E}} \to 0$$

where $\tilde{\Omega}^1_{X,\mathcal{E}} = \mathcal{A}_X(\mathcal{E})^\vee$

$\mathfrak{g}^\vee_{\mathcal{E}} = (p_* \Omega^1_{\mathcal{E}/X})^G$ = coadjoint bundle associated to $\mathcal{E}$.

$\tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$ - the Weil algebra of $\mathcal{E}$, is just $\tilde{\Omega}^{\bullet}(P_{\mathcal{E}})$
- filtered $\Omega^{\bullet}_X$ algebra, quismorphic to $\Omega^{\bullet}_X$ as complex.

$\mathcal{E} \to Y$ principal G-bundle $\Rightarrow \tilde{\Omega}^{\bullet}_{X,\pi^*\mathcal{E}} = \Omega^{\bullet}_X \otimes_{\pi^{-1}\Omega^{\bullet}_Y} \pi^{-1}\tilde{\Omega}^{\bullet}_{Y,\mathcal{E}}$.

Key point : $\tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$ has a natural bigrading :

- Notice that $\Lambda^{\bullet}\tilde{\Omega}^1_{X,\mathcal{E}}$ has a natural differential d' -
one way to see this is to notice that
$\Lambda^i\tilde{\Omega}^1_{X,\mathcal{E}} = (p_* \Omega^i_{\mathcal{E}/X})^G$ which has natural differential, exterior diff. along fibers
(this is deRham complex along fibers.)

Equivalently recall that Atiyah algebra $\mathcal{A}_X(\mathcal{E})$ had a natural $\mathbb{C}$-linear Lie bracket
$[,] : \Lambda^2\mathcal{A}_X(\mathcal{E}) \to \mathcal{A}_X(\mathcal{E})$, which after dualizing gives $d' : \tilde{\Omega}^1_{X,\mathcal{E}} \to \Lambda^2\tilde{\Omega}^1_{X,\mathcal{E}}$,
extend by Leibnitz : $(d')^2 = 0 \iff$ Jacobi !

$\Rightarrow$ - $d' : \tilde{\Omega}^1_{X,\mathcal{E}} \to \Lambda^2\tilde{\Omega}^1_{X,\mathcal{E}} = F^2\tilde{\Omega}^2_{X,\mathcal{E}} \subset \tilde{\Omega}^2_{X,\mathcal{E}}$

- define $d'' = d - d' : \tilde{\Omega}^1_{X,\mathcal{E}} \to \tilde{\Omega}^2_{X,\mathcal{E}}$
But by construction d, d' coincide on Ω^1_X

$\Rightarrow$ d'' descends to a map $\alpha : \mathfrak{g}^\vee_{\mathcal{E}} = \tilde{\Omega}^1_{X,\mathcal{E}}/\Omega^1_X \to \tilde{\Omega}^2_{X,\mathcal{E}}$.

By property (ii) we know that $F^1/F_2 = [\tilde{\Omega}^1_{X,\mathcal{E}} \to \mathfrak{g}^\vee_{\mathcal{E}}]$
so $\tilde{\Omega}^2_{X,\mathcal{E}}/F_2 \cong \mathfrak{g}^\vee_{\mathcal{E}} = \tilde{\Omega}^1_{X,\mathcal{E}}/\Omega^1_X$

$\to$ $d''(v) \bmod F^2 = v \bmod \Omega^1_X$ and
$\alpha : \mathfrak{g}^\vee_{\mathcal{E}} \hookrightarrow \tilde{\Omega}^2_{X,\mathcal{E}}$ is an inclusion, and it splits

$\tilde{\Omega}^2_{X,\mathcal{E}} = \Lambda^2\tilde{\Omega}^1_{X,\mathcal{E}} \oplus \alpha(\mathfrak{g}^\vee_{\mathcal{E}})$

Consider the free comm. graded algebra with generators $\tilde{\Omega}^1_{X,\mathcal{E}}$ in deg 1, $\mathfrak{g}^\vee_{\mathcal{E}}$ in deg 2 : i.e. just the algebra $\Lambda^\bullet \tilde{\Omega}^1_{X,\mathcal{E}} \otimes \mathrm{Sym}^\bullet \mathfrak{g}^\vee_{\mathcal{E}}$.

We have a canonical map $\tilde{\alpha} : \Lambda^\bullet \otimes S^\bullet \to \tilde{\Omega}^\bullet_{X,\mathcal{E}}$,
$\tilde{\alpha} | \tilde{\Omega}^1_{X,\mathcal{E}} = id$, $\tilde{\alpha} | \mathfrak{g}^\vee_{\mathcal{E}} = \alpha$.

