R. Dijkgraaf - Geometry & Matrix Models I

8/~~4~~/03

Big Picture

DV
N=1 SUSY gauge theories
Seiberg: holomorphy:
F-terms $\int d^2\theta \, W(\phi)$
- capture IR dynamics / vacuum structure

string coupling $\Longrightarrow$

effective geometry (SW)
- Riemann surfaces, or more generally topological strings on $\boxed{\text{CY 3-folds}}$ (B-type)

strings!

go back by geometric engineering of gauge theory

perturbation theory $\Downarrow$

anomalies

DO
Feynman super graphs,
chiral ring
"analyticity": good convergence properties

planar diagrams $\Longrightarrow$

$\boxed{\text{matrix model}}$

large N: geometry appears as master field summing large N

double scaling limit $\Downarrow$

singular CY geometry

noncritical (super) strings

$\Downarrow$

Kontsevich matrix models

exactly solvable via integrable hierarchies of free fermions + <u>branes</u>

very strong link of integrable hierarchies with CY geometry!

"quantum CY manifolds"
CY given by $H=0$
Branes in CY, to all orders in $g_{string} = \hbar$
$\Longleftrightarrow$ solving Schrödinger operators $\hat{H}\psi = 0$
NC geometry $[x,y] = i\hbar$

(2d Super CFT: F-terms fix data ... maybe for massless (conformal) N=1 theory same works?
look at everything as perturbation of conformal fixed point...)

$D=4$ $N=1$ SUSY: $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \gamma^\mu_{\alpha\dot\beta} P_\mu = P_{\alpha\dot\beta}$

$\alpha, \dot\beta = 1,2$ spinor indices

Super space: $(x^\mu, \theta^\alpha, \bar\theta^{\dot\beta})$

$$Q_\alpha = i\frac{\partial}{\partial\theta_\alpha} - \frac{1}{2}\bar\theta^{\dot\beta}\frac{\partial}{\partial x_{\alpha\dot\beta}}$$

$$D_\alpha = \partial_\alpha + \frac{1}{2}\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}$$

Note that D_α is conjugate to ∂_α: $D_\alpha = e^{-u}\partial_\alpha e^{u}$

$u = \frac{1}{2}\theta^\alpha\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}$ ... convenient gauge $\Rightarrow$

$$\bar{D}_\alpha = \frac{\partial}{\partial\bar\theta_\alpha}.$$

Natural cohomology problem: $\bar{Q}_\alpha^{\,2} = 0$

Chiral ring = $\bar{Q}$ cohomology $\overline{\bar{Q}_\alpha \cdot \mathcal{O} = 0} / \mathrm{Im}\,\bar{Q}_\alpha$

– Lowest components of chiral superfields

$\bar{D}\,\Phi(x,\theta) = 0$ (no $\bar\theta$ dependence) – holomorphic objects.

Note $\partial_\mu \mathcal{O} = [\bar{Q}_{\dot\alpha}, [Q_\alpha, \mathcal{O}]] \simeq 0$ in cohomology

SUSY vacuum $\bar{Q}|0\rangle = 0 \Rightarrow$ correlation functions

$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n)\rangle = \langle 0|\mathcal{O}_1 \ldots \mathcal{O}_n|0\rangle$ are

1. independent of x_i & of representatives of class in chiral ring

2. If theory is massive (no massless light dofs of freedom) + "cluster decomposition" $\Rightarrow$ diagonalize chiral ring:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n\rangle = \langle\mathcal{O}_1\rangle \ldots \langle\mathcal{O}_n\rangle$$

property of particular vacuum not of theory

– depends on base ...

Chiral fields Φ – Lagrangian has two types of terms depending on all θ's.

F-terms: $\int d^2\theta\, d^4x\, W(\phi_1, \ldots \phi_n)$

W tree-level superpotential: function of chiral superfields ...

__Gauge theory:__ gauge group G, ϕ lies in rep R of G, require $W(\phi)$ G-invariant. … invariant polynomial (typically) of a G-rep.

· Gauge field: can describe as super gauge potential – Lie algebra-valued function $V(x, \theta, \bar\theta)$ $\Longrightarrow$ Super field strength (curvature): spinor, two components

$$W_\alpha = -i \bar D^2 e^{-V} D_\alpha e^{V}.$$

– __chiral__ superfield $\bar D W_\alpha = 0$, same role as matter fields Φ

In components $W_\alpha = \underbrace{\lambda_\alpha}_{\substack{\text{"gaugino"}\\ \text{adj-valued spinor}}} + \theta^\beta \underbrace{F_{\alpha\beta}}_{\substack{\text{self-dual part}\\ \text{of connection } F^+_{\mu\nu}}} + \dots$

Any chiral operator $\mathcal{O} \Rightarrow$

$$[W_\alpha, \mathcal{O}] = [\bar Q_{\dot\alpha}, \partial_{\dot\alpha\alpha} \mathcal{O}] \simeq 0 \text{ in cohomology:}$$

commutators are all $\bar Q$-exact!

$\Rightarrow$ $W_\alpha W_\beta + W_\beta W_\alpha \simeq 0$: anticommute up to $\bar Q$-exact term.

Glueball superfield: $S = \frac{1}{32\pi^2} \mathrm{Tr}(W_\alpha W_\beta)\, \varepsilon^{\alpha\beta}$

eg to write super Y-M action this is an F-term:

$$\int d^4x\, d^2\theta\; 2\pi i\, \tau \cdot S + \text{complex conj} \quad \text{c.c.}$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad \text{complex gauge coupling.}$$

__Basic problem:__ Given chiral multiplets ϕ, $W(\phi)$ superpotential compute effective superpotential $W_{eff}(S)$

— Super analogue of det of Dirac operator in background connection

[Chiral ring: consider only gauge invariant operators $\mathcal{O}$..]

Deformed $N=2$ theory

$G = U(N)$, $\phi \in$ adjoint rep (ϕ $N\times N$ $\mathbb{C}$-matrix)

θ-term $\int d^2\theta \, \mathrm{Tr}\, W(\phi)$. $W = \sum_{k=1}^{n+1} \frac{1}{k} t_k \phi^k$, $t_{n+1} = 1$

n moduli $t_1, \dots, t_n$

Assume Isolated critical points $W'(\phi) = \prod_{i=1}^{n} (\phi - a_i)$ 

so in small fluctuations ~ Gaussian:

massive scalar at each of these (crit. pts)

Classical vacua: minimize W functional on ϕ gauge invariantly.

Write $\phi = U\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}U^{-1}$, need all $W'(\lambda_i) = 0$

– so eivals distribute among crit points of W 

– partition $N = N_1 + \dots + N_n$

So unbroken gauge group at a critical point is product $U(N_1) \times \dots \times U(N_n)$

$S_i = \mathrm{Tr}\, W_i^2$ in factor $U(N_i)$: objects in gauge field

– compute $W_{eff}(\underbrace{S_1, \dots S_n}_{\text{glueballs}}; \underbrace{t_1, \dots, t_n}_{\text{moduli}})$ effective action $\Rightarrow$ $2n$ parameters

S_i = trace of W in $U(N_i)$ factor : gauge field in i^{th} factor. t_i ordered, S_i not ordered.

$$W_{eff} = \sum_{i=1}^{n} \Big[N_i \frac{\partial \mathcal{F}_0(s,t)}{\partial s_i} + 2\pi i \underbrace{\tau}_{\text{bare coupling}} S_i \Big]$$

$\mathcal{F}_0$ = Prepotential, universal function: doesn't know about $U(N)$ or N_i : just depends on degree of W!

Facts • $\mathcal{F}_0^{pert} = \sum_{\substack{\text{planar}\\\text{diagrams}}} \frac{1}{\# \mathrm{Aut}\,\Gamma} \, \mathrm{weight}(t_k)\, S^h$ h = # holes

for Feynman diagrams with action $W(\phi) = \sum \frac{1}{k} t_k \phi^k$

$\Gamma =$  ....

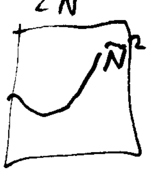
• Associated matrix model ϕ $\tilde{N} \times \tilde{N}$ matrix
$\tilde{N}$ not connected to ranks of gauge groups . . . .

$$Z = \frac{1}{\mathrm{vol}\, U(\tilde{N})} \int d\phi \, \exp\left[\frac{1}{g_s} \mathrm{Tr}\, W(\phi)\right]$$

. . . gauged matrix model: ϕ descendant of original gauge field ϕ . . . $U(\tilde{N})$ - invariant: want to think of ϕ as gauge invariant quantity.

holomorphic integral w general complex polynomial - integral might not make sense:

$2\tilde{N}^2$



pick cycle / contour in space of complex matrices where integral makes sense . . . depends on choice of contour . . . integral on space of complexified matrices, avoiding nondiagonal matrices . .

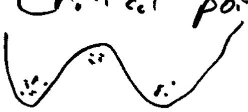
Weff: Quantum field ϕ, have action
$\int \mathcal{D}\phi \, e^{\int d^4x\, d^2\theta\, \mathrm{Tr}_{\tilde{N}}(\sum t_k \phi^k)}$ + coupling $(W^\alpha \ldots)$

SS

$e^{\int d^4x\, d^2\theta\, W_{eff}(S)}$ effective action, depends only on S: all other ells in chiral ring are zero: $SU(N)$ chiral ring is generated by S, eg $\mathrm{Tr}\, W^3 = \bar{Q}(\ldots)$

Weff depends only on chiral ring in general, correlation functions in vacua . . .

