A. Polishchuk - Perverse Sheaves 9/16/96

Goresky-MacPherson - Intersection Homology Theory I Topology 19(1980) 135-162
II InvMath 72 (1983) 77-129

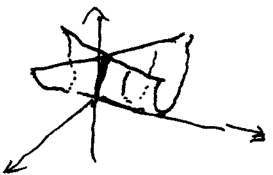
& B-B-D-(G) Asterisque 100

Stratified pseudo-manifold: $X = X_n \supset X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$.

topological spaces s.t. i) $X_n - X_{n-2}$, $X_{n-k} \setminus X_{n-k-1}$ are topological manifolds of dims n, $n-k$ respectively ($\forall k$) $= S_{n-k}$ strata

ii) $X - X_{n-2}$ dense in X

iii) local normal triviality: $x \in S_{n-k}$ $\exists$ open neigh. U,
L compact stratified mfld of dim $k-1$ $(L = L_{k-1} \supset L_{k-3} \supset \dots)$
and an isomorphism $U \simeq B_{n-k} \times cL$, compatible with [illegible] filtration:
$[cL$ cone $= [0,\infty) \times L / (0 \times l \sim 0 \times l')$.$]$
$U \cap X_{n-i} \simeq B \times cL_{k-i-1}$.
(L is set of normal directions)



Example 1. $y^2 z = x^2$

To get loc norm triv must take something like $X \supset \{x=y=0\} \supset \{x=y=z=0\}$

-origin looks different L is two real circles....

Stratified PL-mfld: PL = class of loc. fin simple triangulation, all subspaces are PL-subspaces

Ex. 2 X complex analytic $\Rightarrow$ $\exists$ structure of strat. PL-mfld on X.
PL structure is not canonical -rest can be given canonically.

Assume T is a triangulation of X. R abelian group $\Rightarrow$ chain group
$C_i^T(X; R)$ - $\{\sum r_\sigma \cdot \sigma\}$ σ oriented i-simplex
T' refinement of T: $\cdot\, C_i^T(X,R) \to C_i^{T'}(X,R)$ $\partial : C_{i+1}^T \to C_i^T$
write σ in T wrt T'. . .

Geometric chains: $C_\bullet(X,R) = \varinjlim_T C_\bullet^T(X;R)$

$H_i(C_\bullet(X,R)) = H_i(X,R)$ Borel-Moore

$H_i(C_\bullet^c(X,R)) = H_i^c(X,R)$ usual homology.

Fix a Perversity: $k \mapsto p_k$ $k = 2, 3, \dots n$, (p_k natural #'s)
s.t. $p_2 = 0$, $p_k \leq p_{k+1} \leq p_k + 1$. e.g. 0-perversity $p_k = 0$
or top perversity (t-perversity) $t_k = k-2$.

$I_p C(X, R) \subset C_i(X, R)$ consists of chains
$\xi \in C_i$ s.t. $\left.\begin{array}{l} \dim(|\xi| \cap X_{n-k}) \leq i - k + p_k, \\ \dim(|\partial\xi| \cap X_{n-k}) \leq i-1-k+p_k. \end{array}\right\} \forall k$

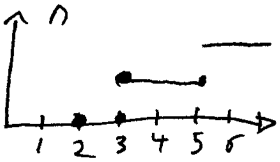
This is a subcomplex $\Rightarrow$ groups $I_p H_i(X; R)$, <u>intersection homology</u> wrt p.
When $X_{n-2} = \emptyset$ this changes nothing.

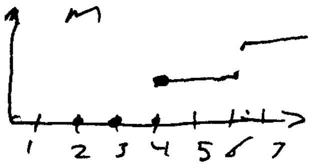
1. $\xi \in I_p C_i \Rightarrow |\xi| \not\subset X_{n-2} =: \Sigma$. If $\xi = \xi'$ outside $\Sigma \Rightarrow \xi = \xi'$.

2. X normal if each pt x has neigh. U s.t. $U \setminus U_{sing}$ connected.
$\tilde{X} \to X$ normalization $\Rightarrow I_p H_i(X, R) \simeq I_p H_i(\tilde{X}, R)$.

3. Suppose givens p, p' s.t. $p_k = p'_k$ for all k s.t. $X_{n-k} \setminus X_{n-k-1} \neq \emptyset$
$\Rightarrow I_p C_\bullet(X, R) = I_p C_\bullet(X, R)$

Middle preversities:





If all strata are even dimen then m, n give same IC
X normal $\Rightarrow I_{top} H_\bullet(X, R) \xrightarrow{\sim} H_\bullet(X, R)$ quism of complexes

X orientable $\Rightarrow H^{n-i}(X, R) \xrightarrow{\sim} I_0 H_i(X, R)$ $(\cap [X])$.
(Poincaré dual simplex in barycentric subdivision)
$\Rightarrow$ strongest intersection condition holds, get lowest possible perversity.

If p, q complementary, i.e. $p + q = t$ $(p_k + q_k = k-2)$
$\Rightarrow$ intersection pairing $I_p H_i(X; R) \times I_q H^c_{n-i}(X, R) \longrightarrow I_t H^c_n(X; R) \to R$
perfect duality.
In particular if all strata are even dim $\Rightarrow$ $n=m$: $p = q$ is self-dual.

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$I_pH_i(X,R)$ - independent of PL-structure & stratification

To show indep of PL-structure give sheaf definition . .

$\boxed{\text{Calculations}}$ $\boxed{1.}$ $(X\times\mathbb{R})_i = X_{i-1}\times\mathbb{R}$. what is $I_pH_i(X\times\mathbb{R})$?

$\cong I_pH_{i-1}(X)$

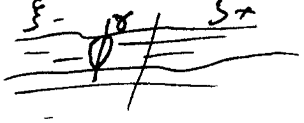
$$I_pC_\bullet(X,R) \hookrightarrow I_pC_{\bullet+1}(X\times\mathbb{R},R), \quad \xi \longmapsto \xi\times\mathbb{R}$$

To show quism, sufficient to show quotient complex has no homologies . .

$$I_pC_{\bullet+1}(X\times\mathbb{R})/I_pC_\bullet(X,R) \ni \underbrace{\xi,\ \partial\xi}_{\text{cycle}} = \eta\times\mathbb{R},\ \eta\in I_pC_i(X)$$

we'd like $\Rightarrow$ $\xi=\partial\mu+\gamma\times\mathbb{R}$ for some $\mu\in I_pC_{i+2}(X\times\mathbb{R})$, $\gamma\in I_pC_i(X)$.

How to show this? 1) If $|\xi|\subset X\times[0,\infty)\subset X\times\mathbb{R}$ & $\partial\xi=0$ then ξ is a boundary : sweep out $i+1$ cycle of which it's the boundary by dragging it up the $\mathbb{R}$ direction $\xi=\partial\mu$ (we used the fact it was closed).