<u>Lemma:</u> $\tilde{\alpha}$ is an isomorphism of graded commutative algebras.
Proof Let $F^\bullet$ be the filtration by powers of the augmentation ideal in $\Lambda^\bullet \otimes S^\bullet$, then $\tilde{\alpha}$ is a filtered homomorphism, and $gr\,\tilde{\alpha} : gr_F \Lambda^\bullet \otimes S^\bullet \to gr_F \tilde{\Omega}^\bullet_{X,\mathcal{E}}$ is an isomorphism $\square$

[Note $\Lambda^\bullet \otimes S^\bullet$ has natural differential from d, α ..]

$*$ Set $\tilde{\Omega}^{a,b}_{X,\mathcal{E}} = \alpha \left(\Lambda^{a-b} \tilde{\Omega}^1_{X,\mathcal{E}} \otimes S^b \mathfrak{g}^\vee_{\mathcal{E}} \right) \subset \tilde{\Omega}^{a+b}_{X,\mathcal{E}}$.

$\Rightarrow \tilde{\Omega}^k_{X,\mathcal{E}} = \bigoplus_{a+b=k} \tilde{\Omega}^{a,b}_{X,\mathcal{E}}$, $\quad F^i \tilde{\Omega}^\bullet_{X,\mathcal{E}} = \bigoplus_{a \geq i} \Omega^{a,b}_{X,\mathcal{E}}$

$d = d' + d''$, $\quad d' : \tilde{\Omega}^{a,b} \to \tilde{\Omega}^{a+1,b}$, (exterior dif. along fibers)
$d'' : \tilde{\Omega}^{a,b} \to \tilde{\Omega}^{a,b+1}$

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<u>Example</u> $X = pt$, $\mathcal{E}$ trivial
$\tilde{\Omega}^{a,b}_{X,\mathcal{E}} = \Lambda^{a-b} \mathfrak{g}^\vee \otimes S^b \mathfrak{g}^\vee$ <u>Classical Weil algebra</u>:

The horizontal differential $d' : \Lambda^a \mathfrak{g}^\vee \otimes S^b \mathfrak{g}^\vee \to \Lambda^{a+1} \mathfrak{g}^\vee \otimes S^b \mathfrak{g}^\vee$
- Chevalley complex (Lie algebra cohomology) with coeffs in $S^b \mathfrak{g}^\vee$: $H^*(\mathfrak{g}, S^b \mathfrak{g}^\vee)$
The vertical differential $d'' : \Lambda^a \mathfrak{g}^\vee \otimes S^b \mathfrak{g}^\vee \to \Lambda^{a-1} \mathfrak{g}^\vee \otimes S^{b+1} \mathfrak{g}^\vee$ is the Koszul differential.

Every Ad-invariant polynomial on $\mathfrak{g}$ gives us a polynomial on any vector bundle associated to a principal $\mathfrak{g}$-bundle via Ad ... $\Longrightarrow$

$\forall i \geq 0$ we have a canonical map $W^i : (S^i \mathfrak{g}^\vee)^G \longrightarrow S^i \mathfrak{g}^\vee_{\mathcal{E}} = \tilde{\Omega}^{i,i}_{\mathcal{E}}$
$\cap$
$F^i \tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$

– Chern-Weil homomorphism.

Lemma The image of $W^\bullet$ consists of cycles wrt the differential in $\tilde{\Omega}^\bullet_{X,\mathcal{E}}$, i.e.
$W^\bullet : (S^\bullet \mathfrak{g}^\vee)^G[-2\bullet] \longrightarrow \tilde{\Omega}^\bullet_{X,\mathcal{E}}$ is a morphism of dga's.

Proof statement is local in X $\rightarrow$ may assume $\mathcal{E}$ trivial $\Rightarrow$ follows from previous example $\square$.

1) Recall the Topological Chern Classes: G topological group $\Rightarrow$ has classifying space BG, classifies principal G-bundles topologically: BG topological space, characterized uniquely up to homotopy by properties:

- $\exists$ a contractible topological space ΔG, and map $\Delta G \to BG$ principal G-bundle.
- For Every $\mathcal{E} \to X$ principal G-bundle, $\exists\, f_{\mathcal{E}} : X \to BG$ s.t. $f^*_{\mathcal{E}} \Delta G \simeq \mathcal{E}$.