Critical points of $\mathrm{Tr}\, W(\phi)$: factor $\tilde{N} = \tilde{N}_1 + \ldots + \tilde{N}_n$



eigenvalues of matrix $\phi = U \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^{-1}$

't Hooft limit : $g_s = \hbar \to 0$ (quantum mechanics interpretation / NC geometry)
filling fractions $\tilde{N}_i \to \infty$ $S_i = \tilde{N}_i \hbar$
"Quantum numbers $\tilde{N}_i$" $\rightsquigarrow$ S_i classical versions of quantum numbers

$$Z \underset{\substack{\text{semiclassical}\\ \text{expansion}}}{\sim} \exp \sum_{g \geq 0} g_s^{2g-2} F_g(S_1, \ldots S_n; t_1, \ldots t_n)$$

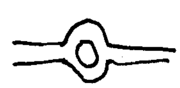
$S_1 \ldots S_n$ $\quad$ S_i now parametrize moduli space of <u>critical points</u> in the large N limit.

At finite N S_i are quantized, discrete .. but
finite g_s $\quad$ become real variables in large N,
answer in fact holomorphic in S_i's ...
- expansion around a particular choice of N_i

<u>Supermatrices</u> [D-branes & anti D-branes together ...]

$\phi \in U(N_+ | N_-)$

Action $\mathrm{Str}\, W(\phi)$ super trace



ribbon diagram expansion : $\mathrm{Tr}\, 1 = N$ for every whole
in super version : $\mathrm{STr}\, 1 = N_+ - N_-$

- will be equivalent to <u>ordinary</u> matrix model
with $N' = N_+ - N_-$ ....

Eigenvalues : $\phi = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U^{-1}$

matrix model $Z = \int \prod d\lambda_i \cdot \underbrace{\prod_{i<j} (\lambda_i - \lambda_j)^2 \quad \exp \frac{1}{g_s} \sum W(\lambda_i)}_{\exp \frac{1}{g_s} S_{eff}(\lambda_1 \ldots \lambda_n)}$

$$S_{eff}(\lambda_1 \ldots \lambda_n) = \sum W(\lambda_i) + 2 g_s \sum_{i<j} \log(\lambda_i - \lambda_j)$$

At saddle point : $W'(\lambda_i) = 2 g_s \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$

Put "probe" eigenvalue (nondynamical) $x = \lambda_{N+1}$

$S_{eff}(x)$ in solution of saddle pt eqn for other λ_i's $\Rightarrow$

$$y(x) = \frac{dS_{eff}}{dx} = \underset{\text{classical}}{W'(x)} + \underset{\text{1-loop}}{2 g_s \sum \frac{1}{x - \lambda_i}}$$

Resolvent of matrix model

$$w(x) = \frac{1}{\hat{N}} \operatorname{Tr} \frac{1}{x-\phi} = \frac{1}{\hat{N}} \sum \frac{1}{x-\lambda_i}$$

So write $y = W'(x) + 2(g_s \hat{N})\, w(x)$

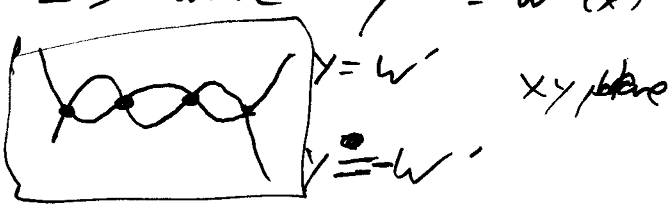
$S = \sum S_i$ overall 't Hooft parameter

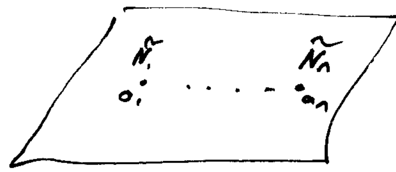
Pictures: $g_s = 0$ no quantum corrections: $y = W'(x)$

[in supermatrix version: 2 kinds of evals, $\pm$ charged

$\Rightarrow$ $y = \pm W'(x)$ = charge • potential]

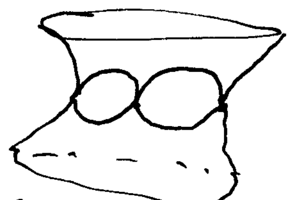
$\Rightarrow$ write $y^2 = W'(x)^2$





x-plane

$\Rightarrow$ RS with nodal points



Introduce quantum correction - introduce angular variable $\Rightarrow$

Claim: Loop equations

$g_s = \hbar \neq 0$ $\quad \tilde{N}_i \longrightarrow \infty$

$$y^2 = W'(x)^2 + f_{n-1}(x) \longrightarrow \text{polynomial of degree } n-1$$

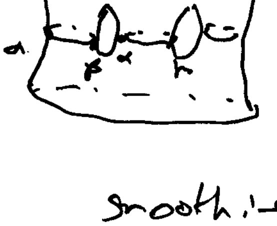
$$f(x) = \sum_0^{n-1} b_k x^k \qquad n \text{ new variables } b_k$$

natural 1-1 corresp

i.e. n extra variables S_k we're

looking for encode the quantum corrections b_k . . .

$\Rightarrow$



hyperelliptic curve

$$y^2 = W'^2 + f = P_{2n}(x) \text{ polynomial}$$

$$= \prod_{i=1}^{n} (x-\alpha_i)(x-\beta_i)$$

α_i, β_i parametrize branch cuts:

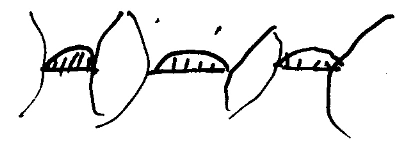
smoothing out of our double points a_i

Rolling in g_s correction gives interaction of eigenvalues, they now behave like fermions & fill up potential well : e'value density $\rho(x)$

$$= \frac{1}{\tilde{N}} \sum \delta(x-\lambda_i)$$

becomes a smooth function

– get profile



Proof of loop equation: $\omega(x)^2 = \frac{1}{\tilde{N}^2}\left(\sum \frac{1}{x-\lambda_i}\right)^2$

$\sum \frac{1}{x-\lambda_i}$ simple pole at each critical point

The square has some double poles but more pairs of single poles, which dominate in large $\tilde{N}$

$$\omega(x)^2 \sim \frac{1}{\tilde{N}^2} \sum_{i \neq j} \frac{2}{(x-\lambda_i)(\lambda_i-\lambda_j)} = \frac{1}{\tilde{N} g_s} \sum_i \frac{-W'(\lambda_i)}{x-\lambda_i}$$

$$\left(\text{Since } W'(\lambda_i) = \sum_{j \neq i} \frac{2 g_s}{\lambda_i - \lambda_j}\right) \qquad = \frac{1}{\tilde{N} g_s} \sum_i \frac{-W'(x) + W'(x) - W'(\lambda_i)}{x-\lambda_i}$$

$$= \frac{1}{g_s \tilde{N}} \omega(x) W'(x) + \underbrace{\frac{1}{\tilde{N}} \sum_i \frac{W'(x) - W'(\lambda_i)}{x-\lambda_i}}$$

very sensitive to how e'vals are distributed $\longleftarrow$ Polynomial $\frac{1}{4s} f_{n-1}(x)$

$\sim \sum \tilde{N}_i \; W''(a_i)$ to first approximation

$S = g_s \tilde{N}$

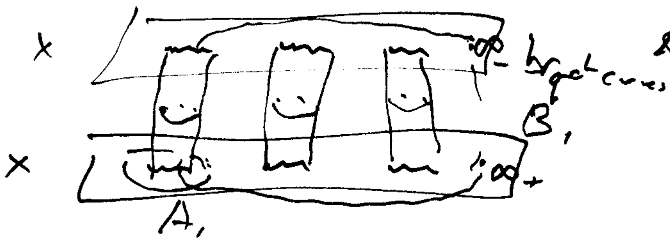
$\Rightarrow$
- Algebraic curve $y^2 = W'(x)^2 + f(x)$
- meromorphic 1-form $y\,dx = d\,S_{eff}(x)$

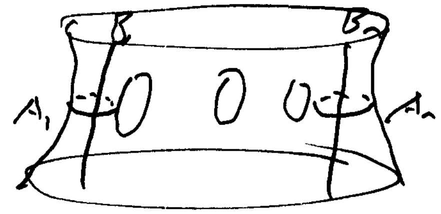
variation of eff action wrt probing e'value ...

holomorphic on affine curve but with pole at ∞

$$y = \sum t_k x^{k-1} + \underbrace{g_s \sum \frac{1}{x-\lambda_i}} \rightsquigarrow g_s \tilde{N} \frac{1}{x} + O\left(\frac{1}{x^2}\right)$$

$\Rightarrow$ <u>Special geometry</u>





A, B cycles

Compute periods $\oint_{A_i} y\,dx$: at finite N

y had simple pole with residue g_s around each eigenvalue

$\Rightarrow \oint_{A_i} y\,dx = g_s \hat{N}_i = S_i$. $\quad$ [$y = p_x$ momentum conjugate to x]

Bohr-Sommerfeld quantization wrt $\hbar$

So coefficients $b_k \longleftrightarrow S_k$ = symmetric polys in b_k.