2) ξ unbounded in $\mathbb{R}$ - cut it, $\xi=\xi_++\xi_-$

$$\partial\xi_+ = [t,\infty)\times\eta + \{t\}\times\gamma$$

$$\xi_+ + [t,\infty)\times\gamma = \partial\mu_+,$$

$$\xi_- + (-\infty,t]\times\gamma = \partial\mu_-.$$

$$\Rightarrow \xi+\mathbb{R}\times\gamma = \partial(\mu_+-\mu_-)$$

■ (1)

$\boxed{\text{Calculation 2}}$ L^{k-1} Compact PL-mfld,

$$cL = [0,\infty)\times L/(0,l)\sim(0,l'),\ l,l'\in L)$$

$(cL)_i = cL_{i-1}$, $i\geq 1$, $(cL)_0$ = Vertex of cone.

$$\xi\in I_pC_i(L) \xrightarrow{?} c\xi\in I_pC_i(cL)$$: when is cone over intersection chain an intersection chain

$$c\xi\in I_pC_i(cL) \iff \left[\begin{array}{l} i>k-p_k, \\ i=k-p_k,\ \partial\xi=0 \end{array}\right.$$ (by checking ∩ condition with the vertex)

So we need to <u>truncate</u> $I_p C_\cdot(L)$:

$\tau_{\geq r}\,(C_\cdot \xleftarrow{\partial} C_r \xleftarrow{\partial} C_{r+1} \leftarrow \dots)$

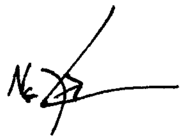
$= (0 \leftarrow \ker \partial_r \xleftarrow{\partial} C_{r+1} \xleftarrow{\partial} C_{r+2} \leftarrow \dots)$

The map $c : \xi \mapsto c\xi$ defines a morphism of complexes

$c : \tau_{\geq k-p_k} I_p C_{\cdot -1}(L) \to I_p C_\cdot(cL)$

Claim/Prop c induces an isomorphism on homologies.

Proof Again c is an embedding, take quotient complex & show it's acyclic -- similar to $X \times \mathbb{R}$ case



Take small neigh. N_ε of vertex, so it doesn't contain any simplexes. $\Rightarrow$ $\xi \cap N_\varepsilon$ is conical $= c\gamma \cap N_\varepsilon$

$i < k - p_k \Rightarrow \gamma = 0$, ξ lies outside N_ε, do same sweep.

We define a sheaf $\underline{IC}_i$ on X so that $\underline{IC}_i(U) = IC_i(U; R)$

- by def a presheaf. To show it's a sheaf :
  ξ is completely described by $(|\xi|, |\partial\xi|, H_i^{e}(|\xi|, |\partial\xi|; R))$
  which triples form a sheaf
- $U \subset X$ gets induced PL structure.

$\Rightarrow$ 1. $\underline{IC}_i$ is a sheaf

2. $\underline{IC}_i$ is a soft sheaf : $F \subset X$ closed,

res : $\underline{IC}_i(X) \twoheadrightarrow \underline{IC}_i(F)$

$\Rightarrow H^i(U, \mathcal{F}) = 0$ $i > 0$ $\mathcal{F}$ soft.

Section over closed set - defined over some open neigh. of it.

$F \subset U \subset X$ take fine triangulation of U, T,

take barycentric subdiv of it T'.

Star $st\,v = \bigcup_{\sigma \in T'} (\sigma \ni v)$.

- take F' = union of stars intersecting F.

$\xi \cap F'$ satisfies intersection condition,

get required chain restricting to given one on F.

$\mathcal{F}_i$ soft $\qquad \mathcal{F}_i \to \mathcal{F}_{i+1} \to \dots$

$\Rightarrow \quad \mathbb{H}^i(X, \mathcal{F}_\bullet) = H^i(\dots \to \mathcal{F}_i(X) \to \mathcal{F}_{i+1}(X) \to \dots)$

hypercohomology degenerates

$\Rightarrow \quad \mathbb{H}^i(X, \underline{IC}_{n-\bullet}) \simeq I_pH_{n-i}(X)$

Hypercohomology indep of PL structure or stratification

$\Rightarrow$ this is invariant definition.

Moreover if $f: \mathcal{F}_\bullet \to \mathcal{G}_\bullet$ is a quism

then $\mathbb{H}(X, \mathcal{F}_\bullet) \simeq \mathbb{H}(X, \mathcal{G}_\bullet)$

Denote $U_k = X \setminus X_{n-k}$

$S_{n-k} = X_{n-k} \setminus X_{n-k-1}$

We now claim:

$j_k : U_k \hookrightarrow U_{k+1}$

Theorem 1) $\underline{IC}_{n-\bullet}|_{U_2} \simeq$ orientation sheaf on U_2.

2) $\underline{H}^i(\underline{IC}_{n-\bullet})|_{S_{n-k}} = 0$ for $i > p_k$

3) $\underline{IC}_{n-\bullet}|_{U_{k+1}} \longrightarrow Rj_{k*}(\underline{IC}_{n-\bullet}|_{U_k})$ is isomorphism on $\underline{H}^i$ with $i \le p_k$.

Proof 1. Obvious. 2, 3 local statements (check at every point). Use local normal triviality at every point of strata $\Rightarrow$ everything is product of ball & cone ... ■

$\Rightarrow$

Corollary $\underline{IC}_{n-\bullet}$ doesn't depend (up to quism) on PL structure.

Proof – by induction on U_k – obvious for U_2.

2, 3 imply that $\underline{IC}_{n-\bullet}|_{U_{k+1}} \underset{qu}{\simeq} \tau_{\le p_k}(Rj_{k*}(\underline{IC}_{n-\bullet}|_{U_k}))$

& τ preserves quisms. ■

[Note: in Pf of Thm use softness in computing Rj_{k*} is j_{k*} applied to terms in complex...]

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Homological algebra (à la S. Gelfand/Y. Manin - Homological Algebra)

1. Quasi-isomorphism (quism), e.g. resolution
$$\begin{tikzcd} A \arrow[r, "qu"] \arrow[dr] & I_\bullet \arrow[d, phantom, "\| \ qu"] \\ & I'_\bullet \end{tikzcd}$$

Ideology of derived category

a. A - object of abelian category should be identified with all its resolutions

b. All standard functors should be redefined (Hom, $\otimes$, Γ ...)

c. New functors should be exact. (& defined on complexes/quism)
→ need notion of exact sequence on derived category

Definition/Theorem Let $\mathcal{A}$ be an abelian category.
$Com(A)$ - complexes in $\mathcal{A}$. → ∃ category $\mathcal{D}(A)$, & functor
$Q : Com(A) \longrightarrow \mathcal{D}(A)$ s.t.
1) Q : quisms ⟶ isoms
2) ∀ category $\mathcal{D}'$, $Q' : Com \longrightarrow \mathcal{D}'$ sending quisms → isoms ⟹
∃! functor $F : \mathcal{D}(A) \longrightarrow \mathcal{D}'$ s.t. $Q' = F \circ Q$

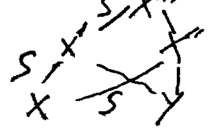
$\mathcal{B}$ a category, S class of morphisms ⟹ category $\mathcal{B}[S^{-1}]$ "formally invert arrows in S". $Ob\, \mathcal{B}[S^{-1}] = Ob\, \mathcal{B}$, morphisms are chains $X \xleftarrow{s} \bullet \rightarrow \bullet \xleftarrow{s} \bullet \rightarrow \bullet \xleftarrow{s} \cdots \xleftarrow{s} \bullet \rightarrow Y$ (universal such.) - localized category

Not obviously abelian - how do we add such chains?

