R. Dijkgraaf - Intersecting Branes & D-Modules

Note Title 7/19/2007

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4d SUSY gauge theories	2d CFT (free fermions)
$\mathcal{N}=4$: S-duality $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g} \longmapsto -\frac{1}{\tau}$	modular invariance $SL_2\mathbb{Z}$ on T^2
$\mathcal{N}=2$ Seiberg-Witten	spectral curve Σ_g
$\mathcal{N}=1$ matrix models	effective geometry: complex curve appears at large N

Gravitational couplings in all of these!

$F_1(\tau) \int R \wedge R$ term in gauge theory

(response to curvature R of
four manifold)

On 2d side $F_1 = \log \det \bar{\partial}_\Sigma$!
relates to genus 1 topological string.

Nekrasov: $U(1)$-equivariant theory on
$\mathbb{R}^4$ $\xleftrightarrow[\text{engineering}]{\text{geom.}}$ topological string F_g's

$\updownarrow$ 2d picture: F_g relate to "quantum curve"

Gauge theory: operators $\sum t_i O_i$
$\longleftrightarrow$ CFT side: integrable hierarchies,
e.g. KP, Grassmannian...

Try to introduce new viewpoint on these
from intersecting branes!

<u>Example</u> $\mathcal{N}=4$ SYM for $U(N)$
on M^4, hyper-kähler manifold,
in Vafa–Witten topological twist.

$$Z = \sum_n d_n \, q^{n - c/24} \quad \text{partition func}^{\text{n}}$$

$q = e^{2\pi i \tau}$, $n = ch_2$ = instanton #

d_n = "Euler #" of instanton moduli space

$c = N\chi$, where χ = regularized Euler # of M.

Z should be modular form (for $SL_2\mathbb{Z}$ in $U(N)$ case)

d_n expected to be integers:
put D_4 brane on $M \times S^1$,
& compute $\mathrm{Tr}\,(-1)^F$ on a subsector of

the Hilbert space labelled by instanton number n, $\mathcal{H}_n$. n should be integer.

M = ALE space:

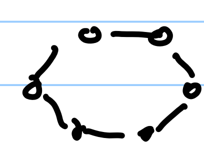
$\Gamma \subset SU_2$ $\qquad M = \widetilde{\mathbb{C}^2/\Gamma}$ noncompact h-k manifold.

eg $\Gamma = \mathbb{Z}_k$, A_{k-1} singularity.

Boundary of $M = S^3/\Gamma$.

Pick flat connection at ∞ using McKay correspondence:

$\Gamma \subset SU_2 \longleftrightarrow$ affine Lie algebra $\hat{\mathfrak{g}}$

eg $\mathbb{Z}_k \longleftrightarrow$  $\hat{A}_{k-1}$

Flat bundles on S^3/Γ given by n-dim rep of Γ, $\rho \in \mathrm{Hom}(\Gamma, U(n))$

$\longleftrightarrow$ d_i = labels of Dynkin diagram = dims of irreps of Γ.

ρ $\longleftrightarrow$ integrable rep V_ρ of $\hat{\mathfrak{g}}$
N-dim rep at <u>level</u> N

... level-rank duality.

Nakajima: action of $SU(k)_N$ (affine Lie algebra at level N)
on $\bigoplus_n H^{\text{middle}}(\mathcal{M}_n)$.

Partition function of gauge theory with fixed boundary condition ρ

$$Z_\rho(\tau) = \chi_\rho(q) = \mathrm{Tr}_{V_\rho}\left(q^{L_0 - \frac{c}{24}}\right)$$

Vafa-Witten partition fn = character (vector-valued modular form)

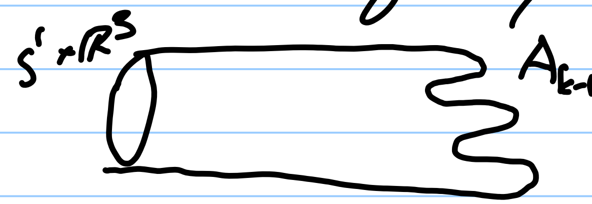
I. Frenkel: if we work <u>equivariantly</u> wrt gauge group $U(N)$ at ∞
$\longrightarrow$ should also have action