$\oint_{B_i} y\,dx = 2\int_{a_i}^{\infty} d\,S_{eff}(x)$: effect of taking out one eigenvalue from system

$= \dfrac{\partial F_0}{\partial \hat{N}_i g_s} = \dfrac{\partial F_0}{\partial S_i}$ $\quad$ variation of free energy wrt changing $\hat{N}_i$

$\Rightarrow$ Special geometry : $\quad y^2 = W'^2 + c$ $\quad$ fn of $2n$ variables
$\qquad\qquad t \qquad s$

Study variation only wrt S's . S's are "normalizable moduli", t's are "nonnormalizable"
$\quad$ ↳ want to integrate/minimize over.

- Multimatrix models $\phi_1, \ldots \phi_{p-1} \Rightarrow$ curves of form
  $y^p = P(x)$ $\quad$ p-fold covers :
  take quiver action for A_{p-2} $\quad \phi_1 \rightleftarrows \phi_2$ $\quad$ & bifundamentals
  matrix chain
  - bifundamentals .. work for arbitrary ADE diagrams.
  Can put arbitrary potentials at each node.
- Double scaling limit : tune W & critical points, merge $\boxed{\text{merge}}$ cuts
  can get any P we want : $\quad y^2 = x^{2n+1} \ldots$
  $y^p = x^q + \ldots$ $\quad$ (p,q) minimal models coupled to gravity.

Multimatrix model:

$\sum W_i(\phi_i) + Q_{ij}\,\phi_i\,\tilde{Q}_{ji}$ arrows go both ways

eigenvalues charged by elt of Lie algebra of ADE
charge lattice = root lattice;
quiver models are those that come from Coulomb gas
picture in terms of eigenvalues.

$$\frac{\partial F_0}{\partial t_k} = \langle \mathrm{Tr}\,\phi^k \rangle \text{ in the matrix model}$$

$$= \langle \mathrm{tr}\, W^2 \phi^k \rangle \text{ in the gauge theory}$$

– think of F_0 as depending on ∞ many t_k
leading ones determine RS / universality class
all higher ones formal variations, formal asymptotic expansion.

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Matrix Model $Z = \int d\phi \, \exp \frac{1}{g_s} \mathrm{Tr}\, W(\phi)$ ϕ complex matrix

$\left(W = \sum_{k=1}^{n+1} t_k \phi^k \right)$ $\xRightarrow[N\to\infty]{}$ alg. curve

$y^2 = W'(x)^2 + f_{n-1}(x)$ loop eqns

Moduli : t_k , $S_k = \oint_{A_k} y \, dx$ A-periods

$\eta = y\,dx$ mero 1-form : $\frac{\partial \eta}{\partial S_k} = \sum M_{kl} \frac{x^l}{y}$ $l < n$

differentials of 1st kind, hol at ∞

$\frac{\partial \eta}{\partial t_c} \sim \frac{x^{n+l}}{y} dx$ 2nd kind mero at ∞

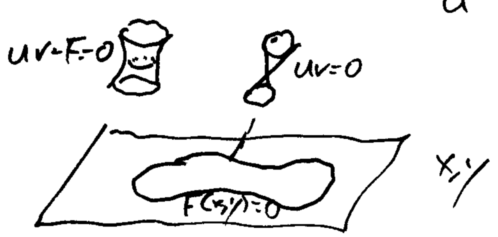
$\Rightarrow$ distinction of s, t variables.

CY 3fold picture : noncompact hypersurface in $\mathbb{C}^4$

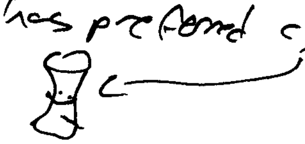
$X : uv + F(x,y) = 0$

Tian-Yau : these noncompact CY admit Ricci flat metrics conical (flat) at ∞ .

$\Omega = \frac{du\, dx\, dy}{u}$ holomorphic 3-form on X



fiber $uv+F=0$ has preferred cycle



$dx \wedge dy = \oint \frac{du}{u} dx\,dy$: reduce to complex 2-dim picture, with standard symplectic form!

(x,y) phase space

$F(x,y)$: Hamiltonian , curve is "Fermi level" level set of $F(x,y)$

$\Rightarrow$ want to quantize $[x,y] = i g_s$

$g_s \longleftrightarrow \hbar$, N = quantum #

Period on 3-cycle $\int_{C_3} \Omega = \int_{F\geq 0} dx\, dy = \oint_{F=0} y\, dx$ periods
- take real slice

$F \geq 0$ in real picture

So $y\,dx$ comes from classical mechanics picture.

$N \to \infty$, $\hbar \to 0$ $\hbar N$ fixed : classical limit.

Loop equations work for all genus :
diffeomorphisms of ϕ $\delta\phi = \phi^{n+1}$ $n \geq -1$
$L_n = \phi^{n+1}\frac{\partial}{\partial\phi}$ redefinitions of matrix integral

$\Rightarrow$ Ward identities $0 = \int d\phi \frac{\partial}{\partial\phi_{ij}} \left\{ \phi_{ji}^{n+1} e^{\frac{1}{g_s}W} \right\}$

$$= \left\langle \sum_p \mathrm{Tr}\,\phi^p\, \mathrm{Tr}\,\phi^{n-p} + \frac{1}{g_s} \mathrm{Tr}(\phi^{n+1}\partial W) \right\rangle$$

Rewrite as $L_n \cdot Z = 0$ Virasoro constraint

$$L_n = g_s^2 \sum \frac{\partial^2}{\partial t_p \partial t_{n-p}} + \sum k t_k \frac{\partial}{\partial t_{n+k}}$$

action of diffeomorphism $\to$ second order diffop on Z.

In Resolvent language : $y(x) = W'(x) + 2g_s\, \mathrm{Tr}\frac{1}{x-\phi}$ $\rightsquigarrow$ comes from Coulomb potential

$$0 = \int d\phi \frac{\partial}{\partial\phi_{ij}} \left\{ \left(\frac{1}{(x-\phi)}\right)_{ji} e^{\frac{1}{g_s}W} \right\}$$

$$= \left\langle \left(\mathrm{Tr}\frac{1}{x-\phi}\right)^2 + \frac{1}{g_s}\mathrm{Tr}\frac{W'(\phi)}{x-\phi} \right\rangle$$

think now of y as quantum field

Recall $y(x)$ = force on probe eigenvalue $\lambda = x$
$y = \partial_x \varphi$ $\varphi(x)$ action of eigenvalue $= \underset{\text{classical potential}}{W(x)} + 2g_s\,\mathrm{Tr}\,\underset{\text{Coulomb}}{\log(x-\phi)}$

$\Rightarrow$

$$y = \partial_x \varphi = \sum k t_k x^{k-1} + 2g_s \frac{\partial}{\partial t_k} x^{-k-1}$$

$\downarrow$

free boson representation; y lives on curve

Tuning t_k's (coupling constants $\longleftrightarrow$ value of y at $x=\infty$)
- - unnormalizable variables in AdS/CFT language.

$\frac{\partial}{\partial t_k} \text{tom} \stackrel{?}{=} \langle \mathrm{Tr}\,\phi^k \rangle$ = response of MM to coupling constants

Stress tensor of quantum field

$$T(x) = (\partial_x \varphi)^2 = \sum L_n x^{-n-2}$$

$y^2 = \langle T(x) \rangle$
should be algebraic fn of x : polynomial in x
so want negative powers to vanish :

$$\langle T(x) \rangle_- = 0 \iff L_n = 0 \quad n \geq -1$$

i.e perturb $y(x)$ so that well behaved at small x,
change curve at ∞ but be smooth inside
- only deform at ∞ $\longleftrightarrow$ loop equations .. interpreted as expectation value of y

ϕ = effective action of single eigenvalue at x

$\Rightarrow$ write $e^{\varphi(x)/2g_s} \doteq \psi(x)$ $\quad e^{-\varphi(x)/2g_s} \doteq \psi^*(x)$

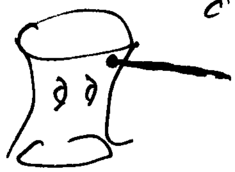
$= e^{W(x)/2g_s} \cdot \det(x-\phi)$ $\qquad$ wave function

[: loop eqns are differential relations, become algebraic in planar limit]

[Double scaling limit $\quad$ send one branch point to ∞
send curve to $y^2 = x$, get pure gravity Virasoro constraints]

$\varphi \leadsto$ WKB action of wave function $\psi(x)$
Bosons are coherent states / planar limit / WKB approx
fermions $\longleftrightarrow$ wave functions

CY picture: put branes in noncompact (u, v) directions, over points on curve x



$\Longleftrightarrow$ see as inserting operator $\psi(x)$

– topological B-branes which are points in the RS, transversal D5 brane (one $\mathbb{C}$-direction)

Aim: extend this to $c=1$ string, minimal models, maybe general $uv + F(x,y) = 0$

[These CY3: appear from mirrors of manifolds with T^2 fibrations
- closed string duals to branes wrapped around 3-manifolds
- dual to CS theory]

