7/03
UCSB

E. Diaconescu - Large N Duality & Gromov-Witten Theory

GW theory — Chern-Simons theory — Extremal transitions

$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{\Uparrow}$

String theory

<u>Plan</u>
- Closed string GW invariants, localization
- <u>Open</u> string GW invariants
- <u>Extremal</u> transitions & CS theory

$\boxed{\text{Closed String GW Invariants}}$ (Kontsevich, K-Manin, Li-Tian, Behrend-Fantechi, ~~Behrend~~ Graber-Pandharipande)

Σ genus g $\xrightarrow{\text{hol map}}$ X projective (quasi-proj) variety (main interest: $K_X = 0$)

- closed string instantons

Want to sum over all such

- Moduli space of hol maps, allow Σ to degenerate

Σ prestable curve $\xrightarrow{f} X$, $Aut(f)$ finite

(don't rec need Σ stable - only f need have fin many automorphisms)

$\Rightarrow \overline{M}_{g,0}(X, \beta) \qquad \beta \in H_2(X)$

Virtual dimension : $f : \Sigma \to X$, infi deformations of (Σ, f) given by $Ext^1(f^*\Omega_X \to \Omega_\Sigma(D), \mathcal{O}_\Sigma) =: T^1$

Obstructions $Ext^2(\;''\;) =: T^2$

$D = \Sigma p_i$

Tangent-obstruction complex

exact

$$0 \to Ext^0(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \to Ext^0(f^*\Omega_X, \mathcal{O}_\Sigma) \to$$
$$\to T^1 \to Ext^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \to$$
$$\to Ext^1(f^*\Omega_X, \mathcal{O}_\Sigma) \to T^2 \to 0$$

$Ext^0(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \to Aut(\Sigma)$, $\quad Ext^0(f^*\Omega_X, \mathcal{O}_\Sigma) = H^0(f^*T_X)$, $\quad Ext^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma) \Rightarrow Def(\Sigma)$, $\quad Ext^1(f^*\Omega_X, \mathcal{O}_\Sigma) = H^1(f^*T_X)$

Virtual dimension of $\overline{M}_{g,n}(X, \beta) = (1-g)(\dim X - 3) - \langle K_X, \beta \rangle + n$

if $\dim X = 3 \quad K_X = 0 \quad n = 0 \Rightarrow$ virtual dim $= 0$, namely expect to <u>count</u> maps

Construct virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{vir}$

algebraic cycle of virtual dimension

so in CY3 case count points in this cycle even though moduli can be positive dimensional.

$$C_{g,\beta} = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} \mathbb{1} \qquad \text{GW invariant}$$

GW potential $F_X(g_s, q)$ $\sum_{g\geq 0} g_s^{2g-2} \sum_{\beta\in H_2X} C_{g,\beta}\, q^{\beta}$

q multiindexed parameter "$q = e^{-\text{area of map}}$"

Examples $X \subset Z$ hypersurface in a (reflexive) toric variety Z

Assume & $H_2(X) \cong H_2(Z)$

$\underline{K_X} = 0 \qquad X \in |-K_Z|$

$M_{0,0}(Z,\beta)$ carries <u>obstruction bundle</u> V

Fibers: $V_{(\Sigma,f,p_i)} = H^0(\Sigma, f^*\mathcal{O}_Z(-K_Z))$

since $H^1(\Sigma, f^*\mathcal{O}_Z(-K_Z)) = 0$ this is actually a bundle

- select maps landing in X (or anything in linear system $|-K_Z|$ -- can replace $-K_X$ by any <u>convex bundle</u> $E \to Z$, ie $H^1(\Sigma, f^*E) = 0 \Rightarrow V^E$)

Using $V \to \overline{M}_{0,0}(X,\beta)$, $e(V) \in A^{\cdot}(\overline{M}_{0,0}(X,\beta))$ Euler class

$$C_{0,\beta} = \int_{[\overline{M}_{0,0}(Z,\beta)]^{vir}} e(V) \qquad \text{genus 0 GW invariant.}$$

2nd Example X <u>toric</u> CY 3-fold (<u>noncompact</u>).