(ΔG simplicial scheme, EG its geometric realization)

Example: $G = GL(n, \mathbb{C})$, $BG = Gr(n, \mathbb{C}^\infty)$, ΔG universal bundle.
Fact: G reductive, $H^\bullet(BG, \mathbb{C}) = (S^\bullet \mathfrak{g}^\vee[-2\bullet])^G$.

The Chern-Weil homomorphism corresponding to $\mathcal{E}$ is the map
$W^i_{\mathcal{E},top} : (S^i \mathfrak{g}^\vee)^G = H^{2i}(BG, \mathbb{C}) \xrightarrow{f^*_{\mathcal{E}}} H^{2i}(X, \mathbb{C})$.

Chern-Weil theory: if X is C^∞ $\Rightarrow$ $W^i_{\mathcal{E},top}$ can be taken with values in $H^i_{dR}(X, \mathbb{C})$ and can be expressed in terms of any connection ∇ on $\mathcal{E}$ as
$W^i_{\mathcal{E},top}(\varphi) = [\varphi(\mathrm{Curv}\, \nabla)]$

2). <u>deRham Chern classes</u> X algebraic, $\mathcal{E}$ principal G-bundle, we have a qu-ism $\Omega^\bullet_X \hookrightarrow \tilde{\Omega}^\bullet_{X,\mathcal{E}}$

$\Rightarrow \mathbb{H}^\bullet(\tilde{\Omega}^\bullet_{X,\mathcal{E}}) \xrightarrow{\sim} \mathbb{H}^\bullet(\Omega^\bullet_X)$,

and even on level of filtration: $\mathbb{H}^\bullet(F^i\tilde{\Omega}^\bullet_{X,\mathcal{E}}) \xrightarrow{\sim} \mathbb{H}^\bullet(F^i\Omega^\bullet_X)$ (*)

$w^i_{\mathcal{E},dR} : (S^i\mathfrak{g}^\vee)^G \to \mathbb{H}^{2i}(F^i\Omega^\bullet_X)$

is composition of $w^i_{\mathcal{E}}$ and (*).

Again $w^i_{\mathcal{E},dR}$ can be interpreted as pullback of forms by a classifying map:

Given G an algebraic group, can ask if there is a <u>simplicial</u> classifying space for G - i.e. a simplicial variety $BG_\bullet$ s.t.

- $\exists\ \Delta G_\bullet$ simplicial & "contractible" variety: i.e. $\forall X_\bullet$ simplicial variety, $\forall F^\bullet \to X_\bullet$ sheaf, $H^\bullet(X_\bullet \times \Delta G_\bullet, p^*F^\bullet) = H^\bullet(X_\bullet, F^\bullet)$

& a map $\Delta G_\bullet \to BG_\bullet$ principal G-bundle.

- $\forall\ \mathcal{E} \to X$ principal G-bundle $\Rightarrow \exists f_{\mathcal{E}} : X \to BG_\bullet$ s.t. $f_{\mathcal{E}}^*(\Delta G_\bullet) = \mathcal{E}$.

$\Delta G_\bullet$ exists: set $\Delta G_n = G^{n+1}$, $BG_n = G^{n+1}/G$ diagonal action

Also $BG = |BG_\bullet|$, $\Delta G = |\Delta G_\bullet|$

Denote $\mathcal{E}_{un} := \Delta G_\bullet \to BG_\bullet$

<u>Lemma</u> G reductive, $w^i_{\mathcal{E}un,DR}$ = isomorphism and for $j \neq i$, $H^j(BG_\bullet, \Omega^i_{BG_\bullet}) = 0$

<u>Proof</u> Use Koszul complex for $\Delta G_\bullet$, BG affine & ΔG contractible, most cohomologies die outside degree 0: 1st term deRham forms comparing with top piece = invariant polys $\square$

<u>Corollary</u> $(S^i\mathfrak{g}^\vee)^G \to H^{2i}(BG_\bullet, F^i\Omega^\bullet_{BG}) \to H^i(\Omega^i_{BG})$ is an isomorphism, + odd dimensional cohomologies vanish.

$W^i_{\mathcal{E},dR}$ admits also interpretation via a connection:
if ∇ is a hol. connection on $\mathcal{E} \to X$ $\Rightarrow$
$\nabla : \tilde{\Omega}^1_{X,\mathcal{E}} \to \Omega^1_X$ splits the Ω^1 extension $P_{\mathcal{E}}$ or equivalently gives a morphism $P_{\mathcal{E}} \to P_0$
$\Rightarrow$ $\tilde{\nabla} : \tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \to \Omega^{\bullet}_X$ left inverse to $\Omega^{\bullet}_X \xrightarrow{qu} \tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$.