$\mathcal{N}=1$ SYM in $D=4$

Simplest case: pure gauge theory, G simple gauge group
.. i.e. connection A_μ, gauginos λ_α, $\bar{\lambda}_{\dot\alpha}$ in adjoint rep

Action $\displaystyle \frac{1}{g^2} \underbrace{\int \mathrm{Tr}\left(F \wedge * F - \frac{i}{2} \bar\lambda \not{D} \lambda\right)}_{\text{YM term}} + \underbrace{\frac{\theta}{4\pi^2} \int \mathrm{Tr}\, F \wedge F}_{\text{topological term}}$

$U(1)_R$ symmetry: $\lambda_\alpha \to e^{i\varphi} \lambda_\alpha$ $\quad \bar\lambda_{\dot\alpha} \to e^{i\varphi} \bar\lambda_{\dot\alpha}$
classical symmetry, broken by instanton effects:

$U(1)_R$ anomaly $\quad \partial_\mu J^\mu = \frac{1}{32\pi^2} \mathrm{Tr}_{\text{adjoint}} F \wedge F$

$\Rightarrow$ integrate this get ~~topological~~ topological # $n \cdot h$ $\quad n \in \mathbb{Z}$
$h = C_2(\text{adj})$ = dual Coxeter # $= N$ for $SU(N)$

$\Rightarrow$ <u>Index</u> $\not{D}$ will be quantized in units $2h$, so fermion #s only ~~defined~~ conserved mod $2h$

$$\langle \underbrace{\lambda \ldots \lambda}_{2h} \rangle = e^{2\pi i \tau} \Lambda_0^{3h} \qquad \tau = \frac{\theta}{2\pi} + \frac{i n \pi}{g^2}$$
$$= \Lambda^{3h}. \quad \Lambda = \text{dynamical scale}$$

$U(1)_R$ symmetry broken by instantons to $\mathbb{Z}/2h$ cyclic exact symmetry of system. Spontaneously broken down to $\mathbb{Z}_2$ by gaugino condensation: bilinear of fermions → Sheaf has expectation value $0 \neq \langle \mathrm{Tr}\, \lambda_\alpha \lambda_\beta \varepsilon^{\alpha\beta} \rangle = \langle S \rangle$ superfield

$\langle S^h \rangle = \Lambda^{3h} = \langle \underbrace{\lambda \dots \lambda}_{2h} \rangle$: $\longleftrightarrow$ quantum cohomology of $\mathbb{CP}^n$

quantum chiral ring

Vacuum states that diagonalize the trace, labeled by k

$$\langle S^h \rangle_k = e^{\frac{2\pi i k}{h}} \Lambda^3 \qquad \text{fractional instantons } n = \frac{1}{h}$$

Strange quantum effect ... follows from one-instanton effect & cluster decomposition $\langle S \rangle^h = \langle S^h \rangle = \Lambda^{3h}$

only in strong coupling.

Veneziano - Yankielowicz : action $\int d^2\theta\, W_{eff}(S)$

$dW_{eff} = 0 \implies \langle S \rangle_k = e^{\frac{2\pi i k}{h}} \Lambda^3$ minima give multiple states

$$W_{eff}(S) = 2\pi i \tau S + h S [1 - \log S/\Lambda_0^3]$$

morally obtained by integrating out ~~...~~ all gauge fluctuations...

G gauge group, matter ϕ in some rep V = gauge field

$$S_{YM} + \int d^4x \Big[\underbrace{\int d^4\theta\, \bar\phi e^V \phi}_{D\text{-terms}} + \underbrace{\int d^2\theta\, W(\phi) + \int d^2\bar\theta\, \bar W(\bar\phi)}_{F\text{-terms}} \Big]$$

matter coupled to background gauge field V.

$W_\alpha = -i \bar D^2 e^{-V} D_\alpha e^V$ e.g. Lie algebra wt 3/2 W general Grassmann polynomial

F-terms $\leftrightarrow$ IR behavior :

F-terms are insensitive to UV completion.

... can think of $W(\phi)$ itself as effective action from integrating out some fields of more full theory.

e.g. $W = \frac{1}{2} m\phi^2 + \frac{1}{3} + \phi^3$

things we calculate independent of $\bar W$ - chiral path integral

so make simple choice $\bar W = \frac{1}{2} \bar m \bar\phi^2$

Expand around $\phi \approx 0$.

Integrate out $\bar{\Phi}$, get action in ϕ only, holomorphic

$$\int d^4x\, d^2\theta \left[-\frac{1}{2m} \phi \left[\underset{(\partial/\partial x)^2}{\Box} - i W^\alpha \underset{\partial/\partial\theta^\alpha}{D_\alpha} \right] \phi + W(\phi) \right]$$

Rescale $x \to \sqrt{m}\, x$, $\theta \to \sqrt{m}\,\theta$. recall $\int d\theta \cdot \theta = 1$

$\Rightarrow$ $d^4x\, d^2\theta$ invariant $\Rightarrow$ get rid of m term altogether.

Try $m \to 0$ $\quad \lim_{x \to y} \phi(x) \left[\Box - i W^\alpha D_\alpha \right] \phi(x) = \bar{D}^2 (\bar{\phi} e^V \phi)$
$= \delta(x-y) \operatorname{Tr} W^2$ $\Big\}$ Konishi anomaly

better to put $m = 1$.

String theory: 1^{st} quantized language

Feynman diagrams  s_i : Schwinger parameters = length of propagator

(think of as string diagram with $\alpha' \to 0$)

Propagation $\dfrac{s}{p^\mu = \frac{\partial}{\partial x^\mu},\ \pi^\alpha = \frac{\partial}{\partial \theta^\alpha}}$ $\Rightarrow \exp -s\left[p^2 + \pi^\alpha W_\alpha + m\right]$

— Hamiltonian acting on wave functions on chiral superspace, $L^2(\mathbb{R}^{4|2})$

$\Rightarrow$ Feynman diagrams give $\displaystyle\int \prod_{\text{edges}} ds_i \prod_{a=1}^{l} d^4\hat{p}_a\, d^2\hat{\pi}_a\, e^{-\sum s_i [p_i^2 + \ldots]}$

where momenta are written

$p_i^\mu = \sum L_{ia} \hat{p}_a^\mu \quad a = 1, \ldots l = \#$ loops

$\pi_i^\alpha = \sum L_{ia} \hat{\pi}_a^\alpha$

Moduli space of Feynman diagrams $\mathcal{M} \ni (s_1, \ldots s_\ell)$

Schwinger space - first integrate out p's, π's get function on moduli space & then integrate it out

Bosonic part: $dp \Rightarrow e^{-\hat{p}_a M_{ab} \hat{p}_b}$ $\quad M =$ period matrix of graph $l \times l$ 

$\Rightarrow \dfrac{1}{(\det M)^{D/2}}$ $\quad D =$ spacetime dim $= \sum_i s_i L_{ia} L_{ib}$

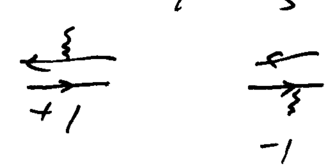
Fermionic part: $\int d\pi \, e^{W\pi} = W$

So get factor of W for each $W\pi$

$\Rightarrow W^{2\ell} = (\mathrm{Tr}\, W^2)^\ell = S^\ell$ ..... at most $2\,W$/edge.

Example $G = U(N)$, ϕ adjoint $\quad W = \frac{1}{2} m \phi^2 + \frac{1}{3} t \phi^3$

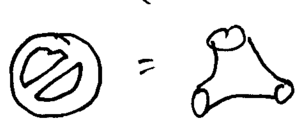
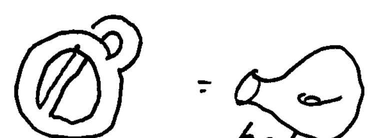
Coupling of gauge field to matter field: $\mathrm{Tr}\, \phi \, [W^\alpha, D_\alpha \phi]$:



Draw Feynman diagram as string diagram

$h = \#$ holes, $g = $ genus

$\ell = \#$ loops $= 2g + h - 1$

 $h=3 \quad g=0$ 

$h=1 \quad g=1$

$W^{2\ell}$, 0, 1 or 2 W's / hole

$h \geq \ell \quad \Rightarrow h = \ell + 1 - 2g \Rightarrow g = 0$: diagram is planar!

nothing to do with large N: $\#$ fermions $\longleftrightarrow$ $\#$ bosons due to SUSY : F-terms have only planar diagrams, part of action depending on chiral superfield W.

$h = \ell + 1$: one more hole than we need.