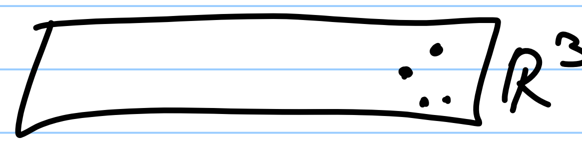
of $U(N)_k$ (i.e. level/rank duality realized in this rep)

Use trick from black hole theory in 4, 5 dims to study this!

replace A_{k-1} by Taub-NUT geometry

$$\begin{array}{ccc} S^1 & \longrightarrow & TN_k \\ & & \downarrow \\ & & \mathbb{R}^3 \setminus \{x_1, \dots x_k\} \end{array}$$





cylindrical end of TN

[Moore: maybe really need to replace ALE by ALF for this ....]

Let's study D4 brane on $TN \times S^1$ (gives $N=4$ theory on TN) in IIA.

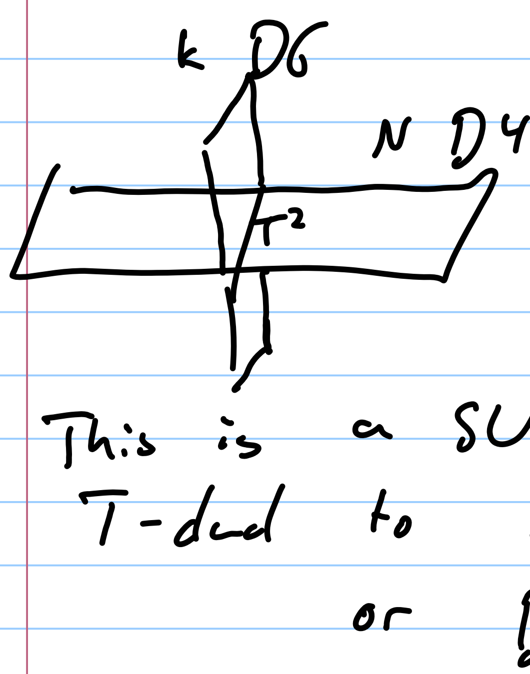
Lift to M-theory $\Rightarrow$ M5 brane on $\underbrace{TN \times T^2}_{N \text{ M5 branes}} \times \mathbb{R}^5$

modular invariance of $N=4$ should come from T^2 conformal invariance on the M5 brane.

" $9 \longleftrightarrow 11$ flip" : look at dual

IIA on S^1_{TN} : intersecting (I-)

brane system: $\underbrace{\mathbb{R}^3 \times T^2}_{N \text{ D4 branes}} \times \mathbb{R}^5$



$\mathbb{R}^3 \times \underbrace{T^2 \times \mathbb{R}^5}_{k \text{ D6 branes}}$

This is a SUSY system,

T-dual to D8 - D0 system

or D9 - D1 system

In principle have $U(N) \times U(k)$ gauge theory, but frozen since noncompact branes

Now have open strings connecting D4 & D6 (interesting degs of freedom):

$N \times k$ 4-6 open strings!

this is a $(0,8)$ 2dim SUSY theory on T^2.

Only massless (topological) modes are chiral fermions in Ramond sector

$\psi_{ia}(z)$, $\psi^{+}_{ia}(z)$

$i = 1 \ldots k$
$a = 1 \ldots N$

Action: $\int d^2 z \, \psi^{+} D_{\underset{U(N)}{A} + \underset{U(k)}{A'}} \psi$

Chiral anomaly cancelled by chiral inflow:

$$\int_{DY} G_2 \wedge CS(A) \qquad CS = \text{Chern-Simons}$$

$$dG_2 = \delta_{T^2}.$$

- chiral anomalies cancelled by Chern-Simons terms on the branes.

