R. Dijkgraaf : Topological M-Theory 8/6/04
(w/ S. Gukov, A. Neitzke, C. Vafa)

Topological strings $D=6$ $SU(3)$ $\simeq$ chiral CFT $D=2$

$\downarrow$ $\downarrow$

Top M-theory $D=7$ G_2 — C-S $D=3$

$\downarrow$ $\downarrow$

Top F-theory $D=8$ $Spin(7)$ — Top Y-M $D=4$

Hints

Topological string partition fn:

- $Z_{top}(x)$ hol fn of x $x \in \mathcal{M}_{cpx}$ moduli of cpx str. space
  well, hol anomaly $\overset{\text{(Witten)}}{\Longrightarrow}$ Z_{top} wave function in Hilbert space obtained by quantizing $H^3(X)$
  ... maybe think of $X = \partial(M^7)$, get particular state?
- 4 dim Black hole partition: $Z_{BH} \sim Z_{top} \times \bar{Z}_{top}$ (Ooguri ..)
  $\Rightarrow$ theory on $X \times S^1$ ---> 7-dim partition fn.
- 7 dim geometry mixes Kähler & complex structure ...
  e.g. S-duality A model $\longleftrightarrow$ B model

Topological string field theory : (Spacetime POV)

BCOV X CY3 :
B-model : cx structures on X
local coords $z^i, \bar{z}^i$. $\Omega = dz^1 \wedge dz^2 \wedge dz^3$
(3,0) form
Variations of cx structure: cl form $\bar\partial$ by Kodaira-Spencer field $\bar\partial_A = \bar\partial + A\,\partial$

in components $\bar\partial_j = A_j^{i}(z,\bar z)\,\partial_i$

"closed string field for B-model"

"field" $A' = \Omega \cdot A \in \Lambda^{2,1}$

... gives infi defs of Ω, $\delta\Omega = A'$.

Gauge condition $\partial A' = 0$

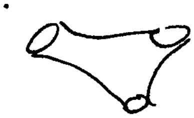
Write $A' = x + \partial\phi$

$x \in H^{2,1}$ massless field, gives moduli (take as harmonic form)

$A' = \partial\phi$ massive modes (offshell fields)

$\phi \in \Lambda^{1,1}$

$\Rightarrow$ KS action $S_{KS} = \int \frac{1}{2}\,\partial\varphi \wedge \bar\partial\varphi + \frac{g_B}{3!}\, A' \wedge (A')^2$



cubic action

$Z_{top}(x) = \int D\phi\, e^{-S_{KS}}$

Reduce KS to 2 dimensions: chiral theory

Similar to 2D chiral CFT

$\int \partial\varphi\,\bar\partial\varphi + g\,(\partial\varphi)^3$

... analog of only left moving conformal fields.

[holomorphic theory ... need to choose contours ... "integral on contour in path space"]

A-model version -- Kähler gravity

fundamental field is Kähler (1,1) form $k = k_{i\bar j}\, dz^i \wedge d\bar z^{\bar j}$

integrability $\Rightarrow dk = 0$

(Bershadsky-Sadov) Gauge $d_c^{\dagger} k = 0$ $(d_c = \partial - \bar\partial)$

Massless piece: $k = x$ $x \in H^{1,1}$ (moduli)

massive offshell fields: $k = d_c^{\dagger}\gamma$ $\gamma \in \Lambda^3$ 3-form

Action $S_{KG} = \int \frac{1}{2}\, d\gamma \wedge d_c^{\dagger}\gamma + \frac{1}{3!}\int k \wedge k \wedge k$

$(k = x + d_c^{\dagger}\gamma \ldots)$

$\rightarrow$ gives Z^A_{top} in limit $\alpha' \to 0$: ignores worldsheet instantons, of form $\sum e^{-\int_\Sigma (t + iB)}$ $\Sigma \to X$ RS.

