T. Pantev - NC Geometry & Framed Sheaves 10/30/03

Heuristics: Deformations of a scheme X should be thought of as deformations of the category of coherent sheaves $Coh(X)$.

[Gabriel: can reconstruct any X from $Coh(X)$]

- relax conditions on geometric object to get better deformation problem

... e.g stacks: deform X as presheaf / Sch

Look instead at families of abelian categories : source of NC Alg Geom

- problem: moduli problems just as nasty. (Artin et al.)

Relax further: deform not $Coh(X)$ but derived category $D^b(X)$

Remark: $D^b(X)$ does not in general reconstruct X

But if X Fano or general type $\Rightarrow$ it does reconstruct.

Seems good setting to do Mori theory in higher dimensions

... analogs of blow ups, flows..

Physics: CY geometry - key features are features of $D^b(X)$ not of X itself!

Freyd Representability: decent abelian category is category of modules of a ring, so think of family of rings.. problems from Morita equivalence..

~~Not~~ satisfactory def of flat family of categories

- Look at categories of modules over A_∞-algebras, & deform the A_∞ algebra.

Bondal: Special class of infinitesimal deformations of $D^b(X)$, coming from deformations of Identity functor

$I : D^b(X) \circlearrowleft$

These are parametrized by $Ext^2(I, I) = HH^2(X)$
in category of functors

$$= Ext^2_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

$$\cong H^0(\overset{\overset{\xi}{\pi}}{\Lambda^2 T_X}) \oplus \underset{}{\overset{\text{ordinary deformations}}{H^1(T_X)}} \oplus H^2(\mathcal{O}_X)$$

(Gerstenhaber - Schack) — three natural pieces of deformation

$H^0(\Lambda^2 T_X)$: bivector fields --- deform multiplication of $\mathcal{O}_X$: deformation quantization

$$f * g = fg + \hbar\, \xi(df \wedge dg) + \dots$$

- NC deformations of X

$H^2(\mathcal{O}_X)$ = tangent space to $H^2(\mathcal{O}_X^*)$ = stacky deformations of X to $\mathcal{O}^*$-gerbes.

What happens to these when apply natural geometric constructions, such as passing to moduli space over it?

Observation: Passage to moduli mixes different types of deformations.

If we have a moduli problem on X e.g moduli of vector bundles, solved by a space M

Deformations of $X \rightsquigarrow$ deformations of M

- eg ordinary defs of M can come from either stacky or NC deformations of X! $Def_X \to Def_M$ neither injective nor surjective in usual geometry!

~~[illegible]~~

<u>Main source of examples</u>:

Moduli of framed sheaves:

S complex surface (e.g $K3$), $\lambda \in H^0(S, \Lambda^2 T_X)$ algebraic Poisson structure $= H^0(S, K_X^{-1})$

Look at moduli of framed sheaves on S: framed along $\{\lambda = 0\}$ --- only curve that survives in NC deformation.

1). Nekrasov-Schwarz, Kapustin-Kuznetsov-Orlov:

$S = \mathbb{P}^2$, $\lambda = f^3$ $\quad f \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ constant
Poisson structure $\quad \lambda|_{\mathbb{C}^2}$ = constant symplectic form

$M = \{ (E, \varphi) \mid E \to \mathbb{P}^2$ torsion free sheaf,
rank $= r$, $c_1 = 0$, $c_2 = k$
$\varphi : E|_{f=0} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ trivialization $\}$

Donaldson: $M \leftrightarrow$ instantons on S^4 of charge k framed at a point

M is hyperkähler, with twistor family $\quad M \subset \mathcal{M}$
$\mathcal{M}|_{\mathbb{P}^1 - \{0,\infty\}} = \mathbb{C}^* \times \mathcal{M}_1$ trivializes $\quad \downarrow \quad \downarrow$
$\mathcal{M}_\infty = \overline{M}$, $\mathcal{M}_0 = M$. $\quad 0 \in \mathbb{P}^1$

- deformation of M to $\mathcal{M}_1$, while $\mathbb{P}^2$ doesn't deform, changing λ just rotates line so doesn't change M.
Can we interpret $\mathcal{M}_1$ as moduli of something?
$\mathcal{M}_1$ = moduli of framed sheaves on $\mathbb{P}^2_\lambda$ NC deformation with same framing, c_1, c_2.