$X = \mathrm{Tot}(E \to B)$ total space of bundle $/B$ <u>Fano</u>

$\dim B = 1 \Rightarrow \mathbb{P}^1$

2 toric del Pezzo $\qquad c_1(E) = -c_1(B)$

Extra condition on E : <u>Concave</u> : any (Σ, f) genus g
$H^0(f^*E) = 0$ — automatic for B toric & P
$E = \mathcal{O}(-K_B)$, ample

$B = \mathbb{P}^1$ take $E = \underline{\mathcal{O}(-1) \oplus \mathcal{O}(-1)}$, $\mathcal{O} \oplus \mathcal{O}(-2)$ ...

Reason :
$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[dr, dotted] & X \arrow[d] \arrow[loop right, "\sigma \text{ 0-sec}"] \\ & B \end{tikzcd}$$
maps factorize to base B,

so $\overline{M}_{g,n}(X, i_*\beta) \simeq \overline{M}_{g,n}(B, \beta)$ $\Rightarrow$ "local" GW invts work on B.

<u>Obstruction bundle</u> on $\overline{M}_{g,0}(B, \beta)$ $\quad V_{(\Sigma, f)} = H^1(\Sigma, f^*E)$
[bundle by concavity]

$$C_{g,\beta} = \int_{[\overline{M}_{g,0}(B,\beta)]^{vir}} e(V) = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1$$

virtual cycle changes from just 3-fold case, <u>not</u> 0-cycle; same moduli space as for X but different fund. cycle.

<u>Localization</u> (T torus), M smooth manifold with T-action, $\alpha \in H_T^*(M)$, Ξ components of the fixed locus (smooth)

Atiyah-Bott :
$$\int_{M_T} \alpha = \sum_{\Xi} \int_{\Xi} \frac{i_\Xi^* \alpha}{e_T(N_\Xi)} \qquad i_\Xi : \Xi \hookrightarrow M$$

$e_T(N_\Xi)$ equiv Euler class of normal bdle.

... expression in fraction field K_T of $H_T^*(pt)$

$$C_{g,\beta} = \int_{[\overline{M}]^{vir}} e(V) \quad \text{calculate via localization :}$$

T acts on $\overline{M}$ so that V is equivariant
$[\overline{M}]^{vir}$ equivariant version of virtual class $\in A_\cdot^T(\overline{M})$
$[\Xi]^{vir}$ virtual classes of fixed loci, induced by $[\overline{M}]^{vir}$

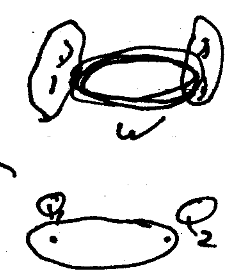
N_{Ξ}^{vir} virtual normal bundle to Ξ (object in $D^b(\Xi)$)

$e_T(V) = \alpha \in A_T^{\cdot}(\overline{M})$.
$$\int_{[\overline{M}]^{vir}} \alpha = \sum_{\Xi} \int_{[\Xi]^{vir}} \frac{i_{\Xi}^{*}\alpha}{e_T(N_{\Xi}^{vir})}$$

<u>Explicit example</u> $X = \mathrm{Tot}(\underbrace{\mathcal{O}(-1) \oplus \mathcal{O}(-1)}_{E} \longrightarrow \mathbb{P}^1)$

$H_2(X) = \mathbb{Z} \cdot (0\text{-section } C = \sigma(\mathbb{P}^1))$

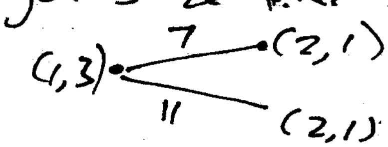
$\overline{M}_{g,0}(X, d[C]) \simeq \overline{M}_{g,0}(\mathbb{P}^1, d)$



T has 2 fixed points on X, $0, \infty$ on 0-section

Fixed loci Ξ on moduli space: $g > 0$ must map to Q_1 or Q_2, rational components can map onto C with some degree w.

Kontsevich: Associate a graph to each fixed locus
Vertices $\longleftrightarrow$ contracted components (over $0, \infty$ fixed points)
edges rational components
Vertex labelled by genus & fixed point, edges by multiplicity

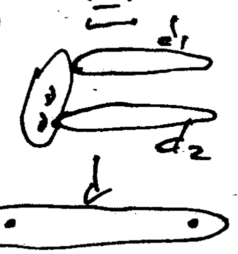


Fixed loci: $\Xi \longleftrightarrow$ graph Γ, product of Deligne-Mumford space
$$\Xi \simeq \prod_{\substack{\text{vertices}\\ v}} \overline{M}_{g(v), val(v)} \Big/ G(\Gamma)$$
val: valence of vertex
$G(\Gamma)$ = aut. of fixed maps.