<u>Hitchin</u> : p-forms in d-dimensions for special p,d
... <u>Stable</u> forms: in $\Lambda^p(M^d)$ all forms are $GL(d,\mathbb{R})$-equivalent ...

today $p = 3, 4$ $\quad d = 6, 7, 8$

1 • 3-form ρ in $d = 6$:
dim $\Lambda^3 = \binom{6}{3} = 20$. $G = GL(6,\mathbb{R})$ dim 36
Stabilizer G_ρ has dim $36 - 20 = 16$.
Claim: $\exists$ open orbit of G of form
$\rho = \alpha + \beta$. $\quad \alpha, \beta$ indecomposable: $\alpha = \omega_1 \wedge \omega_2 \wedge \omega_3$

<u>Case a.</u> α complex, $\beta = \alpha^*$ conjugate:
write as $\rho = \Omega + \bar{\Omega}$. $\Omega = e^1 \wedge e^2 \wedge e^3$, with
e^i basis of $(T^* \otimes \mathbb{C})^{1,0}$. Stabilizer
$G_\rho = SL(3, \mathbb{C})$ ... "Euclidean case"

<u>Case b.</u> (Minkowski case) α, β real
$\rho = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$ basis
$G_\rho = SL_3\mathbb{R} \times SL_3\mathbb{R}$.

Case a : $\rho = \Omega + \bar{\Omega}$. Set $\hat{\rho} = \frac{1}{i}(\Omega - \bar{\Omega})$
$\frac{1}{2}(\rho + i\hat{\rho}) = \Omega$ --- think of $\hat{\rho} = \mathrm{Im}\,\Omega$
$\rho = \mathrm{Re}\,\Omega$.

If $d\rho = 0$ & $d\hat{\rho} = 0 \Rightarrow d\Omega = 0$
& Ω defines <u>integrable</u> <u>complex</u> <u>structure</u>

Action: $\quad S_H(\rho) = \int \frac{1}{2} \rho \wedge \hat{\rho}(\rho) = i \int \Omega \wedge \bar{\Omega}$

Variation $\delta S_H(\rho) = \int \delta\rho \wedge \hat{\rho}$
$\leftrightarrow S_H$ homog of degree 2 in ρ, but <u>not</u> quadratic

$$K_\rho : \overset{\text{tangent}}{T} \longrightarrow \overset{\text{5-form}}{\Lambda^5} = \Lambda^6 \otimes T$$

$$K_\rho(v) = i_v\rho \wedge \rho$$

$\mathrm{Tr}\, K_\rho = 0$, $\sqrt{-\frac{1}{6}\mathrm{Tr}\, K_\rho^2} = \rho \wedge \hat\rho$ ...

get square root of quartic functional in ρ ....

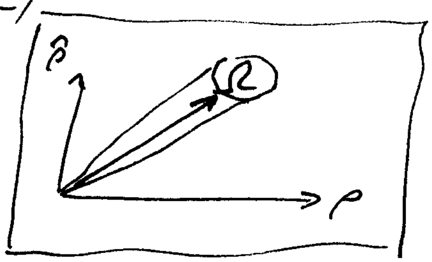
Assume $d\rho = 0$, vary ρ in cohomology class $X = [\rho] \in H^3(X, \mathbb{R})$

$\rho = \rho_0 + d\beta \quad \Rightarrow \quad \delta S_H = \int d\beta \wedge \hat\rho = -\int \beta \wedge d\hat\rho$

$\Rightarrow d\hat\rho = 0$ at critical points of S_H

so critical pts give complex structures!

$H^3(X, \mathbb{C})$



get cone over moduli of complex structures as graph of $\hat\rho$ as function of ρ.

$H^3(X, \mathbb{R})$

Quantum Hitchin theory:

$$Z_H(X) = \int D\beta \, e^{-\frac{1}{g^2 H} S_H(X + d\beta)}$$

Claim: $\quad Z_H \propto Z_{top} \bar{Z}_{top}$ for topological B-model

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(2). $\sigma \in \Lambda^4$ on X^6 : $\dim \Lambda^4 = \dim \Lambda^2 = \binom{6}{2} = 15$.

G_σ has dim $36 - 15 = 21$. Generically $G_\sigma = Sp(6, \mathbb{R})$

---> get symplectic structures!

Stable $\sigma \Rightarrow \sigma = \frac{1}{2} k \wedge k \quad k \in \Lambda^2$

Action $V = \int \frac{1}{3!} k \wedge k \wedge k = \int \frac{1}{3} \sigma \wedge k$

homog of deg $3/2$ as function of σ

$\delta V = \frac{1}{2} \int \delta\sigma \wedge k$.

Assume $d\sigma = 0$ $\quad \sigma = \sigma_0 + \varepsilon\gamma(\sigma)$

Critical points : $dk = 0$ - integrable symplectic structure

$\delta\sigma = k \wedge \delta k = d\gamma$

Use Lefschetz operators $L = k\wedge\_$ & $\Lambda = L^{\dagger}$

$[\Lambda, d] = d_c^{\dagger}$.