— worldsheet theory of heterotic string (via D9-D1 system duality)

$\Longrightarrow$ WZW model for $U(Nk)_1$ level one

$$U(Nk)_1 \to SU(N)_k, \quad SU(k)_N$$

$$\sum \psi_{ia} \psi^{\dagger}_{jc} \qquad \sum \psi_{ia} \psi^{\dagger}_{ib}$$

- conformal embedding:

$$U(Nk)_1 \supset U(1)_{Nk} \times SU(N)_k \times SU(k)_N$$

with charges $Nk = 1 + c_{N,k} + c_{k,N}$.

This is the affine version of Schur-Weyl duality

[recall $\quad (\mathbb{C}^N)^{\otimes k} \circlearrowleft U(N)\times S_k$ dual pair

$$\overset{\shortparallel}{\bigoplus R_{U(N)} \otimes \bar{R}^{\vee}_{S_k}}$$]

Ignore overall $U(1)$ piece, get decomposition

$$Z \simeq \sum_{\rho} \chi_{\rho}(\tau)\, \chi_{\rho^T}(\tau)$$

Frenkel, Nakajima

ρ runs through reps of $SU(N)_k \longleftrightarrow SU(k)_N$

Root lattices of $SU(N)$, $SU(k)$ appear from quantization on $U(1)$ gauge fields, so we get the Cartan dependence of the characters.

More general picture

[e.g TN_1 = "A_0-sing" $= \mathbb{R}^4$]

Study MS brane on $TN \times \Sigma_g$

Σ_g genus g Riemann surface

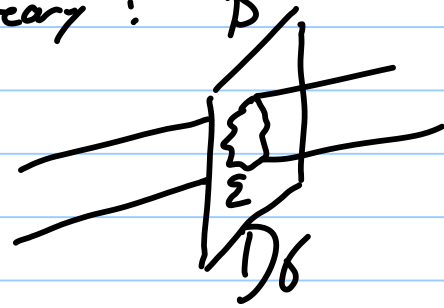
$$\Sigma_g : \quad P(x,y)=0 \qquad (x,y) \in \mathbb{C}^2$$

or replace $\mathbb{C}^2$ by some degeneration of $B = E \times E'$ product of two elliptic curves

M theory geometry: $TN \times B \times \mathbb{R}^3$

$$MS : TN \times \Sigma_g \quad (\Sigma_g \subset B)$$

In IIA theory: $B \times \mathbb{R}^3$
have



$D4 \times \mathbb{R}^3$

$\Sigma = D4 \cap D6$

Free fermions on Σ: set

$F_1 = \log\det \bar{\partial}_\Sigma$ = partition function of D4-D6 system.

Chains of dualities

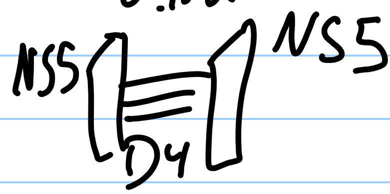
Go to $\mathrm{II}A$: pick a circle: 3 chains

$$TN \times B \times \mathbb{R}^2 \times S^1$$

$$\underbrace{TN \times \Sigma}_{M5}$$

TN circle $\rightarrow$ I-brane system

circle in B $\rightarrow$ within:



$\rightarrow TN \times B \times \mathbb{R}^2$

$\cup$

NS5: $TN \times \Sigma$

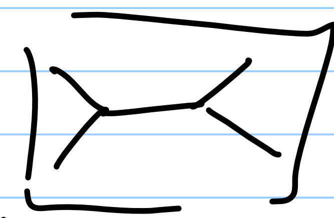
T-dualize transverse to 5-brane $\rightarrow$

$\mathrm{II}B$ on $TN \times CY_\Sigma$

"$\{uv + P(x,y) = 0\}$"

IIB on $TN \times CY_\Sigma$: completely geometric
no branes anymore

mirror symmetry

IIA $TN \times \widetilde{CY}$: toric picture 

" tropicalization of curve

M theory: $TN \times \widetilde{CY} \times S^1$
get GV-invariants

IIA : $\mathbb{R}^3 \times \underbrace{S^1 \times \widetilde{CY}}_{\text{D6 brane}}$: Donaldson-Thomas

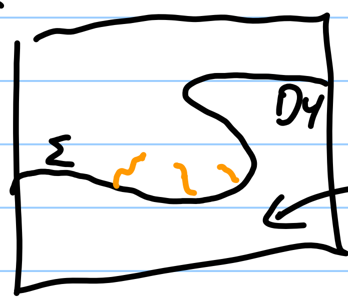
Let's do I-brane parallel of
all these dualities : compute
higher genus corrections to F_1.