Presentation of G: $1 \longrightarrow \prod_e \mathbb{Z}/d_e \longrightarrow G(\Gamma) \longrightarrow \mathrm{Aut}\,\Gamma \longrightarrow 1$

Ordinary Fundamental class of this space agrees with virtual class induced on Ξ

<u>Integrand</u>:



$$0 \to \mathrm{Aut}\,\Sigma \to H^0(f^*T_X) \to T^1 \to \mathrm{Def}\,\Sigma \to H^1 f^*T_X \to T^2 \to 0$$

K-theory class of $[N^{vir}_{\Xi}]^{*} = [H^0(f^*T_X)^m] - [H^1(f^*T_X)^m] + [\mathrm{Def}(\Sigma)^m] - [\mathrm{Aut}(\Sigma)^m]$

m = moving part ($\perp$ to T-fixed part)

Calculate as sum over edges

$$-[N_{\Xi}^{vir}] = \sum_e [H^0(f_e^*T_X)^m] - [H^0(f_e^*T_X)^m] - [H^0(f_0^*T_X)^m] + [H^1(f_0^*T_X)^m] + \Sigma_e (T_X)_{Q_i} + \Sigma_e [H^0(T_\Sigma(-p_e))^m] - \underbrace{\Sigma_e [T_{p_e}\Sigma_0 \otimes T_{p_e}\Sigma_e]}_{\text{deforming away nodes}}$$

. . . .

end up with integral of Hodge & Mumford classes on moduli of curves for each graph, need to sum up

Diaconescu II

Above: local contribution $\prod(\text{edge factor}) \int_{\overline{M}_{g,\mathrm{val}(v)}} \dfrac{c_g(E^\vee \otimes T_{Q_i}X)}{\prod(-\psi_e - \frac{1}{d_e} c_T(T_{Q_i}C))}$

Faber-Pandharipande formula:

GW potential for $X = \mathrm{Tot}(\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1)$ is

$$F_X(g_s, g) = \sum_{d \geq 1} \frac{q^d}{d(2\sin(\frac{d g_s}{2}))^2}$$

- answer indep of torus wts, so do computation for good choice of torus weights:

$$(z, u, v) \longrightarrow (e^{i\lambda_z \varphi_z}, e^{i\lambda_u \varphi_u}, e^{i\lambda_v \varphi_v})$$

with $\lambda_u = 0$, $\lambda_z + \lambda_v = 0 \Longrightarrow$

only graphs with precisely one edge contribute!

· Use identities · $c(\mathbb{E}^\vee)c(\mathbb{E}) = 1$

· $\int_{\overline{\mathcal{M}}_{g,1}} c_g(\mathbb{E})\psi^{2g-2} = b_g$

<u>Example</u> $X = \mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$

Fixed loci 

<u>Open String GW Invariants</u> Katz-Liu, Li-Song, Graber-Zaslow, Fukaya-Oh

X CY 3-fold $\supset L$ Lagrangian cycle

$\Sigma_{g,h}$ genus g + h boundary components.

$f: \Sigma \to \cancel{X}$, $f: \partial\Sigma \to L$

$\beta \in H_2(X, L)$ relative homology class, $\gamma \in H_1(L)$

$\Downarrow q^\beta$ $\quad q \sim e^{-\text{area(map)}}$ $\quad \Downarrow y^\gamma$, y ~ holonomy on boundary of D-brane $U(1)$ connection

· $\boxed{\text{Want:}}$ space $\overline{\mathcal{M}}_{g,h}(X, L, \beta, \gamma)$
· Virtual fundamental cycle ...
· orientation

$$C_{g,h,\beta,\gamma} = \int_{[\overline{\mathcal{M}}_{g,h}(X,L,\beta,\gamma)]^{\mathrm{vir}}} 1$$

$$F_{X,L}(g_s, q, y) = \sum_{g \geq 0} \sum_{\beta,\gamma} g_s^{2g-2+h} C_{g,h,\beta,\gamma}\, q^\beta y^\gamma$$

Try to calculate these assuming localization formula — need list of fixed loci & tangent-obstruction complex

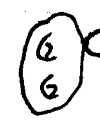
(Assume torus action preserving L)