Def $S \subset Mor\, \mathcal{B}$ is called a localizing class of morphisms if
1) S is closed under composition
2)
$$\begin{tikzcd} \bullet \arrow[r, dashed, "g"] \arrow[d, dashed, "S \ni t"'] & \bullet \arrow[d, "s \in S"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$
$s, f \Rightarrow \exists\, g, t$ st. $sg = ft$, similarly for
$$\begin{tikzcd} \bullet & \bullet \arrow[l, dashed, "g"'] \\ \bullet \arrow[u, dashed, "S \ni t"] & \bullet \arrow[l, "f"] \arrow[u, "s \in S"'] \end{tikzcd}$$
3) $f, g : X \to Y$. $(\exists s \in S ; sf = sg) \Leftrightarrow (\exists t \in S : ft = gt)$

Rule two allows us to replace expression $s^{-1} f$ by $g t^{-1}$, $s, t \in S$ (right ⟷ left).

Lemma S localizing class, then $Mor_{\mathcal{B}[S^{-1}]}(X, Y)$ = equivalence classes of "roofs"
$$\begin{tikzcd} & X' \arrow[dl, "S \ni s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
under equivalence



Composition of roofs:

$$\begin{tikzcd} & & X'' \arrow[dl, dashed] \arrow[dr] & & \\ & X' \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Unfortunately quisms <u>don't</u> form a localizing class ....
→ introduce homotopy category $K(A)$: same objects as $Com(A)$, morphisms = $Mor\ Com(A)/\text{homotopy}$.

There is a functor

$$\begin{tikzcd} K(A) \arrow[r] & D(A) \\ Com(A) \arrow[u] \arrow[ur, "Q"'] & \end{tikzcd}$$

- just need to check that if $f \sim g \Rightarrow f = g$ in $D(A)$.

<u>Some constructions</u> 1. Shift $(K^\bullet[n])^i = K^{i+n}$, $d_{K[n]} = (-1)^n d_K$

2. Cone: $f: K^\bullet \to L^\bullet$ morphism $\Rightarrow Cone(f)^i = K^{i+1} \oplus L^i$, differential $\left(\begin{array}{c|c} -dk^{i+1} & 0 \\ \hline f(k^{i+1}) & dl^i \end{array}\right)$. $\Rightarrow$ maps $L \to C(f) \to K[1]$

Given $f_1, f_2 : K \to L$, $f = f_1 - f_2 \sim 0$ want to see $f = 0$ in $D(A)$.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook] & L \\ K \oplus C(id_L)[-1] \arrow[ur] & \end{tikzcd}$$

. $C(id_L) \sim 0$ so $K \xhookrightarrow{quis} K \oplus C(id_L)[-1]$.

<u>Note</u> We have functors $H^i : D(A) \to A$, $\oplus H^i : D(A) \to gr.A$
<u>not</u> equivalence in general - only if A is semisimple.

$K^i \hookrightarrow K^i \oplus C(id_L)[-1]^i = K^i \oplus L^i \oplus L^{i-1}$

$k \mapsto (k, -f(k), h(k))$ where $f \overset{h}{\sim} 0$

quasi-isom .. above complex is acyclic $\Rightarrow$ quasi-isom...

3. <u>Cylinder</u> $f: K^\bullet \to L^\bullet$: $Cyl\ f = Cone(Cone\ f \to K[1])[-1]$

$Cyl(f)^i = K^i \oplus K^{i+1} \oplus L^i$, $d(k^i, k^{i+1}, l^i) =$
$= (dk^i - k^{i+1}, -dk^{i+1}, f(k^{i+1}) + dl^i)$

__Lemma__ For every $f: K^\bullet \to L^\bullet$ there is a comm. diagram with exact rows in $\mathrm{Com}(\mathcal{A})$

$$\begin{tikzcd}
 & 0 \arrow[r] & L^\bullet \arrow[r, "\pi"] \arrow[d, "\alpha"] & C(f) \arrow[r, "\delta"] \arrow[d, equal] & K^\bullet[1] \arrow[r] & 0 \\
0 \arrow[r] & K^\bullet \arrow[r, "\bar{f}"] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
 & K^\bullet \arrow[r, "f"] & L & & &
\end{tikzcd}$$

Moreover, $\beta\alpha = id_L$, $\alpha\beta \sim id_{\mathrm{cyl}(f)}$

__Def__ A __triangle__ is a diagram $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d, "\alpha"] & L^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r] \arrow[d] & K^\bullet[1] \arrow[d, "\alpha[1]"] \\
K'^\bullet \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & K'^\bullet[1]
\end{tikzcd}$$

morphism:

__Def__ A ~~distinguished triangle~~ in $\mathcal{D}(\mathcal{A})$ is a triangle isomorphic to $K \xrightarrow{\bar{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} K[1]$

$$\begin{tikzcd}
K \arrow[r, "\bar{f}"] \arrow[d] & \mathrm{Cyl}(f) \arrow[r, "\pi"] \arrow[d, "\sim"] & C(f) \arrow[r, "\delta"] \arrow[d, "\sim"] & K[1] \arrow[d, "\sim"] \\
K \arrow[r] & L \arrow[r] & C(f) \arrow[r] & K[1]
\end{tikzcd}$$

__Ex.__ 1. $K \to L \to C(f) \to K[1]$

Isomorphic in homotopy category.

2.

$$\begin{tikzcd}
0 \arrow[r] & K^\bullet \arrow[r, "f"] & L^\bullet \arrow[r, "g"] & M^\bullet \arrow[r] & 0 \\
 & K \arrow[r] \arrow[u, equal] & \mathrm{Cyl}\, f \arrow[r] \arrow[u, "\beta"] & C(f) \arrow[u, "\gamma"] &
\end{tikzcd}$$

exact

isomorphic in $D(A)$

$\gamma(k^{i+1}, l^i) = g(l^i)$ $\Rightarrow$ can be completed to distinguished Δ.

[Note $\mathrm{Cyl}\, f \simeq L$ in homotopy category]

__Theorem__ $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ distinguished Δ $\Rightarrow$ long exact seq of cohomology

$H^i(K) \to H^i(L) \to H^i(M) \to H^i(K[1]) \simeq H^{i+1}(K) \to \ldots$

__Proof__ Use cylinder form of triangle.. ■

__Corollary__ f quism $\Leftrightarrow$ $C(f)$ acyclic

__Theorem__ __quis__ is a localizing class in $K(A)$

[same true for $\mathrm{Com}^*(A)$, $* = \overset{b}{\text{bounded}}$, b. $\overset{+}{\text{below}}$/$\overset{-}{\text{above}}$ etc.)

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Remarks

1. Morphisms $f \in \mathrm{Mor}_{\mathcal{D}}$ are not determined by $H^i f \in \mathrm{Mor}_{\mathcal{A}}$, $i \in \mathbb{Z}$

e.g. an extension in $\mathcal{A}$ $0 \to A \to B \to C \to 0$

$0 \to \overset{0}{B} \to \overset{1}{C} \to 0 \to$ is quis. to $0 \to \overset{0}{A} \to 0$

Now
$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & {} \\ 0 \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
f has no cohomologies $H^0 f = H^1 f = 0$ But $f \neq 0$ in general in $\mathcal{D}(\mathcal{A})$ if the extension is nontrivial.