Count: $\quad \sum N \binom{\ell+1}{1} S^\ell = N \frac{\partial F}{\partial S} \quad$ where $F = \sum a_\ell S^\ell$

Correlation fns in gauge theory are derivatives of these in MM:

$\langle \mathrm{Tr}\, \phi^k \rangle_{GT} = N \frac{\partial}{\partial S} \langle \mathrm{Tr}\, \phi^k \rangle_{MM}$

$\langle \mathrm{Tr}\, \phi^k W^2 \rangle_{GT} = \langle \mathrm{Tr}\, \phi^k \rangle_{MM}$

Period matrix of $y^2 = W'^2 + c$ $\quad \frac{\partial^2 F}{\partial S_i \partial S_j} = (\tau_{eff})_{ij}$

$\binom{\ell+1}{2} S^{\ell-1}\, \mathrm{Tr}\, W_\alpha \, \mathrm{Tr}\, W^\alpha = \tau_{eff}(S)\, (\mathrm{Tr}\, W_\alpha)^2$

$U(1)$ gauge field

R. Dijkgraaf III

8/7/03

Higher genus

$N=1$ systems $\longleftrightarrow$ B-type topological strings on $CY3$ (deformations of $\mathbb{C}$-structure)

$\updownarrow$ mirror

A-type top strings on $\widetilde{CY3}$ (deformation of Kähler structure)

naively only planar diagrams contribute

what is full genus expansion?
quantum K-S theory
cubic closed string field theory

higher genus GW invariants

$$\mathcal{F}_g(S_i, t_i)$$

$g=0$ $\quad W_{eff} = \sum N_i \frac{\partial \mathcal{F}_0}{\partial S_i}$ $\quad \tau_{ij} = \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j}$

$\Rightarrow$ chiral ring $\qquad \Rightarrow U(1)^n$ couplings

$\Leftrightarrow$ S-W period matrix

Ooguri-Vafa: Spinor field strength W_α: introduce anticommuting (Clifford) deformation.

$\{W_\alpha, W_\beta\} = C_{\alpha\beta}\cdot \mathbb{1}$ $\quad$ field C (can take constant)

Algebra of W's, chiral ring becomes noncommutative, not diagonalizable — lose factorization $\Rightarrow$ destruct planar limit. – get higher genus contributions

$\sum \mathcal{F}_g C^{2g}$ $\qquad$ C expansion.

(Loop eqns don't factorize any more?)

$N=1$ story: can couple gauge theory to gravity – put on curved 4-manifold M^4
$\rightarrow$ induced term $\int d^4x\, f(S)\, R_+ \wedge R_+ + c.c.$

f: genus 0; R_+: self-dual part of curvature

Claim $f = N_i N_j \tau_{ij}(s) + \mathcal{F}_1(s)$

"genus 0 part" $\qquad$ genus 1 part

All $\mathcal{F}_g = \mathcal{F}_g^* + \mathcal{F}_g^{pert}$

gauge theory/nonpert. part $\qquad$ matter theory/perturbative

Matrix model with canonical normalization $\quad V = \text{Vol}$ (gauge group)!

$$Z = \frac{1}{V}\int d\phi \, \exp \frac{1}{g_s} \operatorname{Tr} W = \exp F,$$

$$F = \sum g_s^{2g-2} F_g$$

$$F^* = -\log \operatorname{vol} U(\tilde{N}) \qquad (g_s \tilde{N} = S \quad \text{'t Hooft coupling})$$

$$F_g^* = \begin{cases} \frac{1}{2} S^2 \log S & g=0 \quad \text{V-Y term} \\ -\frac{1}{12} \log S & g=1 \\ \chi_g \cdot S^{2-2g} & \end{cases}$$

χ_g = Euler # of $\mathcal{M}_g$

$\Rightarrow$ given by genus expansion of $c=1$ matrix model of selfdual radius $R=1$

$$N_i N_j \, \underset{\substack{\wr\\ -\frac{1}{2} N_i^2 \log S_i}}{\tau_{ij}(s)} \; + \; \underset{\substack{\wr\\ -\frac{1}{12} \log S}}{F_1(s)}$$

first term weak coupling, second strong: difference between them is the lots of freedom disappearing at strong coupling

Critical points $\quad dW_{eff} = 0 \quad \Rightarrow \quad N_i \tau_{ij} = \tau_0$

$\Rightarrow f = N\tau_0 + F_1$

constant .. tension of original branes responsible for system

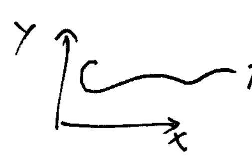
$\Rightarrow$ all interesting data in nonplanar diagrams ..

Quantum CY : full B-topological string, including all genus corrections .. use matrix model as guide

alg curve $\quad y^2 = W'^2 + f = P_{2n}(x) \quad$ deg 2n poly

φ collective field of MM

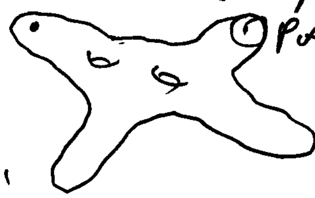
$$Y(x) = \partial_x \varphi = \sum k t_k x^{k-1} + 2 g_s \operatorname{Tr} \frac{1}{x-\phi} \qquad W = \sum t_k \phi^k$$



$F(x,y)=0 \;\rightarrow\; y =$ function of x

Look in B model at $\mathbb{C}$-structure variations of curve $\quad F(x,y)=0$

$\longleftrightarrow$ R.S with asymptotic ends $\longleftrightarrow$ points P_∞ $\{x = \infty$
with local coordinates



Local deformations of the RS:

$$Y = f(x) + \delta f(x) = f(x) + \sum_{k>0} k t_k x^{k-1} + \sum F_k x^{-k-1}$$

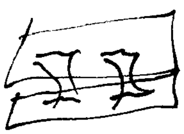
F_k subleading corrections

$\sum_{k>0} k t_k x^{k-1}$ ↓ change behavior of fn at ∞

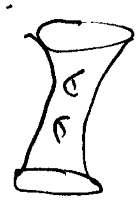
F_k ↓ 0 large x

— local deformations of $\mathbb{C}$-structure,

When does this give good global def of an RS?



 draw as RS with two ends, top & bottom



think of RS as scattering matrix

 x sending deformations at one end to those on other . . .

two ends = two preimages of $x = \infty$

Genus zero:

Full solution from loop equation, genus zero:

$$y(x) = \sum k t_k x^{k-1} + 2 g_s \sum \frac{\partial (F)}{\partial t_k} x^{-k-1}$$

$\frac{1}{x - \phi}$ operator with expectation value! write as diff op

$\frac{\partial F}{\partial t_k}$ derivatives of free energy . .

$$\Rightarrow F_k = \frac{\partial F_0}{\partial t_k} \quad \text{all derivatives of single fn } F_0$$

General perturbation where y^2 extends $\longleftrightarrow$
y^2 a polynomial $\Rightarrow$ negative components $= 0$
$(y^2)_- = 0 \iff$ Virasoro constraints $L_n Z = 0$ $n \geq -1$

Quantum case: all genus: — make Fock space,
quantize field $\varphi(x)$, $\partial \varphi = \sum \alpha_n x^{-n-1}$
mode expansion

Coherent state for creation operators $|t\rangle = \exp \sum t_n \alpha_n |0\rangle$
- in this set action $\alpha_n = \frac{\partial}{\partial t_n}$, $\alpha_{-n} = n t_n$

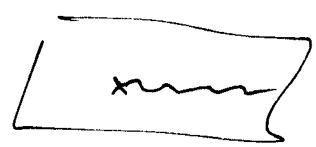
Turns $\partial\phi$ into diff op $\sum k t_k x^{k-1} + g_s^2 \frac{\partial}{\partial t_k} x^{-k-1}$

$\langle t | \partial\phi(x) \dots | \Sigma \rangle$ Operator formalism of CFT

$x = \infty$ one end of RS Σ

Complex structure deformation
makes sense in interior ~ gaugedize
action of (quantum) Virasoro algebra.

Remark double scaling limit : $y^2 = x^{2n+1} + \dots$ odd not even before
collide branch points $(2, 2n+1)$ model
$(2,1)$ topological gravity
$y^2 = x$



1 branch point

only one end ..
write $y = \partial_x \varphi = \sum_{k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} k t_k x^{k-1} + \frac{\partial}{\partial t_k} x^{-k-1}$
$\frac{1}{2}$-integer modes
$\Rightarrow$ Virasoro constraint $L_n Z = 0$ $n \geq -1$
wrt twisted field $\partial\varphi$
$\Rightarrow$ KdV solution
- integrable system with ∞ set of times ..
One set of times $\longleftrightarrow$ one asymptotic end
t_k = modes at ∞.
Time geometry $\rightarrow$ time matrix model

One end $\Rightarrow$ KP hierarchy, (p,q) minimal models $c < 1$
double scaling limit

Two ends $\Rightarrow t_k^{out}$  t_k^{in} One-matrix model
times for each end
$\Rightarrow c = 1$ strings

Topological vertex $\Rightarrow$ 3 sets of times
" 3-Toda

$c=1$ String theory:
world sheet variables
(X, φ) ← Liouville

$X = X + 2\pi R$ compactified

$dx \wedge dy$: RS as Lagrangian in phase space
$y = \partial_x \varphi$ => WKB
wave function $\psi \sim e^{\varphi}$
RS = classical approx of quantum wavefn.

T-duality $R \to \frac{1}{R}$: take self dual radius $R=1$

$$\int d^2\sigma \left[\partial X^2 + \partial\varphi^2 + R\varphi + \mu e^{\varphi} \right]$$

String vertex operators: tachyon momentum states e^{ikX},
winding modes $e^{ik\tilde{X}}$

$R=1$: compactified boson $\Longleftrightarrow$ $SU(2)_{k=1}$ model
=> much bigger symmetry, left & right
$SU(2)_L \times SU(2)_R$. Particles come in
multiplets $(k/2, k/2)$ spin.