So we have $\tilde{\nabla} \circ W^i_{\mathcal{E}}(\varphi) = \varphi(\tilde{\nabla}^{1,1}) \subset (\Omega^{2i}_X)^{\text{closed}}$

$\tilde{\nabla}^{1,1} = \tilde{\nabla}|_{\tilde{\Omega}^{1,1}_{X,\mathcal{E}} = \mathfrak{g}^\vee_{\mathcal{E}}} \xrightarrow{\tilde{\nabla}} \Omega^2_X$: this is just $\operatorname{curv}(\nabla)$
(or rather contraction with curvature).

3. <u>Betti Chern Classes</u> - in cycle case we constructed it first in local cohomology which depended on our cycle, where we had universal cycle class .. so now we need "universal" groups depending on $\mathcal{E}$ ..

$$\begin{cases} W^i_{\mathcal{E}} : (S^i \mathfrak{g}^\vee)^G [-2i] \to \tilde{\Omega}^{i,i}_{X,\mathcal{E}} \subset F^i \tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \subset \tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \\ \mathbb{Z}(i) \hookrightarrow \tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \quad \text{in degree zero} \end{cases}$$

$\Rightarrow$ take fiber product:
Set $U_{\mathcal{E}}(i) = \operatorname{Cone}[\mathbb{Z}(i) \oplus (S^i\mathfrak{g}^\vee)^G[-2i] \to \tilde{\Omega}^{\bullet}_{X,\mathcal{E}}][-1]$

Define universal $\mathcal{E}$-cohomology $H^j_{U_{\mathcal{E}}}(X, \mathbb{Z}(i)) := \mathbb{H}^j(X, U_{\mathcal{E}}(i))$

We have canonical maps $\varepsilon_{\mathbb{Z}} : U_{\mathcal{E}}(i) \to \mathbb{Z}(i)$
$\varepsilon_{\mathfrak{g}} : U_{\mathcal{E}}(i) \to (S^i\mathfrak{g}^\vee)^G[-2i]$.

Also have long exact sequence $(\mathbb{C} \xrightarrow{\text{quism}} \Omega^{\bullet}_X \xrightarrow{\text{quism}} \tilde{\Omega}^{\bullet}_{X,\mathcal{E}})$
from the sequence of the cone
$\mathbb{C}[-1] \to U_{\mathcal{E}}(i) \to \mathbb{Z}(i) \oplus (S^i\mathfrak{g}^\vee)^G[-2i]$

($\mathbb{C}$ is quismorphic to $\tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$ ..)

and its pushout (quotient) by $\mathbb{C} \to \mathbb{C}/\mathbb{Z}(i) \overset{\exp}{=} \mathbb{C}^*(i-1)$

$$\mathbb{C}^*(-1)\ [-1] \to U_{\mathcal{E}}(i) \xrightarrow{\varepsilon_g} (S^i g^\vee)^G[-2i]$$

(kill the $\mathbb{Z}(i)$ on left & right so middle term doesn't change!)

$\Rightarrow$ Properties of $H^j_{U_{\mathcal{E}}}(X, \mathbb{Z}(i))$:

i) $\mathbb{H}^\cdot(U_{\mathcal{E}}(i))$ functorial in X, $\mathcal{E}$

ii) $H^{j-1}(X, \mathbb{C}^*)(i-1) \longrightarrow H^j(X, U_{\mathcal{E}}(i))$ is iso for $j < 2i$, and for $j = 2i$

$$0 \to \underbrace{H^{2i-1}(\mathbb{C}^*)(i-1)}_{\substack{\text{Griffiths Jacobian:}\\ \text{torsion here sits inside}\\ \text{torsion in Griffiths}}} \longrightarrow H^{2i}(X, U_{\mathcal{E}}(i)) \to \underbrace{(S^i g^\vee)^G_{\mathbb{Z},\mathcal{E}}[-2i]}_{\substack{\text{Hodge classes}\\ \text{extra integrality structure comes from}\\ \text{Chevalley's theorem}}} \to 0$$

$$(S^i g^\vee)^G_{\mathbb{Z},\mathcal{E}} = \text{inv. polynomials } \varphi \text{ s.t. } \int_\gamma w^i_{\mathcal{E},top}(\varphi) \in \mathbb{Z} \text{ for } \forall \gamma \in H_{2i}(X, \mathbb{Z})$$