2. Equivalence relation on roofs
$$\begin{tikzcd} & \tilde{X} \arrow[dl, "s"'] \arrow[dr] & \\ X' \arrow[d, "s"'] \arrow[drr] & & X'' \arrow[d] \arrow[dll, "s"'] \\ X & & Y \end{tikzcd}$$

Vite
$$\begin{tikzcd} & Y & \\ X \arrow[ur, bend left] \arrow[d, Rightarrow] & \tilde{X} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & X'' \arrow[ul, bend right] \arrow[dl, Rightarrow] \\ & X & \end{tikzcd}$$
.. add $\tilde{\tilde{X}}$, $\tilde{\tilde{\tilde{X}}}$ using properties of S ...

3. <u>quis</u> in $K(\mathcal{A})$ is a localizing class
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, dashed] \\ \cdot \arrow[r, dashed] & \cdot \end{tikzcd}$$
: in $K(A)$ can replace any morphism by an embeddings (via cylinder)
$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, Rightarrow] & L \arrow[d, Rightarrow, "\text{quis}"] \\ M \arrow[r] & L \oplus M_K \end{tikzcd}$$

4. Embeddings $\mathcal{D}^b(A) \subset \mathcal{D}^+(A) \subset \mathcal{D}(A)$, $\mathcal{D}^b(A) \subset \mathcal{D}^-(A) \subset \mathcal{D}(A)$ are all full (inj. on morphisms):

<u>Lemma</u> $\mathcal{C}$ cat., S loc. class, $\mathcal{B} \subset \mathcal{C}$ full subcategory, &

a. $S \cap \mathcal{B}$ is a loc. class in $\mathcal{B}$

b. If given $X' \xrightarrow{s} X \in \mathcal{B}$, $\exists X'' \in \mathcal{B}$, $X'' \to X'$, $X'' \xrightarrow{s} X$ (or the similar condition with reversed arrows)

<u>Then</u>, $\mathcal{B}[(S \cap \mathcal{B})^{-1}] \hookrightarrow \mathcal{C}[S^{-1}]$ is full.

Derived Functors

$F : \mathcal{A} \to \mathcal{B}$ left exact say. i.e.

$0 \to A \to B \to C \to 0$ exact $\Rightarrow$ $0 \to F(A) \to F(B) \to F(C)$ exact.

We want $\mathcal{D}F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ s.t.

$H^0 \mathcal{D}F|_{\mathcal{A}} = F$, and $\mathcal{D}F$ exact, i.e.

1. commutes with shifts. 2. sends distinguished Δ → distinguished Δ

(i.e. $\mathcal{D}F$ has given extra data, an isom $\mathcal{D}F \circ \text{shift} \simeq \text{shift} \circ \mathcal{D}F$, after which 2. makes sense.).

$A \in \mathcal{A}$ then $\Rightarrow R^iF := H^i\mathcal{D}F(A)$.

Example F exact functor $\Rightarrow$ can set $\mathcal{D}F(K^\bullet) = (FK)^\bullet$

Definition A derived functor is a pair $\boxed{\underset{\text{additive}}{(\mathcal{D}F, \varepsilon_F)}}$, $\mathcal{D}F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$

$$\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r, "Q_A"] \arrow[d, "K^+F"'] & \mathcal{D}^+(\mathcal{A}) \arrow[d, "\mathcal{D}F"] \\
K^+(\mathcal{B}) \arrow[r, "Q_B"'] \arrow[ur, Rightarrow, "\varepsilon_F"] & \mathcal{D}^+(\mathcal{B})
\end{tikzcd}$$

i.e. $\varepsilon_F : Q_B \circ K^+F \to \mathcal{D}F \circ Q_A$

s.t. it is universal for such pairs:

for any $G : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ & ε_G,

$\exists!$ morphism $\mathcal{D}^+F \xrightarrow{\eta} G$

s.t.

$$\begin{tikzcd}
Q_B \circ K^+F \arrow[r, "\varepsilon_G"] \arrow[dr, "\varepsilon_F"'] & G \circ Q_A \arrow[d, "\eta \circ Q_A"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathcal{D}^+F \circ Q_A
\end{tikzcd}$$

If such exists $\Rightarrow$ it is unique. Let's consider existence:

Definition An adapted class of objects to F is a class $\mathcal{R} \subset \mathcal{A}$ of objects s.t.

0) $\mathcal{R}$ is closed under $\oplus$ (finite sums)

1) F sends acyclic$^+$ complexes $/\mathcal{R}$ to acyclic complexes

2) $\forall A \in \mathcal{A} \quad \exists A \hookrightarrow R$ s.t. $R \in \mathcal{R}$.

Example If $\mathcal{A}$ has enough injectives $\Rightarrow$ take $\mathcal{R} = \underline{\text{Inj}}$, adapted to any left-exact functor.

Lemma $K^+(\mathcal{R})$, $S_{\mathcal{R}} = \text{quis} \cap K^+(\mathcal{R})$ is a localizing class and $K^+(\mathcal{R})[(S_{\mathcal{R}})^{-1}] \xrightarrow{\sim} \mathcal{D}^+(\mathcal{A})$

[Note : $\mathcal{D}\mathcal{A}$ is additive : localization of additive category: given fs^{-1}, $f's'^{-1}$ correct to have denominator in common]

Theorem If $\exists$ adapted class for $F \Rightarrow \exists$ a derived functor for F

Proof First define $\mathcal{D}F$ on $K^+(\mathcal{R})$ as $K^+F : K^+(\mathcal{R}) \to K^+(\mathcal{B})$

This respects quisms: $L \xrightarrow[\text{quis}]{f} M$ then

Cone $C(f) \in K^+(\mathcal{R})$ acyclic, $\Rightarrow K^+F(C(f)) \simeq C(K^+F(f))$

acyclic $\Rightarrow K^+F(f)$ is a quism.

$\Rightarrow$ get functor, get $\mathcal{D}F$:

$$\begin{tikzcd} K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \arrow[r] & \mathcal{D}^+\mathcal{B} \\ \mathcal{D}^+(\mathcal{A}) \arrow[u, phantom, "\wr"] \arrow[ur, dashed, "\mathcal{D}F"'] & \end{tikzcd}$$

, ε_F ■

which is universal.

Assume $\mathcal{D}F$ exists. An acyclic object $A \in \mathcal{A}$ for F is such that $R^iF(A) = 0$ for $i > 0$ (e.g. any adapted object.)

Prop $\exists$ adapted class for $F \Rightarrow$ {acyclic objects for F} is an adapted class. ("Acyclic models")

Example X loc. compact, paracompact topological space.

Soft sheaves are acyclic for $\Gamma : Sh(X) \to Ab$, global sections. - in fact are an adapted class. Injectives are soft and acyclic, so we just need to know our class isn't too big, ie all softs are still acyclic:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \overset{?}{\Longrightarrow} \Gamma(\mathcal{G}) \twoheadrightarrow \Gamma(\mathcal{H})$$

(proceed by induction to higher H^i)

Suppose $s \in \Gamma(\mathcal{H})$. Take covering $X = \bigcup_i^{\infty} K_i$ compacts, s.t. $s|_{K_i}$ comes from $t_i \in \Gamma(K_i, \mathcal{G})$.