Ground ring: Introduce a_i, b_i $i=1,2$
$P_k(a) \cdot Q_k(b)$ polynomials in a_i & b_i

introduce 2×2 matrix $M_{ij} = a_i b_j$, $\det M = 0$ $\quad M = \begin{pmatrix} x & u \\ v & y \end{pmatrix}$
write fns in M $\quad xy - uv = 0$

Witten: ring deforms at $\mu \neq 0$ to relations
$xy - uv = \mu$
$\longleftrightarrow$ Physical states are $\mathbb{C}[x,y,u,v]/(xy-uv-\mu)$

Tachyons: vertex operator e^{ikX} $\longrightarrow$ modes x^k, y^k
(for $k>0$ $k<0$)

<u>Relation to topological strings</u>:
(Mukhi-Vafa, Ooguri-Vafa) $c=1$ at $R=1$
$\Longleftrightarrow$ topological string on conifold $uv + xy = \mu$

$[R = n \iff uv + (xy)^n = \mu]$

general string correlation fns $\longleftrightarrow$ deformations of equation: $uv + xy = \mu + f(x,y,u,v)$

turn on arbitrary vertex operators $\longleftrightarrow$ deform the CY3

genus 0 $\longleftrightarrow$ classical deformation theory

higher genus $\longleftrightarrow$ quantum deformations.

Scalar field X dressed by Liouville

x, y tachyon of momentum ± 1

u, v dual tachyon (winding modes) of momentum ± 1,

... any CY3 with conifold singularity ($\longleftrightarrow$ some f) correspond to some particular correlation fns in $c=1$ theory.

<u>Easier problem</u>: only look at deformations of tachyon momentum states, built out of x, y.

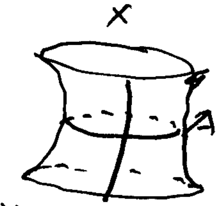
ie perturb conifold

$$uv + xy = \mu + \sum t_n x^n + \tilde{t}_n y^n + \ldots$$

... forget about uv, just have curve $F(x,y) = xy - \mu$, deform this curve.

<u>Genus zero</u>: $xy = \mu$ curve

complexified phase space.



Deform curve:

$$y = \frac{\mu}{x} + \ldots \qquad x = \frac{\mu}{y} + \ldots$$

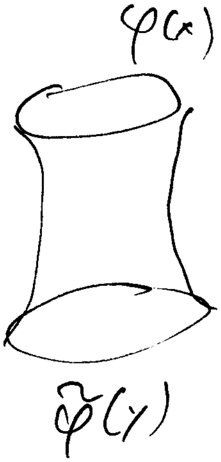
$$y = \frac{\mu}{x} + \underbrace{\sum k\, t_k x^{k-1}}_{\text{leading terms large } x} + \underbrace{\frac{\partial F_0}{\partial t_k} x^{-k-1}}_{\text{subdominant terms / response / 1-pt fncts}}$$

$$x = \frac{\mu}{y} + \sum k\, \tilde{t}_k y^{k-1} + \frac{\partial F_0}{\partial \tilde{t}_k} y^{-k-1}$$

Write $y = f(x)$, $x = g(y)$, $g = f^{-1}$ compatibility

$\Rightarrow$ solves $F_0 = F_0(\mu, t_k, \tilde{t}_k)$

- solution is semiclassical (dispersionless) soln of Toda hierarchy



$y = \partial_x \varphi(x)$ $\quad x = \partial_y \tilde{\varphi}(y)$ Same field expressed in other variables : think of φ as classical field living on RS

$$y = \partial_x \varphi(x) = \frac{\mu}{x} + \partial_x \varphi^{qu}(x), \text{ say}$$

field φ is deforming the geometry, turning it on changes shape of curve! φ nonlinear transformation properties ("gravitational field")

Linear approx ! $\partial_y \tilde{\varphi} = \partial_x \varphi \frac{dx}{dy}$

So $\partial_x \varphi$ transforms as 1-form to linear approximation

- but to higher order . . . .

<u>W-constraints</u> [D- Moore - Plesser]

$y = \partial_x \varphi = \sum \alpha_n x^{-n-1}$, $x = \partial_y \tilde{\varphi} = \sum \tilde{\alpha}_n y^{-n-1}$

What is relation of $\alpha_n, \tilde{\alpha}_n$?

$$\tilde{\alpha}_n = \oint y^n x \, dy = +\oint x \, d\frac{y^{n+1}}{n+1} = -\oint \frac{y^{n+1}}{n+1} dx$$

$$= -\oint \frac{(\partial_x \varphi)^{n+1}}{n+1} dx \longrightarrow W\text{-algebra}$$

So single modes of outgoing field $\tilde{\varphi}$ given by W-algebra transformation of φ

<u>Higher genus</u> : include fermions : fermionize $\partial\varphi$

$\psi = e^{\varphi}$ $\quad \psi^+ = e^{-\varphi}$, $\partial\varphi = \psi^+\psi$

In terms of fermions :

RS gives S-matrix $S : \mathcal{H} \to \mathcal{H}$

Claim: In topological string, include B-branes of ~~faces~~
[$uv - F(x,y) = 0$] localized at fixed x, y on curve, with one noncompact direction:

$F=0$ e.g. write $v=0$, u arbitrary
- transverse brane.

Building matrix model holographically! $\tilde{N}$ brane

Put transversal branes

$\Rightarrow$ bifundamentals coupling branes:

$$\mathrm{Tr}\, W(\phi) + Q\phi\tilde{Q} + Qx\tilde{Q} \quad : U(\tilde{N}) \text{ fundamentals}$$

x = vev of ϕ = mass of field : [only one flavor here]

→ integrate out $Q, \tilde{Q}$ ⇒ effective action contribution: $\mathrm{Tr}\left[W(\phi) + \log(x-\phi)\right]$

$$\log(x-\phi) = \sum \tfrac{1}{k} x^{-k} \phi^k$$

So start with superpotential $W(\phi) = t_k \phi^k$, including these Q's corresponds just to shifting values of parameters t's in superpotential!

$$t_k \rightsquigarrow t_k + \frac{1}{k} x^{-k}$$

$$\int d\phi\, e^{\frac{1}{g_s}\mathrm{Tr}\, W(\phi)} \underbrace{e^{\mathrm{Tr}\,\log(x-\phi)}}_{\substack{\text{new term} \\ = \det(x-\phi)}}$$

In terms of coherent states

$$\partial\varphi = \sum k t_k x^{-k-1} + \ldots \qquad |t\rangle = \exp \sum t_k \alpha_{-k} |0\rangle$$

after shift! $\exp \sum \frac{1}{k} x^k \alpha_{-k} |0\rangle = e^{\varphi(x)} |0\rangle$
$= \psi(x) |0\rangle$

ie just action of fermion..
introducing fundamentals $\longleftrightarrow$ putting in fermions
so branes $\longleftrightarrow$ fermions!

n fermions at pos. $x_1, \dots x_n$:
$t_k = \frac{1}{k} \sum_{i=1}^{n} x_i^{-k} = \frac{1}{k} \mathrm{Tr}\, X^{-k}$ Miwa coordinates

$X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ Σ RS : $\langle t \mid \Sigma \rangle$ after shift of t's $\Rightarrow$ $\dfrac{\langle t \mid \psi(x_1) \dots \psi(x_n) \mid \Sigma \rangle}{\langle t \mid \psi \dots \psi \mid 0 \rangle}$

geometry

$\rightsquigarrow$

$y = \partial_x \varphi$ closed strings / Bosons
deform at ∞'s
complicated transformation between patches

insert branes : fermions e^{φ}
can move freely from end to end :
fermions are <u>global</u> <u>objects</u> ,
should be <u>free</u>

$y^2 = x$ geometry

n branes $\longleftrightarrow$
$\langle \psi(x_1) \dots \psi(x_n) \mid \Sigma \rangle$

get correl fns directly from geometry :
eg 1-pt function $\Psi(x) = \langle \psi(x) \rangle \longleftrightarrow$ eigenfunction of $[\partial_x^2 + x]\, \Psi = 0$ Airy function

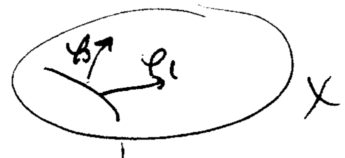
n branes $\Rightarrow$ matrix Airy fn $\Rightarrow$ Kontsevich solution.

[branes as macroscopic loop operators in $c=1$ matrix model ? ?

Dijkgraaf, lecture 4.

Continuing: connect $c \leq 1$ strings to cplx geom,
B-type top. strings.

CY_3



B-branes
(holo. curves)
Euclidean
D1-brane

what is semiclassical action?
reduction to 2d of holo.
CS theory.

$$\frac{1}{g_s}\int d^2z \, \mathrm{Tr}[\phi_0 \bar{D}\phi_1].$$

$$\bar{D} = \partial_{\bar{z}} + [A_{\bar{z}}, -].$$

N branes $\rightsquigarrow$ ϕ_0, ϕ_1 are $N\times N$ matrices.

spin depends on normal bundle.

deforming brane B to B':

$\phi_i(z,\bar{z})$ (non-holo. in general)

$B \subset_3$ 3-cycle.

$$\int_{C_3} \Omega = \int d^2z \, \phi_0 \bar{\partial}\phi_1.$$

Interesting simple case: $\mathbb{P}^1 \subseteq K3$ or ~~[illegible]~~ $T^*_{\mathbb{P}^1}$,
cross w/ $\mathbb{C}$. ϕ_1 is 1-form dir, ϕ_0 is $\mathbb{C}$-dir.