Tangent-obstruction complex of $f: \Sigma_{g,h} \to X$.

$$0 \to \mathrm{Aut}\,\Sigma \to H^0(\Sigma, \partial\Sigma; f^*T_X, f_\partial^* T_L) \to \mathbb{T}^1 \to$$
$$\to \mathrm{Def}\,\Sigma \to H^1(\Sigma; \partial\Sigma; f^*T_X, f_\partial^* T_L) \to \mathbb{T}^2 \to 0$$

Here $f_\partial = f|_{\partial\Sigma}$, pair $(f^*T_X, f_\partial^* T_L)$: Riemann-Hilbert bundle

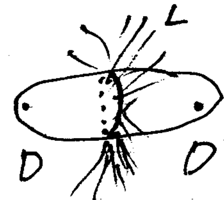
- totally real subbundle of $(f^*T_X)|_L$

Get new divisors on boundary:  "closed string degenerations"

\+  real codim 1 boundary!

<u>Example</u> $X = \mathrm{Tot}\,(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\sigma} \mathbb{P}^1) \supset L = \{|z| = 1,\ v = \overline{zu}\}$

(z, u, v) coords $\qquad L \simeq S^1 \times \mathbb{R}^2$



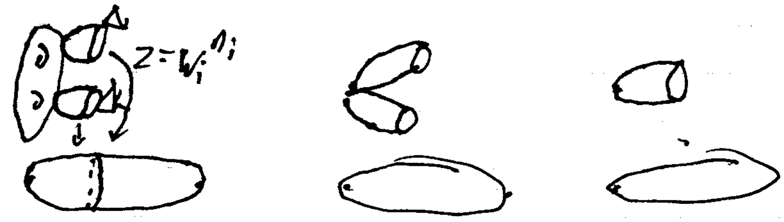
$L = \mathbb{R}^2$-fibration over meridian of zero-section

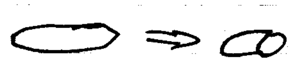
Take $\beta = d \cdot [D] \in H_2(X, L)$

$f : \Sigma_{g,h} \to X \qquad f_*[\Sigma_{g,h}] = \beta, \qquad f_*[\partial(\Sigma_{g,h})_i] = n_i \cdot [\partial D]$

i^{th} boundary component

Torus action: rotate z, u, v with weights $\lambda_z + \lambda_u + \lambda_v = 0$.

Fixed loci: 

- "chops" of closed string fixed loci 

$\mathcal{T}$ = sheaf of sections of RH $(f^*T_X, f_\partial^* T_L)$

$\Rightarrow$ complex

$$0 \to \mathrm{Aut}\,\Sigma \to H^0(\mathcal{T}) \to \mathbb{T}^1 \to \mathrm{Def}\,\Sigma \to H^1(\mathcal{T}) \to \mathbb{T}^2 \to 0$$

$$f_i = f|_{\Delta_i}, \quad \mathcal{T}_i = \mathcal{T}|_{\Delta_i}$$

Normalization sequence

$$0 \to \mathcal{T} \to f_0^* T_X \oplus \bigoplus_i T_i \longrightarrow \bigoplus_i (f^* T_X)_{p_i} \to 0$$

$$C_{g,h,d,n_i} = \sum_{\Xi} \int_{[\Xi]} \frac{1}{e_T(N_\Xi^{vir})}$$

Look at T-action on tangent-obs complex ⇒ fixed part ⊕ moving part

$$[N_\Xi^{vir}] = [H^0(\mathcal{T})^m] - [H^1(\mathcal{T})^m] + [(\mathrm{Def}\ \Sigma)^m] - [(\mathrm{Aut}\ \Sigma)^m]$$

$$\int_{[\Xi]} \frac{1}{e_T(N_\Xi^{vir})} = \frac{1}{h! \prod_i n_i} \frac{\prod_i^h e_T(H^1(\Delta_i, \mathcal{T}_i)^m)\ e_T(\mathrm{Aut}(\Delta_i, p_i))}{\prod_i e_T(H^0(\Delta_i, \mathcal{T}_i)^m)} \int_{\overline{M}_{g,h}} \frac{\mathcal{G}(\mathbb{E}^\vee \otimes T_{Q_i} X)}{\prod_i (-\psi_i + \frac{1}{n_i} e_T(T_{Q_i} C))}$$