Set $L_n = \bigcup_i^n K_i$, construct $r_i \in \Gamma(L_i, \mathcal{G})$ s.t. $r_i \mapsto s|_{L_i}$ and $r_{i+1}|_{L_i} = r_i$: this would be sufficient since $\Gamma(X, \mathcal{G}) = \varprojlim \Gamma(L_i, \mathcal{G})$

(from compactness) : section can be glued from open covers, any L_i has finite cover........ ?

Suppose we've constructed r_{i-1}/L_{i-1}, t_i/K_i such that

$$\begin{tikzcd} r_{i-1}|_{L_{i-1}\cap K_i} - t_i|_{L_{i-1}\cap K_i} \in \Gamma(L_{i-1} \cap K_i, \mathcal{F}) \\ \Gamma(K_i, \mathcal{F}) \arrow[u] \end{tikzcd}$$

$\Rightarrow$ construct r_i ■

$\Rightarrow$ Can compute $R\Gamma$ using soft sheaves, as well as Rf_* & $R\Gamma_c$.

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$F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{C}$ left exact

$G \circ F : \mathcal{A} \to \mathcal{C}$ left exact as well

<u>Thm</u> Assume $R_F \subset \mathcal{A}$ adapted to F, $R_G \subset \mathcal{B}$ to G, $F(R_F) \subset R_G$. Then $D(G \circ F) \simeq D(G) \circ D(F)$.

<u>Truncation functors</u> $K^\bullet : \cdots \to K^i \to K^{i+1} \to \cdots$

$$(\tau_{\leq i} K^\bullet)^n = \begin{cases} K^n, & n < i \\ \ker d, & n = i \\ 0, & n > i \end{cases}$$

$$(\tau_{\geq i} K^\bullet)^n = \begin{cases} 0, & n < i-1 \\ K^{i-1}/\ker d, & n = i-1 \\ K^n, & n \geq i \end{cases}$$

$0 \to \frac{K^{i-1}}{\ker d} \hookrightarrow K^i \to K^{i+1} \longrightarrow$ vs. $\cdots \to K^{i-2} \to K^{i-1} \to \ker d \to 0 \ldots$

$\tau_{\leq i}$ preserves H^n, $n \leq i$, annihilates H^n $n > i$ — similarly $\tau_{\geq i}$

Both preserve quisms $\Rightarrow$ induce functors $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$

Get distinguished Δ $\tau_{\leq i} K \to K \to \tau_{\geq i+1} K \to \tau_{\leq i} K[1]$

* Denote $\mathcal{D}^{\leq i} = \{K \mid H^n K = 0 \text{ for } n > i\}$, $\mathcal{D}^{\geq i}$, $\mathcal{D}^{[a,b]}$ etc.

<u>Prop.</u> 1. $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$

2. $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ strictly full ($\mathcal{A}$ in deg 0.)

<u>Proof</u>

$$\begin{tikzcd} & \tilde{K} \arrow[dl, "quis"'] \arrow[dr] & \tau_{\leq 0} \tilde{K} = K' \arrow[l, "\sim"'] \arrow[d] \\ \mathcal{D}^{\leq 0} \ni K & & L \in \mathcal{D}^{\geq 1} \end{tikzcd}$$

$\Rightarrow$ get map $K' \to L$

$$\begin{tikzcd} \cdot \arrow[r] & K'^{-1} \arrow[r] \arrow[dd] & K'^{0} \arrow[r] \arrow[dl] \arrow[dd] & 0 \arrow[dd] \\ & \ker d \arrow[dr, hook] & & \\ {} \arrow[r] & L^{-1} \arrow[r] \arrow[u] & L^{0} \arrow[r, "d"] & L^{1} \arrow[r] & \cdots \end{tikzcd}$$

- factors through $K' \longrightarrow \tau_{\leq 0} L \longrightarrow L$

$$\tau_{\leq 0} L \downarrow \text{acyclic}$$

$\Rightarrow$ no H^*, map is 0 in $\mathcal{D}$.

Proof (2): A_1, A_2 concentrated in dg 0,

$$\begin{tikzcd} & K \arrow[dl, "\sim"'] \arrow[dr] & \\ A_1 & & A_2 \end{tikzcd}$$

May replace K by $\tau_{\leq 0} K$, $H^0 K \simeq A_1$

$$\begin{tikzcd} \arrow[r] & K^{-1} \arrow[r] \arrow[d] & K^0 \arrow[d] \arrow[r] & H^0 k \arrow[dl] \\ & 0 \arrow[r] & A^2 & \end{tikzcd}$$

factors through $H^0K \simeq A_1$ so is induced by a morphism $A_1 \to A_2$. ■

Ext : $\quad \mathrm{Ext}^i_A(A,B) = \mathrm{Hom}_{\mathcal{D}(A)}(A, B[i])$,

vanishes for i negative by Prop (1).

Explicit (Yoneda) construction of Ext :

$$0 \to \big(B = K^{-i} \to K^{-i+1} \to \dots \to K^0\big) \to K^1 = A \to 0 \quad \text{acyclic}$$

gives element in Ext^i : $(\to \dots)$ is quis to A and maps to $B[i]$: $(0 \to K^{-i} \to \dots \to K^0 \to 0)$

$$\begin{tikzcd} (0 \to K^{-i} \to \dots \to K^0 \to 0) \arrow[d, phantom, "\wr"] \arrow[dr] & \\ A & B[i]. \end{tikzcd}$$

All elements in Ext^i come this way : Ext is equivalence classes of such under morphism which are Id on B, A.

Composition : $\mathrm{Ext}^i(B,C) \times \mathrm{Ext}^j(A,B) \to \mathrm{Ext}^{i+j}(A,C)$

$$\mathrm{Hom}(B[j], C[i+j]) \times \mathrm{Hom}(A, B[j]) \longrightarrow \text{composition}$$

Return to proof of ~~Theorem~~: $X = X_n \supset X_{n-2} \supset \dots$,

$U_k = X \setminus X_{n-k}$, $S_{n-k} = X_{n-k} \setminus X_{n-k-1}$.

1. $\underline{IC}_{n-\bullet}|_{U_2} \simeq$ orientation sheaf (quism)

2. $\underline{H}^i(\underline{IC}_{n-\bullet})|_{S_{n-k}} = 0$ for $i > p_k$

3. $\underline{IC}_{n-\bullet}|_{U_{k+1}} \longrightarrow Rj_{k*}(\underline{IC}_{n-\bullet}|_{U_k})$ is isom on H^i, $i \leq p_k$ $\quad (j_k : U_k \hookrightarrow U_{k+1})$

Proof Local statements, suffices to prove on stalks.

$F^\bullet$ complex of sheaves, $\underline{H}^j(F^\bullet)_x = H^j(F^\bullet_x)$

$$= H^j\big(\varinjlim_{x \in U} F^\bullet(U)\big) = \varinjlim_{x \in U} H^j(F^\bullet(U))$$

Thus suffices to study $IC^\bullet$ complexes (not sheaves) for a fundamental system of neighbourhoods.