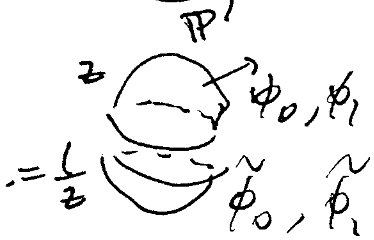
ϕ_i has worldsheet spin i.

get gauged, chiral FT, conformally invariant.

Integrating out $\phi_1 \rightsquigarrow$ constraint $\bar{\partial}\phi_0 = 0 \Rightarrow \phi_0(z,\bar{z}) = \Phi$
constant $N\times N$ mat

To get matrix model, change geometry.

$\Longrightarrow$ n isolated curves.

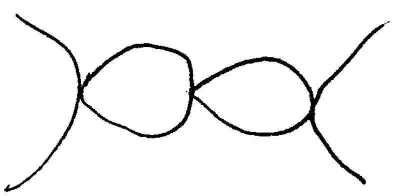


$\tilde{\phi}_0 = \phi_0, \quad \underbrace{\tilde{\phi}_1 = z^2 \phi_1}_{\text{usual for 1-form}} + z\, W'(\phi_0).$

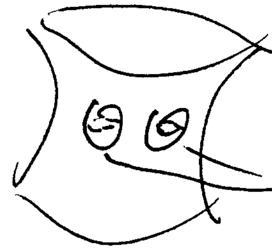
so replacing $\mathcal{O}\oplus\mathcal{O}(-2)$ by 3-fold given by eq.

$$uv + y^2 + W'(x)^2 = 0.$$

This is singular.



$\xleftarrow{\text{resolve}}$



$$Z = \langle 0 | e^{\frac{1}{g_s}\operatorname{Tr} W(\phi_0)} | 0 \rangle = \int d\Phi\, e^{\frac{1}{g_s}\operatorname{Tr} W(\Phi)}.$$

<u>conifold</u> $W' = x$ single sing $\rightsquigarrow$ Gaussian model.

$$y^2 = x^2 + \mu \Leftarrow \int d\phi\, e^{\frac{1}{g_s}\operatorname{Tr}\phi^2}$$

Can get this by studying branes + antibranes:
$U(N_+|N_-)$ supermatrix. (equiv. to $U(N)$, $N = N_+ - N_-$).

But there is more here than just $U(N)$.

expand around $\phi = \left(\begin{array}{c|c} \phi_+ & 0 \\ \hline 0 & \phi_- \end{array}\right)$ bosonic blocks gauge condition

$$\Psi = \Psi^\dagger = 0 \qquad \phi = \left(\begin{array}{c|c} \phi_+ & \Psi \\ \hline \Psi^\dagger & \phi_- \end{array}\right).$$

toral ghosts

$$A, B = \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right) \qquad A, B \ \underline{\text{bosons}}$$

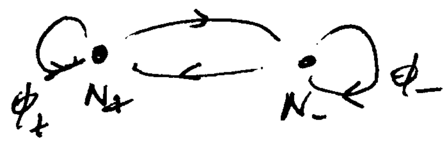
(ghosts have opposite statistics)

extra term in action $\quad s\,\mathrm{Tr}\,A[\phi, B]$.

action then is

$$S = \mathrm{Tr}\left(W(\phi_+) - W(\phi_-) + A_i \phi_+ B^i - B^i \phi_- A^i\right)$$

$\hat{A}_1$ -quiver model!



$W = \phi^2$ get model by Witten–Klebanov, Klebanov–Strasser?

If you int. out $\phi_\pm$, set $M_{ij} = A_i B_j$, then $S = \det(M_{ij})$.

~~[illegible]~~

$U(N) \to U(N_1) \times U(N_2)$.

Payoff: many more deformations.

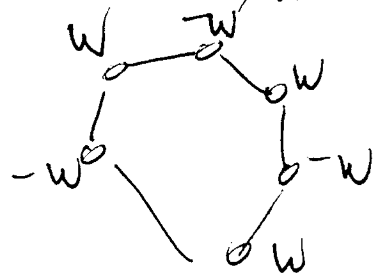
$$\mathcal{O}_{(k,k)} = \mathrm{Tr}\,(A_{i_1} B_{j_1} \ldots A_{i_k} B_{j_k})$$

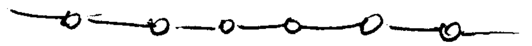
Transform according to $\left(\frac{k}{2}, \frac{k}{2}\right)$ spin rep.

1:1 corr. with all states in $c = 1$ at $R = 1$!

what about for other R ?

For $R = n$, take $\mathbb{Z}/n$ quotient, get $\hat{A}_{2n-1}$.



Can also get A_∞ 

Another MM closely related

Suppose only want to compute tachyon momenta.

oupling t_k wrt. $\tilde{t}_k$

$$T_k = X^k$$
$$T_{-k} = Y^k \qquad \text{in conifold. } uv + xy = \mu.$$

e.)

$$M_{ij} = A_i B_j = \begin{pmatrix} x & -v \\ u & y \end{pmatrix}.$$

Normal matrix model (i.e. $[X, X^\dagger] = 0$).

2-matrix model X, Y ($X = M_{11}$, $Y = M_{22}$ morally)

$$\int dX\, dY\, e^{\frac{1}{g_s} \mathrm{Tr}(XY + \sum t_k X^k + \tilde{t}_k Y^k)}.$$

equiv. to

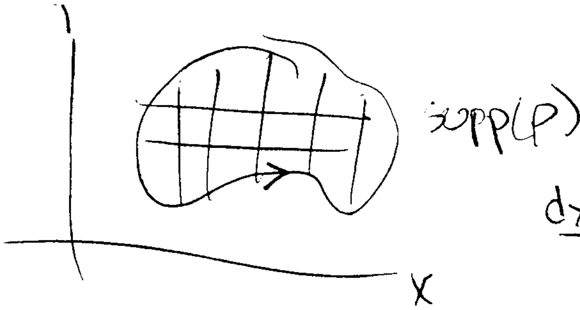
$$\int \left(\prod dx_i\, dy_j\, \Delta(x) \Delta(y)\right) e^{\frac{1}{g_s} \sum x_i y_i + \underbrace{\sum t_k x_i^k}_{W(x)} + \underbrace{\sum \tilde{t}_k y_i^k}_{\tilde{W}(y)}}.$$

...eigenvalue distrib. $\rho(x, y)$.

saddle pt approx. $y_i + W'(x_i) = \sum_{j \neq i} \frac{g_s}{y_i - x_j}$ (?)

got a bit lost, but

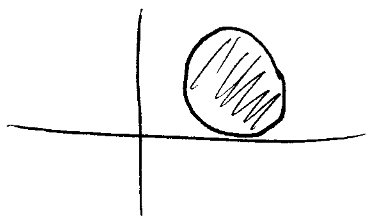
conclusion is $\rho(x,y)$ is char. fn. of some set, $\int \rho(x,y)\, dx\, dy = N$.

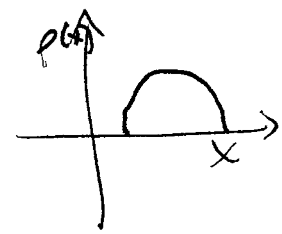


$\frac{dx \wedge dy}{g_s}$.

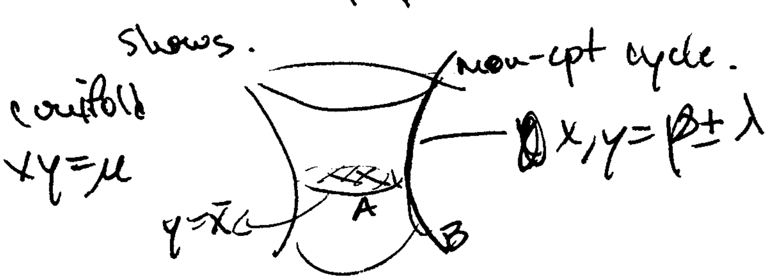
boundary curve is $y = f(x)$ for a holo. fn f.
[papers of Wiegmann & co].

Gaussian case, $W(x,y)$ is quadratic, then curve is circle,





What is connection to usual QM?
Nice paper of Kazakov, Kostov & ??
shows.



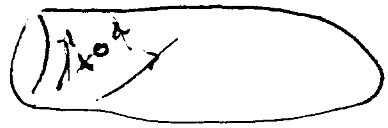
In 2-matrix model,
$\mu = g_s N$.

~~Free Fermions~~?

MQM(μ).	normal MM
$\langle N \rangle = \int_B y\,dx = \frac{\partial \mathcal{F}^{QM}}{\partial \mu}$	$\langle \mu \rangle = \int_A y\,dx = \frac{\partial F}{\partial N}$

(duality)
"Legendre trans."
(Kazakov et al showed these give special sol'n. of Toda).