Result is a function of the torus wts, not a number!
— need to impose boundary conditions on the moduli space
→ choose framing of normal bundle to ∂D in L

$$(u,v) = (e^{-i(a-1)\theta}, e^{ia\theta}) \qquad z = e^{i\theta} \quad a \in \mathbb{Z}$$

Want T-action to preserve framing: $\frac{\lambda_v}{\lambda_2} = a$, $\frac{\lambda_u}{\lambda_2} = 1 - a$

⇝ disc generating function

$$\sum_{d \geq 1} \sum_{m \geq 1} \frac{1}{m^2} N_d(a) q^{md} y^{mg} \qquad \text{with } N_d \text{ } \underline{\text{integers}}$$

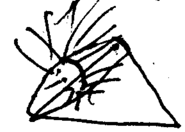
similar for higher genus .... multicover formula.

for $a=0$ $\quad F_{X,L}(g_s, q) = \sum_d \frac{q^d y^d}{2d \sinh \frac{d g_s}{2}}$

Graber-Zaslow:

$$X = \mathrm{Tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$$

$L = S^1 \times \mathbb{R}^2$



Diaconescu III

Extremal transitions between CY 3-folds (Clemens):

Y_μ family of CY 3-folds s.t.

- Y_μ is smooth $\mu \neq 0$
- Y_0 has only isolated ordinary double pts (conifold singularity)

Can find a crepant resolution of singularities $X \longrightarrow Y_0$
(X smooth $K_X = 0$)

Imprecise question: What is the relation between ~~t~~ GW potentials
$F_X(g_s, q_X)$, $F_Y(g_s, q_Y)$ Y some generic smooth Y_μ.

- the topology changes, Y_μ has vanishing cycles $L_1 \dots L_{\nu}$
which become singular pts & then become exceptional
$(-1,-1)$ curves on X. Get r relations on vanishing cycles

$h^{2,1}(Y) = h^{2,1}(X) + \nu - r$
$h^{1,1}(Y) = h^{1,1}(X) - r$ $\therefore$ fewer curves on X

Li, Ruan – symplectic cut construction
$\varphi: H_2(X) \longrightarrow H_2(Y)$. $C^Y_{g,\beta} = \sum_{\substack{\alpha \\ \varphi(\alpha)=\beta}} C^X_{g,\alpha}$

Physics alternative

- local transition (Gopakumar, Vafa)
- Transitions between (noncompact) toric CY3
- " " CY hypersurfaces

Conifold transition: Y_μ: $xy + zw = \mu$ $\subset \mathbb{C}^4$
$X \to Y_0$ resolution, $X = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1)$

Vanishing cycle L on Y_μ given by sphere
$x = \overline{y}$ $z = -\overline{w}$ [assume $\mu \in \mathbb{R}_+$]

Y_μ is diffeomorphic to $T^* S^3$
No curves on Y_μ, only one effective curve on X, the 0-section

$$F_X = \sum_{d \geq 1} \frac{q^d}{d(2 \sin \frac{d g_s}{2})^2}$$

Physics - encode F_X in L, by considering L with more structure!
N D-branes wrapped on L ...
Pick a $U(N)$ bundle $E \to L$, $E = \mathbb{C}^N \times L$

Fields: $A \in \mathcal{A}(E)$ $U(N)$ connections, with topological field theory action of Witten: Chern-Simons action
$$S_{CS} = k \int \mathrm{Tr}\left(A dA + \tfrac{2}{3} A^3\right)$$

= String field theory action for an open string theory, with boundary conditions on L.
... sum over maps from open strings ending on Y degenerate to ribbon graphs, ... calculate we get CS theory

The free energy of the topological open theory = free energy of CS theory
= ln (CS quantum invariant of S^3 assoc to $U(N)$)
$$Z_{N,k}(S^3) = e^{i\pi \frac{N(N-1)}{8}} \cdot \frac{1}{(N+k)^{N/2}} \sqrt{\frac{N+k}{N}} \prod_{j=1}^{N}\left(2\sin\frac{j\pi}{N+k}\right)^{N-j}$$

take large N expansion: let $x = \frac{2\pi}{k+N}$ $\lambda = e^{iNx}$
$$F_{CS}(x,\lambda)\Big|_{x=\frac{2\pi}{k+N},\ \lambda = e^{\sqrt{-1}Nx}} = \ln Z_{N,k}(S^3)$$