How do we compute the F_g's ?

IIB picture: turn on flux of G_5 (self-dual 5-form field) — chase this through duality chain (on $TN \times CY_\Sigma$)

($G_5 = \Omega_{CY} \times \omega_2$ graviphoton)

D6



$B = \mathbb{C}^2 \ni (x,y)$

flux gives NS B-field of form $dx \wedge dy$

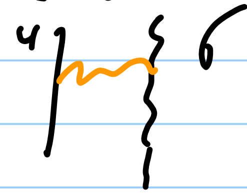
$\Rightarrow$ get system of 4-6 open strings in B-field background.

Claim: Fermions are still free, but <u>non commutative</u>

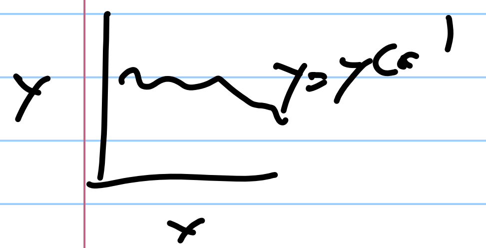
g_s = topological string coupling constant

$B = \frac{1}{g_s}\, dx \wedge dy$

in bulk of D6



$\rightsquigarrow$ gauge field $A = \frac{1}{g_s} y(x)\, dx$



Seiberg-Witten differential on Σ.

$\Rightarrow$ partition function has classical piece $F_0 \sim A^2$

$$\sim \frac{1}{g_s^2} \sum \oint_{A_i} y\,dx \oint_{B_i} y\,dx$$

D-modules : B-field constructs

NC algebra $\mathcal{A} = \mathbb{C}\langle x,y\rangle / [x,y] = g_s$

Open strings give NC algebra

Kapustin-Witten: another D-brane in background of $\mathcal{A}$ gives a D-module

... ie 4-6 strings are module over

$\mathcal{A}$ = Weyl algebra = $D_{\mathbb{C}} = \mathbb{C}[x]\langle \frac{\partial}{\partial x}\rangle$

Intersecting strings give a $\mathcal{D}$-module M
which is "Lagrangian" $\longleftrightarrow$ holonomic.

Suppose $P \in \mathcal{D}$: $P\psi_0 = 0$

$\longleftrightarrow$ $\mathcal{D}/\mathcal{D}P$ analog of functions on the curve

$= \{ D\psi_0 , \ D \in \mathcal{D} \}$

Semiclassically P becomes a function $P(x,y) = 0$
- classical energy levels of the Hamiltonian,

To solve $P\psi_0 = 0$, $\psi_0 \in \mathcal{F}$ function space

$\longleftrightarrow$ $\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{F})$

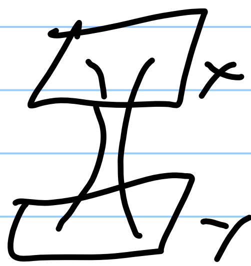
<u>example</u> "$c=1$ string": $P = xy - \mu = x\frac{\partial}{\partial x} - \mu$

$\psi_0 = x^\mu$ solution

$M = \{ \sum_{n \geq 0} a_n x^n + \sum b_n y^n \}$

Now we want to study free fermions with $\psi(x)$ not a section of a line bundle but rather in a D-module M.

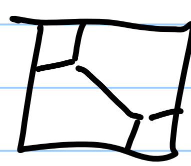
Localize M at $x=\infty$, $y=\infty$,
change of variables is Fourier transform



$\psi(x)$ transform as wave functions, but still free.

$\Longrightarrow F^{top} =$ "$\log \det \bar{\partial}_\Sigma$"
in NC fashion

I have computation for <u>compact</u> genus 2 curve in $E \times E$



related to top. vertex
$F_1 =$ wt 10 modular form for Sp_4