- integrality assumption related to X.

iii) If $\pi : Y \to X$ s.t. $H^*(X, \mathbb{Z}) \xrightarrow[\sim]{\pi^*} H^\cdot(Y, \mathbb{Z})$ then $\pi^* : H^\cdot_{U_{\mathcal{E}}}(X, \mathbb{Z}(i)) \xrightarrow{\sim} H^\cdot_{U_{\mathcal{E}}}(X, \mathbb{Z}(i))$

iv) The same formulas as in the Deligne cohomology define (homotopy assoc, commutative) product $U_{\mathcal{E}}(i) \otimes U_{\mathcal{E}}(j) \to U_{\mathcal{E}}(i+j)$.

Take again $\mathcal{E}_{un} = \Delta G_\cdot \to BG_\cdot$

<u>Lemma</u> $\varepsilon_{\mathbb{Z}} : \mathbb{H}^{2i}(BG_\cdot, U_{\mathcal{E}}(i)) \xrightarrow{\sim} H^{2i}(BG_\cdot, \mathbb{Z}(i))$

(analog of cohomology with supports where fundamental class lives ..)

$\square$

Now consider $\hat{X}_{\mathcal{E}} = \Delta G_\cdot \times \mathcal{E} / G$

$$\begin{tikzcd} & \hat{X}_{\mathcal{E}} \arrow[dl, "\pi_{BG_\cdot}"'] \arrow[dr, "\pi_X"] & \\ BG_\cdot & & X \end{tikzcd}$$

Now $\pi_{BG}^{*}\, \mathcal{E}_{un} \simeq \pi_X^{*}\, \mathcal{E}$, also

$$\pi_X^{*} : H^{\bullet}(X, \mathbb{Z}) \to H^{\bullet}(\hat{X}_{\mathcal{E}}, \mathbb{Z})$$

is an isomorphism because fibres of π_X are $\Delta G_{\bullet}$, contractible.

$\Rightarrow$ by iii) $\quad \pi_X^{*} : \mathbb{H}^{\bullet}(U_{\mathcal{E}}(i)) \xrightarrow{\sim} \mathbb{H}^{\bullet}(\hat{X}_{\mathcal{E}}, U_{\pi_{BG}^{*}\mathcal{E}_{un}}(i))$

Define $W_{\mathcal{E},u} : H^{\bullet}(BG_{\bullet}, \mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}(i))$ as the composition $H^{2i}(BG_{\bullet}, \mathbb{Z}(i)) \underset{\varepsilon_{\mathbb{Z}}}{\cong} H^{2i}(BG_{\bullet}, U_{\mathcal{E}_{un}}(i))$

$\xrightarrow{\pi_{BG}^{*}} \mathbb{H}^{2i}(\hat{X}_{\mathcal{E}}, U_{\pi_{BG}^{*}\mathcal{E}_{un}}(i)) \overset{\pi_X^{*}}{\cong} \mathbb{H}^{2i}(X, U_{\mathcal{E}}(i))$.

Notice that $\varepsilon_{\mathbb{Z}} \circ W_{\mathcal{E},u} = W_{\mathcal{E},top}(i)$ specializes well

3) <u>Deligne Chern classes</u>: $\Omega_X^{\bullet} \hookrightarrow \tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$ induces

$\mathcal{D}(i)_X \xrightarrow{\sim} \mathcal{D}(i)_{X,\mathcal{E}} = \mathrm{cone}(\mathbb{Z}(i) \oplus F^i\tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \to \tilde{\Omega}^{\bullet}_{X,\mathcal{E}})[-1]$

But $w^i_{\mathcal{E}} : (S^i\mathfrak{g}^{\vee})^G[-2i] \to F^i\tilde{\Omega}^{\bullet}_{X,\mathcal{E}} \quad \Rightarrow \quad \exists!$ map

$U_{\mathcal{E}}(i) \to \mathcal{D}(i)_{X,\mathcal{E}}$ which is identity on $\mathbb{Z}(i)$ and on $\tilde{\Omega}^{\bullet}_{X,\mathcal{E}}$ and is $w^i_{\mathcal{E}}$ on $(S^i\mathfrak{g}^{\vee})^G[-2i]$

$\Rightarrow$ Chern classes in Deligne cohomology.