<u>Gopakumar-Vafa</u>: $\underset{\text{C-S}}{F_{CS}(x,\lambda)\Big|_{x=g_s,\ \lambda=q}} = \underset{\text{G-W of resolution}}{F_X(g_s, q)}$

Transitions between toric CY 3-folds: (W.B. Florea, A. Grassi)
$X = \mathrm{Tot}(\mathcal{O}(K_B) \to B)$ B toric del Pezzo
e.g. dP_2, dP_3 2,3-pt blowups of $\mathbb{P}^2$

$B = dP_2$ e_1, e_2 ~~exc.~~ curves on $B \Rightarrow (-1,-1)$ curves on X

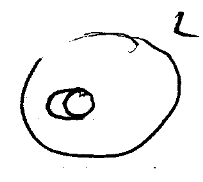
Y_μ hypersurface in $Z = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2)$
$U z_1 + V z_2 z_3 = \mu$ $\mu = 0 \Rightarrow$ ODP at $U = V = z_1 = z_3 = 0$
& $U = V = z_1 = z_3 = 0$.

Y has no effective holomorphic curves

Need more than just D-branes wrapping the 2 vanishing cycles (get just 2 decoupled CS theories)

Witten: missing ingredient – holomorphic curves with boundary on L_1, L_2

— need to add open string instanton corrections to the CS theory.



Simplified model: isolated disc ending on L
– compact version of disc ending on $\mathbb{R}^2 \times S^1$ last time.

$\Rightarrow$ instanton effect $\quad S(A) = S_{CS}(A) + \frac{1}{g_s} q \, \mathrm{Tr}\, \mathrm{Hol}(A, \partial D)$

$(q = e^{-\mathrm{area}(D)})$ $\quad F = F_{CS} + \log \langle e^{\frac{1}{g_s} q \mathrm{Tr}\, \underbrace{\mathrm{Hol}(A,\partial D)}} \rangle_{CS}$

$\overset{!!}{\checkmark}$

– Wilson line in the CS theory

∂D is knot in S^3 – make sense of this via knot invariants!

$$\langle e^{\frac{1}{g_s} q \mathrm{Tr}\, V} \rangle_{CS} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{q}{g_s}\right)^n \langle (\mathrm{Tr}\, V)^n \rangle_{CS}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{q}{g_s}\right)^n \sum_{\substack{R \text{ rep of} \\ U(N)}} C_n(R) \underbrace{\langle \mathrm{Tr}_R V \rangle_{CS}}$$

$\shortparallel$

$U(N)$ Jones poly of ∂D in rep R

Here A is off-shell: not flat, doesn't satisfy eqns of motion – or flat, cont arbitrary connection! – ~~cont~~ singularities on Wilson line, don't specify connection but integrate over all connections.

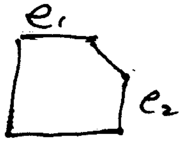
Must add many corrections: multi-covers, higher genus maps, families

Set up an enumerative problem: $\overline{\mathcal{M}}_{g,h}(Y, L; \beta)$ stable open string maps + virtual fund class, do ...

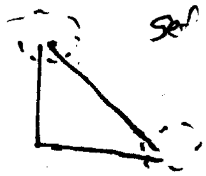
Diaconescu V

Topological Vertex

(Aganagic-Klemm-Marino-Vafa, Diaconescu-Grassi)

e

e_1, e_2

local $\mathbb{P}^2$ $\Longrightarrow$ 

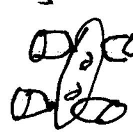
e_i: exceptional curves

get two linked circles + degenerations

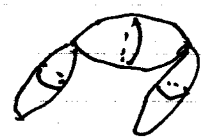
Closed string graphs & exceptional curves get isolated
- end up with far fewer graphs.
Want a more local ~~we~~ way to see CS theory appearing, without having to go to a transition...

X arbitrary toric 3-fold + config of T-invariant $\mathbb{P}^1$'s form trivalent graph

Introduce noncompact Lagrangian cycles $S^1 \times \mathbb{R}^2$ locally dividing each $\mathbb{P}^1$ into two discs
- cutting isolates a piece of the graphs



open string graphs: genus g over vertex + a bunch of discs.



- Want gluing algorithm to generate all closed string graphs by sewing open graphs
- find Chern-Simons expression for local contributions of open graphs

Write formula in terms of winding vectors, components give number of maps with given winding numbers around the boundary circles & three framings