$x \in S_{n-k}$, $x \in U \cong \mathbb{R}^{n-k} \times cL$,

Orientation sheaf = sheaf of top degree homologies. $\underline{IC}(U_2)$ is just usual chain complex of $\mathbb{R}^n$ locally - compute this get resolution of orientation sheaf

Calculation of IC for $\mathbb{R}^n_x$ and for $c(\ )$ gives steps 2, 3:

$$\begin{tikzcd}
\tau_{\leq p_k} IC_{k-1-\bullet}(L) \arrow[r] \arrow[d, "\text{quis}"] \arrow[dr, phantom, "\circlearrowleft"] & IC_{k-1-\bullet}(L) \arrow[d, "\text{quis"] \\
IC_{n-\bullet}(\mathbb{R}^{n-k} \times cL) \arrow[r, "\text{restriction}"'] & IC_{n-\bullet}(\mathbb{R}^{n-k+1} \times L)
\end{tikzcd}$$

*

$\cong$ - dropping vertex of cone.

$**\quad U \longrightarrow U \cap U_k = U \setminus X_{n-k}$

$\underline{IC}|_{U_k}$ is soft so adapted to Rj_{k*},

$\Rightarrow$ take just j_* of each term in complex,

$$\underline{H}^i(\underline{IC}_{n-\bullet}|_{U_{k+1}})_x \to \underline{H}^i(j_{k*}\underline{IC}_{n-\bullet}|_{U_k})_x$$

Only need to prove for $x \in S_{n-k}$ since the above are same for $x \in U_k$.

$$\begin{tikzcd}
H^i(\underline{IC}_{n-\bullet}|_{U_{k+1}}(U)) \arrow[r] \arrow[d, equal] & H^i(j_{k*}\underline{IC}_{n-\bullet}|_{U_k}(U)) \arrow[d, equal] \\
H^i(IC_{n-\bullet}(U)) \arrow[r] & H^i(IC_{n-\bullet}(U \cap U_k))
\end{tikzcd}$$

$\to$ use above diagram * and ** ... ∎

Motivation for more Homological Algebra

We could replace $\underline{IC}$ by complex coming say from a local system over U_2 instead of orientation ... etc!

get lots of IC complexes $\subset \mathcal{D}(Sh_X)$

$\boxed{\text{abelian category}}$? how do abelian categories sit in $\mathcal{D}(\mathcal{A})$?

Triangulated Category : $\mathcal{D}$ - additive category, equipped with additive automorphism $T : \mathcal{D} \to \mathcal{D}$ ("shift"), $T^n X =: X[n]$ with notion of triangles: class of diagrams of form

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "u"'] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & K[1] \arrow[d, "u[1]"] \\
K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & K'[1]
\end{tikzcd}$$

morphisms

— distinguished Δ's are given class of such.

Axioms. TR1 a) $X \xrightarrow{id} X \to 0 \to X[1]$ is dist.

b) Any Δ isom to a dist. one is dist.

TR2 $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ dist $\Longrightarrow$

$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ dist

TR3

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

$\Rightarrow$ can fill in $Z \downarrow Z'$ to make commutative.

TR4 Octahedron axiom.

upper cap

$$\begin{tikzcd}
X' \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] & & Z \arrow[ll] \arrow[ll, phantom, "*" below] \\
 & Y \arrow[ur] \arrow[dl] & \\
Z' \arrow[rr, "{[1]}"'] & & X \arrow[ul, bend right] \arrow[uu]
\end{tikzcd}$$

$\circlearrowleft$ $\circlearrowleft$ $*$

$$\begin{tikzcd}
X' \arrow[dd, "{[1]}"'] & & Z \arrow[ll] \arrow[dl] \\
 & Y \arrow[ul] \arrow[dr] & \\
Z' \arrow[ur] \arrow[rr, "{[1]}"'] & & X \arrow[uu]
\end{tikzcd}$$

$\circlearrowleft$ $*$ $*$ $\circlearrowleft$

$\longrightarrow$ lower cap

[1] denotes morphism of deg 1 - e.g. $X' \longrightarrow Z'[1]$

$*$ denotes dist Δ. $\circlearrowleft$ commutative

The axiom says any upper cap can be completed by a lower cap. (external rectangles are in common) + the 2 morphism (through Z, Z') from Y to Y' agree + same for $X X'$ $Y' \to Y$ — commutative octahedron.

- Analog of property for abelian groups $A < B < C$,

$B/A < C/A \Rightarrow C/A / B/A \simeq C/B$.

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[Note: TR1 c. $X \xrightarrow{f} Y$ can be completed to a dist Δ $X \xrightarrow{f} Y \to Z \to X[1]$ for any morphism f.]

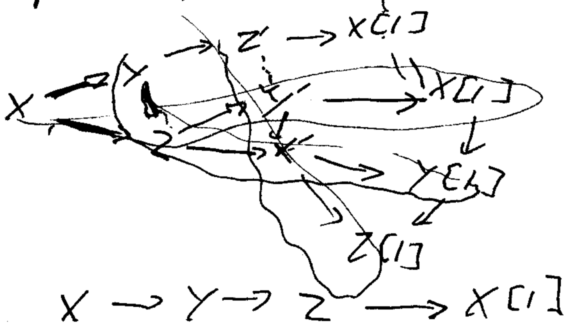
Inside an upper-cap helix (3-dim structure) can ~~find~~ find a lower cap:

$$\begin{tikzcd}
X' \arrow[dr] \arrow[dd, "{[1]}"'] & & Z[1] \arrow[ll, "{[1]}"'] \\
& Y[1] \arrow[ur, "*"] \arrow[dl] & \\
Z' \arrow[rr, "*"] & & X[1] \arrow[ul] \arrow[uu]
\end{tikzcd}$$

∂ $\circlearrowright$ – lower. So only need in axiom upper $\Rightarrow$ lower, not also lower $\Rightarrow$ upper.

<u>Remark</u>: Any dist. Δ $X \to Y \to Z \to X[1]$ completing a given $X \to Y$ is unique up to isomorphism

$\Rightarrow$ implies upper cap is determined up to iso from
$$\begin{tikzcd}
Y \arrow[r] & Z \\
& X \arrow[ul] \arrow[u]
\end{tikzcd}$$
$\circlearrowright$



$\Leftrightarrow$ octagon starting from

$Z' \dashrightarrow Y'$ is the octagon axiom.

<u>Prop.</u> $X \to Y \to Z \to X[1]$ dist Δ, $U \in \mathcal{D}$

$\Rightarrow$ $\mathrm{Hom}(U,X) \to \mathrm{Hom}(U,Y) \to \mathrm{Hom}(U,Z) \to \mathrm{Hom}(U,X[1]) \to \ldots$

is a long exact sequence. (true even without TR4).

<u>Proof</u>

Sufficient to prove $\mathrm{Hom}(U,X) \to \mathrm{Hom}(U,Y) \to \mathrm{Hom}(U,Z)$ is exact – rest are turnings of it hence also dist Δ's.