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2. with Katzarkov, Orlov:
Start with C curve of genus g, $S = \mathbb{P}(K_C \oplus \mathcal{O}_C) \supset D$ divisor at ∞
$\cup$
has unique Poisson structure $\quad \operatorname{tot}(K_C)$
$\lambda \in H^0(S, K_S^{-1}) = \mathbb{C}$, $\{\lambda = 0\} = 2 \cdot D$.
Look at sheaves framed along D:
fix $F \to C$ stable v. bundle on C of rank r

$M = \{ (E, \varphi) \mid E \to S \quad \varphi : E|_C \xrightarrow{\sim} F \}$
Theorem: M is holomorphic symplectic, with natural 1-parameter (twistor type) deformation $\quad \mathcal{M} \supset M$

& the other fiber $\cong$ moduli of framed sheaves on S_λ $\quad \downarrow \quad \downarrow \quad \mathbb{C} \ni 0$

Example Take $F = 0$ $\quad M$ = moduli of Higgs bundles on C

Proposition: $\left\{ \begin{array}{l} E, \varphi \text{ framed sheaf } E|_D \xrightarrow{\sim} F \\ E \text{ is globally generated over } C \end{array} \right\}$

$$\wr\!\wr$$

hol symplectic $\left\{ \begin{array}{l} W \to C \text{ vector bundle}, \quad 0 \to W \to V \to F \to 0 \\ \& \ \theta : W \to V \otimes K_C \end{array} \right\}$

"Higgs-type map"

[1-parameter deformation:
make θ satisfy Leibniz rule rather than being $\mathcal{O}$-linear]

This projects to moduli of $\{W \to C, \ 0 \to W \to V \to F\}$ - so, & Higgs map is the cotangent bundle!

So we're deforming this cotangent bundle to an affine bundle

has natural 1,1 form $\omega \in H^1(\Omega^1)$

$\Downarrow$

deform: to twisted cotangent bundle

$$\{W \to C, \ 0 \to W \to V \to F, \ \theta\} = T^*\{W \to C, 0 \to W \to V \to F\}$$

Construction: $E, \varphi \Rightarrow \pi^* \pi_* E \twoheadrightarrow E$

$$0 \to \ker(ev) \to \pi^* \pi_* E \to E \to 0$$

$$\Rightarrow \quad 0 \to \pi_* \ker \xrightarrow{=0} \pi_* E \xrightarrow{id} \pi_* E \to R^1\pi_* \ker \to 0$$

$$\Downarrow$$

$\Rightarrow$ means along fibers our bundle $\ker(ev)$ is $\oplus$ of $\mathcal{O}(-1)$ many times!

$\ker(ev) = \pi^* A \otimes \mathcal{O}(-D)$ $\quad$ A vector bundle

$$\Rightarrow \quad 0 \to \pi^* A \to \pi^* \pi_* E(D) \to E(D) \to 0$$

$\Downarrow$ push down

$$0 \to A \to \pi_* E \otimes (\mathcal{O} \oplus K_C^{-1}) \to \pi_*(E(D)) \to 0$$

two pieces!

1) $: A \to E$

2) $: A \to E \otimes K_C^{-1}$

$0 \to W \xrightarrow{(2)} V \to F \to 0$

So let $V := \pi_* E, \quad W = A \otimes K_C, \quad W \xrightarrow{(1)} V \otimes K_C$

Same vector bundle data is also Koszul dual to t-f sheaves on the twisted cotangent bundle (other Koszul dual) — so moduli of D-bundles with ~~S~~ stable framing are also moduli of twisted cotangent bundle.
((Choose polarization on curve

Twisting element $\frac{1}{2} c_1(K)$ is first differential in spectral sequence $H^i(S^j T) \Rightarrow H^0(\mathcal{D})$

So $\frac{1}{2} c_1(K)$ ~~controls twisted cotangent~~ bundle of C which is $\boxed{(T^*_{\hbar} C)^{[1]} = T^*_{\frac{1}{2}c_1(K)} C}$

$$H^p(S^q T_X) \longrightarrow H^{p+1}(S^{q-1} T_X)$$ cup with (1,1) class $-\frac{1}{2} c_1(K_X)$

For twisted diffops $\mathcal{D}(\mathcal{L})$ get $c_1(\mathcal{L}) - \frac{1}{2} c_1(K_X)$

Pf: $\mathcal{D}(\Omega^{\frac{1}{2}})$ have ~~no higher cohomology~~ differentials zero.
Same left & right modules for this
Differential is difference of left & right module structures on jets.

$$H^\bullet(\mathcal{D}(\Omega^{\frac{1}{2}})) = H^\bullet(S^\bullet(T_X))$$