$\Rightarrow \quad \delta k = d_c^{\dagger}\gamma + d(\Lambda\gamma)$

$\cancel{k = k_0} + \varepsilon \quad V = \int \frac{1}{2}\, d\gamma \wedge d_c^{\dagger}\gamma + \frac{1}{6}k^3 \Rightarrow$ Kähler gravity

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(3) 3-form Φ or 4-form G on M^7

$\phi = \phi_{ijk}\, dx^i \wedge dx^j \wedge dx^k$. $\dim \Lambda^3 = 35$. $\dim GL_7\mathbb{R} = 49$

$\dim G_\phi = 14$, generically get $G_\phi = G_2$

$\Rightarrow \phi = c_{ijk}\, e^i \wedge e^j \wedge e^k$ with c_{ijk} structure coeffs of octonions

Equivalently write $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$

e^a, ω^a selfdual 2-forms

$\phi = \sum_a e^a \wedge \omega^a + e^1 \wedge e^2 \wedge e^3$

$\Rightarrow$ metric $ds^2 = g_{ij}\, dx^i dx^j = \sum e^i \otimes e^i$

$*\phi = G$, $*G = \phi$ $\qquad \Lambda^2 \ni \phi_i = \phi_{ijk}\, dx^j dx^k$

$\Rightarrow$ matrix $B_{ij} = \phi_i \wedge \phi_j \wedge \phi$

$(\det B)^{1/9} \in \Lambda^7$ is volume form

$\mathcal{L} = \sqrt{g} = (\det B)^{1/9} = \phi \wedge *_\phi \phi = G \wedge *_G G$

$S_H = \int \sqrt{g} = \int G \wedge *G$ has degree $7/4$ in G, $7/3$ in ϕ

$\delta S_H = \frac{7}{4} \int \delta G \wedge *G$.

Vary G in fixed cohomology class : $G = G_0 + dC$,

$[G_0] = x \in H^4$

$\delta S_H = 0 \Rightarrow dG = 0,\ d*G = 0 \Longleftrightarrow$ metric with G_2 holonomy

Quantum theory : $Z_H(x) = \int DC\, e^{-\frac{1}{\lambda^2}\int G \wedge *G^3}$ $\qquad G = x + dC$

Quantization condition : $\left[\frac{G}{2\pi}\right] \in H^4(M,\mathbb{Z})$

Hamiltonian Formulation : $M = X \times \mathbb{R}$

Hitchin! ρ, σ closed 3, 4 forms on $X \Rightarrow$

G_2 metric on $X \times I$ using variation $\frac{\partial \rho}{\partial t} = dk$,

$\frac{\partial \sigma}{\partial t} = k \wedge \frac{dk}{dt} = -d\hat{\rho}$. $\Rightarrow \phi = \rho + k \wedge dt$

time-independent, ground state $\Rightarrow dk = 0, d\hat{\rho} = 0$ so

X is CY.

Coordinates : "q" : $\gamma \in \Lambda^3 / \Lambda^3_{exact}$,

Momentum "p" : $\rho \in \Lambda^3_{closed}$. $\langle \rho, q \rangle = \int \rho \wedge \gamma$.

Hitchin Lagrangian ~~...~~ $= \int_X \rho \wedge \dot{\gamma} + \rho \wedge \hat{\rho} + k^3$

$\qquad p\dot{q} \qquad \frac{p^2}{T} \qquad \frac{q^{3/2}}{-V}$

Hamiltonian $H = T + V$

$\doteq \lambda^2 \int_X \rho\hat{\rho} - \frac{1}{\lambda^2}\int_X k^3$

B model $\qquad$ A-model

complex volume $\qquad$ Kähler volume

- $M = X \times S^1$, radius$(S^1) = \varepsilon$. Limit $\varepsilon \to 0, \lambda^2 \to \infty$

  with $\varepsilon / \lambda^2 = \frac{1}{g_A^2}$ fixed

  $Z_{qm} = \mathrm{Tr}\left(e^{-\frac{\varepsilon H}{\hbar}}\right) \xrightarrow[\hbar\beta = \varepsilon]{\hbar \to 0} Z_{cl} = \int dp\, dq\, \exp(-\beta H)$

  so this is classical limit

  ...factorizes into A, B parts

- Limit $g_B \to 0$ $\quad Z_H \Rightarrow Z_{top}^B(X)$ KS action.