To prove composition is zero suffices to show $X \to Y \to Z$ with $X \xrightarrow{0} Z$, but we have

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] & Z & \\
X \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "f"] \arrow[ur, "0"] & 0 \arrow[r] \arrow[u, dotted] & X[1]
\end{tikzcd}$$

$\swarrow$ can complete

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[r] \arrow[d] & U[1] \arrow[r, phantom, "\simeq"] \arrow[d, dashed, "\beta"] & U[1] \\
Y \arrow[r] & Z \arrow[r] & X[1] \arrow[r] & Y[1]
\end{tikzcd}$$

$\beta[-1] \in \mathrm{Hom}(U,X)$ which gives α $\Rightarrow$ exactness at middle term ■

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f\ \wr"'] & Y \arrow[r] \arrow[d, "g\ \wr"'] & Z \arrow[r] \arrow[d, dashed, "h"'] & X[1] \arrow[d, "\wr"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd} \Rightarrow \quad \Big\downarrow \text{ is also } \simeq$$

(if $X \xrightarrow{f} X'$; $Y \xrightarrow{g} Y'$)

– use five lemma with previous proposition to show $\mathrm{Hom}(U,Z) \xrightarrow{h_*} \mathrm{Hom}(U,Z')$ is iso for any U

hence $Z \xrightarrow{h} Z'$ is isom

This gives the desired isomorphism of any two dist$\triangle$ completions of $X \to Y$ – but not uniquely . . $\square$

We call the resulting isomorphism class of $X \xrightarrow{f} Y \to \underline{\underline{Z}} \to X[1]$ the cone of f.

Proposition Assume
$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r] & X[1] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Then the following are equivalent:

a. $v' g u = 0$

b. $\exists f : X \to X'$ s.t. the first square commutes

b′ $\exists h$ s.t. 2nd square commutes

c. $\exists$ morphism of $\triangle$'s

If these (equivalent) conditions are satisfied, and $\mathrm{Hom}^{-1}(X, Z') = 0$ $(= \mathrm{Hom}(X, Z'[-1])$ then $\exists! f$ s.t. first $\boxed{\circlearrowleft}$, $\exists! h$ s.t. second $\boxed{\circlearrowleft}$.

Proof e.g. $a \Rightarrow b$: Consider $gu : X \to Y'$, goes to zero when composed with v' $\Rightarrow$ comes from X' ($\Rightarrow f \in \mathrm{Hom}(X,X')$).
similarly for other equivalences.

Uniqueness part: suppose $\mathrm{Hom}^{-1}(X, Z') = 0$

$$\underset{\overset{\|}{0}}{\mathrm{Hom}^{-1}(X,Z)} \to \underset{\overset{\in}{f}}{\mathrm{Hom}(X,X')} \to \mathrm{Hom}(X,Y') \to \mathrm{Hom}(X,Z')$$

$\Rightarrow$ f unique ■

Corollary $X \xrightarrow{u} Y \to Z \xrightarrow{d} X[1]$ dist.

$\mathrm{Hom}^{-1}(X,Z) = 0$ $\Rightarrow$ i) $\mathrm{Cone}(u)$ unique up to unique isom

ii) d is unique $\in \mathrm{Hom}(Z, X[1])$ s.t. $X \xrightarrow{u} Y \to Z \xrightarrow{d} X[1]$ exact ■

__Theorem__ $\mathcal{A}$ abelian category $\Rightarrow K(\mathcal{A}), \mathcal{D}(\mathcal{A})$ are triangulated categories (with $[1]$, Δ's as before).

__Proof__ T1, T3 obvious from functoriality of triangles

T2 follows from cylider / cone sequences we did.

Octagon: replace $X \to Y \to Z$ by $X \hookrightarrow Y \hookrightarrow Z$

- moreover ask it to be a direct summand, then complete the octagon explicitly - - - - —— - ■

$K(\mathcal{A})$ triangulated $\Rightarrow \mathcal{D}(\mathcal{A})$ triang. by localization

__Definition__ Assume $\mathcal{D}$ triangulated category, S localizing class. S is __compatible__ with triangulation if

1) $s \in S \Leftrightarrow s[1] \in S$
2) in TR 3
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$
$f, g \in S \Rightarrow \exists h \in S$.

- Satisfied by __qis__ in $K(\mathcal{A})$

__Proposition__ S loc class compatible with triangulation $\Rightarrow \mathcal{D}[S^{-1}]$ has canonical triangulation: shift descends etc..

__Other examples__: 1. Filtered derived category: $\mathcal{A}$ abelian,

$F\mathcal{A}$ - filtered objects in $\mathcal{A}$ (finite filtrations)

$$0 \to X \to Y \to Z \to 0 \text{ exact } \overset{\text{def}}{\Longleftrightarrow} 0 \to Gr_n X \to Gr_n Y \to Gr_n Z \to 0 \text{ exact all } n.$$

$K(F\mathcal{A})$ homotopy category, replace __qis__ by __filtered qis__:

$f : K \to L$ s.t. $Gr_n f$ is qis all n.

2. __Constructible sheaves__ (ℓ-adic) X scheme/k

Abelian category underlying: 1. $A = \mathbb{Z}/\ell^n\mathbb{Z}$, consider constructible sheaves of A-modules in étale topology with collection of Zariski loc closed subsets where it's locally constant.

F_n - $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf, $F_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n \simeq F_n$

systems of sheaves:

$$\Rightarrow \text{def } \mathcal{D}_c^b(X, \mathbb{Z}_\ell) = \varprojlim_n \mathcal{D}^b_{ctf}(X, \mathbb{Z}/\ell^n)$$

H^i constructible $\mathbb{Z}/\ell^n$ sheaf

finite Tor dimension

Finite Tor-dim necessary for $K_n^\bullet$ $\boxed{K_{n+1}^\bullet \overset{L}{\otimes}_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \simeq K_n}$ to be bounded (in inverse limit

<u>Prop</u> $\dashrightarrow \mathcal{D}_{n+1} \xrightarrow{\text{exact}} \mathcal{D}_n \xrightarrow{\text{exact}} \cdots$ system of triangulated categories. $K, L \in \mathcal{D}_n$ $\operatorname{Hom}(K, L)$ finite

$\Rightarrow \varprojlim \mathcal{D}_n$ is a triangulated category

$\operatorname{Hom}(K,L)$ finite - need finiteness of Galois cohomology

$\Rightarrow k$ alg. closed.

$S_{n+1} \to S_n \to \cdots$ proj system of nonempty finite sets $\Rightarrow \varprojlim S_n \neq \emptyset$.

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Let $\mathcal{D}$ be a triangulated category, $\mathcal{C} \subset \mathcal{D}$ full subcategory with $\operatorname{Hom}^{-i}(\mathcal{C}, \mathcal{C}) = 0$, $i>0$: i.e.

$\forall X, Y \in \mathcal{C}$, $\operatorname{Hom}^{-i}(X,Y) := \operatorname{Hom}(X, Y[-i]) = 0$ $i>0$ $(*)$

(e.g. complexes in deg. 0 for derived category..)

$X \to Y \to Z$ in $\mathcal{C}$ and $\mathcal{C}$ happens to be abelian $\Rightarrow$ two notions of exactness: that it can be completed to dist. Δ & that it's exact in $\mathcal{C}$'s abelian structure...