Free fermions



$\phi \to \infty$

computing reflection coeff. in Liouville theory. String enters fr. left & gets bounced back.

$$x = e^{ix^0 + \phi}$$
$$y = e^{-ix^0 + \phi}$$

$S : \mathcal{H} \to \mathcal{H}$ $\mathcal{H}$ Fock sp. of free boson (?).

$$\partial\varphi = \sum \alpha_n x^{-n-1}.$$

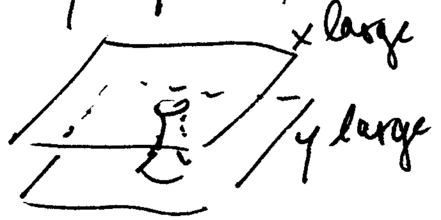
fermions $\psi, \psi^\dagger$

$$S\psi_p S^{-1} = R_p \psi_p \underset{\emptyset}{=} \tilde{\psi}_p, \quad p = \frac{n+\frac{1}{2}}{R} = n + \tfrac{1}{2} \text{ (take } R=1).$$

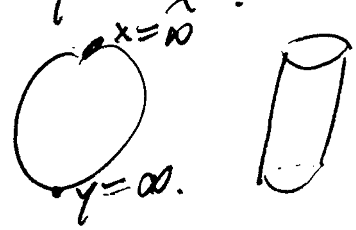
computed long ago. $R_{n+1/2} = \sin(n+1+\mu)\,\Gamma(n+1+\mu)$.

Claim "S = Fourier transform."

geometry $xy = \mu$, so think of $y = \frac{\mu}{x}$.



i.e.



S-matrix connecting in & out states.

$$\tilde{\psi}(y) = y^{\mu} \int dx \cdot x^{\mu} e^{xy} \psi(x).$$

reflected fermion

Redefine ψ to be $x^{-\mu}\psi(x)$, replace $\tilde{\psi}$ by $y^{\mu}\tilde{\psi}(y)$.

Then $\tilde{\psi}(y) = \int dx\, e^{xy} \psi(x)$.

How to think of this in terms of deforming curve?

$|t\rangle = \exp \sum t_n \alpha_{-n} |0\rangle$. coherent st. in terms of bosons.

$Z = \langle \tilde{t} | S | t \rangle$. $S \in GL_\infty$.

So Z is sol. of 2-Toda hierarchy.

Connection to geometry:

$$(*) \begin{cases} y = \frac{\mu}{x} + \partial_x \phi. \end{cases} \quad x = \frac{\mu}{y} - \partial_y \tilde{\phi}$$ appearance of bosons.

Shift $\phi \rightsquigarrow \phi - \mu \log x$

$\tilde{\phi} \rightsquigarrow \tilde{\phi} + \mu \log y$.

(*) becomes $y = \partial_x \phi$, $x = -\partial_y \tilde{\phi}$ i.e. $\phi, \tilde{\phi}$ are Legendre transforms.

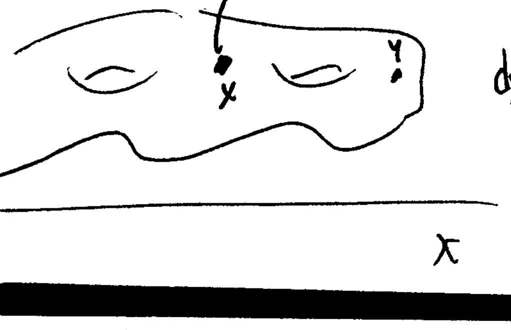
$$\Psi(x) = e^{\phi(x)/g_s}$$

Creates a brane at position x. Writing same object in local coord. y, must transform by Legendre.

"quantum version of Legendre transform is Fourier transform!"

$$e^{\tilde{\phi}(y)/g_s} = \int dx\, e^{[xy - \phi(x)]/g_s}$$

Lesson $\langle \Psi(x) \rangle$ —— 1-pt fn of brane

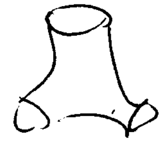


$dx \wedge dy$

move brane fr. x to y, how does it transform?

Expect $\tilde{\Psi}(y) = \Psi(x(y)) \left(\frac{dx}{dy}\right)^{1/2}$

B-model

 $\rightsquigarrow$  $H(u,v)$.

CY is locally $wz + H(u,v) = 0$.
[their notation, different!].

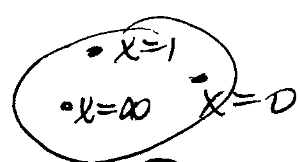
$e^u, e^v \in \mathbb{C}^*$
i.e. $u \mapsto u + 2\pi i$, $v \to v + 2\pi i$ periodicity.

symplectic form $du \wedge dv$.

Curve in $\mathbb{C}^* \times \mathbb{C}^*$!

Simplest ex: curve $\Leftrightarrow$ vertex

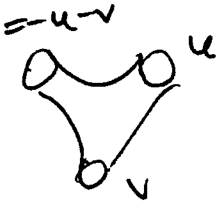
$$H(u,v) = e^u + e^{-v} - 1. \qquad \text{Let } x = e^u.$$



3 punctures

1 puncture = Min. model (KP).

$=0$ is Fermi curve w/ ~~3 ext~~ asymp. ends.



$v = \partial_u \varphi$
$w = \partial_v \tilde{\varphi}$
$u = \partial_w \hat{\varphi}$

partition fn.

$$Z = \langle t^u_k, t^v_k, t^w_k \mid V \rangle$$
$$V \in \mathcal{H}^{\otimes 3}.$$

If you put brane $\psi(u)$ at pos. u, want to move to v, ..

Think of $\Psi(x)$ as wavefunction; $\Psi(x) = e^{\varphi(x)/g_s}(1 + O(g_s))$ is WKB approx.

Then transform is very nontrivial under coord. transform.

$F(x,y)=0$ Fermi surface. Think of it as Hamiltonian, $y = g_s \frac{\partial}{\partial x}$. $F = xy - \mu$. ~~[illegible]~~

- Aganagic, Klemm, Mariño, Vafa.

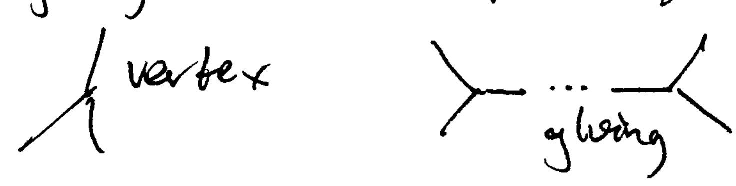
obs. about toric CY_3:

$$\begin{array}{ccc} T^3 & \longrightarrow & CY_3 \\ & & \downarrow \\ & & B^3 \end{array}$$

<u>case</u> CY_3 has T^2 action instead. singularities encoded in planar graph, which indicates pos. of degen. of T^2. and type.

A-model
Σ (instantons)
use localization under T^2 to get instantons hugging graph!! Put in D-branes spiking edges of graph. Build up closed fr. open string instantons.

$\rightsquigarrow$ gluing laws for top. string.



There is also B-model version.
(via Mirror sym).

se... linear canonical trans. $Sp(2,\mathbb{R})$. rep'n in $[x,y]=g_s$.

Here, need action of $SL(2,\mathbb{Z})$, gen. S, T.

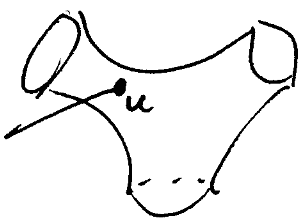
$$S = \int du\, e^{uv/g_s}\, \Psi(u)$$

$$T\Psi(u) = e^{u^2/g_s}\, \Psi(u).$$

$$(ST)^3 = 1.$$

<u>Claim</u> ST relates branes at u, v.

<u>Q</u> Is there a matrix model here?



$$= \langle \Psi(u) \rangle = e^{\sum_{n>0} \frac{1}{n} \frac{e^{nu}}{2\sinh\left(\frac{n g_s}{2}\right)}}$$

$$= e^{\frac{1}{g_s} Li_2(e^u) + \mathcal{O}(g_s)}$$

$$= \prod_{n>0} \left(1 - e^{u} e^{n g_s}\right).$$

again

$H = e^u + e^{-v} - 1$. Try to solve $H\Psi = 0$.

$$e^u \Psi(u) + \Psi(u+g_s) - \Psi = 0.$$

So this 1-pt fn. $\langle \Psi(u) \rangle$ of the brane satisfies Schrödinger eqn.

__Question__ geometry given by $H(u,v)=0$.

Should ~~??~~ we study QM ? i.e. $H\Psi = 0$?

Why related to brane expectations ?

__Example__ (2,1)-model coupled to gravity.

$$y^2 = x.$$

$$\rightsquigarrow \quad \underbrace{(\partial_x^2 + x)}_{\text{Lax op. of KdV}} \Psi = 0. \qquad \Psi = \int dy \, e^{xy + y^{3/2}}$$

BA fn. Airy fn.

What about n branes at $(x_1, \ldots, x_n)$?

$$\int_{n\times n} dY \, e^{\mathrm{Tr}(XY + Y^3/2)}$$

Promoting wavefunctions of QM to matrix thing.

$$\begin{tikzcd}
 & \text{geometry } H(u,v)=0 \arrow[dl, "\text{deform curve } H=0"'] \arrow[dr, "\text{add branes}"] & \\
\text{curve def} & & n\times n \text{ matrix model} \arrow[ll, "\text{holography}"']
\end{tikzcd}$$