A abelian $\subset \mathcal{D}^b(A)$ and $f: X \to Y$ morphism in A, then

$$\operatorname{Cone}(f) := (\overset{\bullet -1}{X} \xrightarrow{\bullet} \overset{0}{Y}) \hookleftarrow (\overset{(-1)}{\ker f} \to \overset{(0)}{0})$$
$$\downarrow$$
$$0 \to \operatorname{coker}(f) \to 0$$

$$(X \to Y) / (\ker f \to 0) \xrightarrow{\sim} (X/\ker f \to Y) \xrightarrow{\text{quis}} (0 \to \operatorname{coker} f)$$

$\Rightarrow$
$$\begin{tikzcd}
\ker f[1] \arrow[dr] \arrow[dd] \arrow[rr, phantom, "*"'] & & \operatorname{coker} f \arrow[ll, "{[1]}"'] \\
& \operatorname{Cone} \arrow[ur] \arrow[dl, "{[1]}"'] & \\
X \arrow[rr] \arrow[rr, phantom, "*"] & & Y \arrow[ul] \arrow[uu]
\end{tikzcd}$$

Based on this we'll give conditions on $\mathcal{C} \subset \mathcal{D}$ as above ($\mathcal{C}$ satisfying $*$):

$f \in \mathcal{C}$ is <u>admissible</u> if given
$$\begin{tikzcd} & S \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"', "*"] & & Y \arrow[ul] \end{tikzcd}$$
$(X, Y \in \mathcal{C})$

$\Rightarrow \exists$
$$\begin{tikzcd} N[1] \arrow[dr] & & C \arrow[ll, "{[1]}"', "*"] \\ & S \arrow[ur] & \end{tikzcd}$$
with $N, C \in \mathcal{C}$.

<u>Proposition</u> Assume $\mathcal{C}$ satisfies $(*)$, stable under $\oplus$. Then [$\mathcal{C}$ abelian, all exact triples in $\mathcal{C}$ completable to distinguished Δ in $\mathcal{D}$] $\Longleftrightarrow$ [every morphism in $\mathcal{C}$ is admissible]

<u>Proof</u> $(\Leftarrow)$: <u>First remark</u> $X \to Y \to Z \to X[1]$ dist, $X, Y, Z \in \mathcal{C} \Rightarrow f, g$ admissible, $Z = \operatorname{coker} f$, $X = \ker g$ (note ker, coker defined in any category (additive) as universal objects..)

$(\Leftarrow)$: every morphism admissible, we want to show $\mathcal{C}$ abelian. assume $f: X \to Y$ in $\mathcal{C}$

$\Rightarrow$
$$\begin{tikzcd} N[1] \arrow[d] \arrow[dr] & & C \arrow[ll, "{[1]}"'] \\ & S \arrow[ur] \arrow[dl, "{[1]}"'] \arrow[u, phantom, "*"] \arrow[d, phantom, "*"] & \\ X \arrow[rr] & & Y \arrow[ul] \arrow[uu] \end{tikzcd}$$
is lower cap, so by TR 4
$$\begin{tikzcd} N[1] \arrow[d, "{[1]}"'] & & C \arrow[ll, "{[1]}"'] \arrow[dl, "{[1]}"] \\ & I \arrow[ul] \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Y \arrow[uu] \end{tikzcd}$$
$$\Longrightarrow$$

$Y \to C$ is in $\mathcal{C}$, $N \to X$ in $\mathcal{C}$.

The right cap is an "admissibility diagram" for $Y \to C$ (as in definition of admissibility)

In admissibility diagram f mono $\Leftrightarrow N = 0$, f epi $\Leftrightarrow C = 0$.

$$\begin{tikzcd} N[1] \arrow[dr] & & C' \arrow[ll, "{[1]}"'] \\ & I[1] \arrow[ur] \arrow[dl, "{[1]}"'] & \\ Y \arrow[rr] & & C \arrow[ul] \end{tikzcd}$$

$\boxed{\text{Claim}}$ $Y \to C$ epi in above diagram whenever f admissible.

i.e.
$$\begin{tikzcd} \operatorname{Hom}(C, T \in \mathcal{C}) \arrow[r, hook] \arrow[d] & \operatorname{Hom}(S, Y) \\ \operatorname{Hom}(S, T) \arrow[d] & \\ \operatorname{Hom}^{-1}(N, T) & \end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Hom}^{-2}(N,T)=0 \arrow[d] & \\
 & \mathrm{Hom}(C,T) \arrow[r, hook] \arrow[d] & \mathrm{Hom}(Y,T) \\
\mathrm{Hom}^{-1}(X,T) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(S,T) \arrow[ur, "\alpha"'] \arrow[d] & \\
0 & \mathrm{Hom}^{-1}(N,T)=0 &
\end{tikzcd}$$

α is mono since $\mathrm{Hom}^{-1}(X,T)=0$

1. $\boxed{\begin{array}{l} \Rightarrow \ Y \to C \text{ epi}, \\ \text{similarly } \ N \to X \text{ mono}. \end{array}}$

2. Next if f is an admissible epimorphism then $C=0$:
   — show $\mathrm{Hom}(C,T)=0$:
   $\mathrm{Hom}(C,T) \simeq \mathrm{Hom}(S,T) \simeq \ker(\mathrm{Hom}(Y,T) \xrightarrow{f^*} \underset{0}{\underset{\|}{\mathrm{Hom}(X,T)}})$ (f epic)
   $\Longrightarrow S \in \mathcal{C}[1]$

3. Similarly f admissible mono $\Rightarrow N=0$.
   $\Longrightarrow S \in \mathcal{C}$.

Back to our lower cap, apply to $Y \to C$ admissible
$\Rightarrow I[1] \in \mathcal{C}[1] \Rightarrow I \in \mathcal{C}$, $I = \ker(Y \to C)$.
Dually on other side of diagram $\Rightarrow I \simeq \mathrm{coker}(N \to X)$ //
— so axiom of abelian category is satisfied. . .

Now we need to show exact $0 \to X \to Y \to Z \to 0$ can be completed to Δ : apply admissibility of $X \to Y$.

($\Longrightarrow$) Can construct the upper cap back from lower cap we construct explicitly from axioms of abelian category . . ■

Definition $\mathcal{C} \subset \mathcal{D}$ is admissible if it satisfies the (equivalent) conditions of the proposition.

Definition ~~A~~ t-structure on $\mathcal{D}$ is a pair $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0} \subset \mathcal{D}$ of strictly full subcategories ($A \simeq B \in \text{subcat} \Rightarrow A \in \text{subcat}$)
[denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$]
if :

1) Hom $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 1}) = 0$

2) $\mathcal{D}^{\le 0} \subset \mathcal{D}^{\le 1}$, $\mathcal{D}^{\ge 0} \supset \mathcal{D}^{\ge 1}$

3) $\forall X \in \mathcal{D}$ $\exists$ dist Δ $\quad \underset{\in \mathcal{D}^{\le 0}}{A} \to X \to \underset{\in \mathcal{D}^{\ge 1}}{B} \to A[1]$.

Heart of $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$: $\mathcal{C} = \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$.

Example $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$, $\mathcal{D}^{\le 0} := \{K \mid H^i K = 0 \ i > 0\}$ etc.
we've proven all above properties (3 uses $\tau_{\le i}, \tau_{\ge i}$).

$(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ t-structure ⇒ new one from $(\mathcal{D}^{\le n}, \mathcal{D}^{\ge n})$
Perverse sheaves will come from gluing of t-structures.

Theorem $\mathcal{C}$ is admissible.
Prop $\forall n$ $\exists$ functor $\tau_{\le n}: \mathcal{D} \to \mathcal{D}^{\le n}$, right adjoint to $\mathcal{D}^{\le n} \hookrightarrow \mathcal{D}$: $X \in \mathcal{D}$, $Y \in \mathcal{D}^{\le n}$, $\mathrm{Hom}(Y, X) \simeq \mathrm{Hom}(Y, \tau_{\le n} X)$
& dually $\tau_{\ge n}$ left adjoint to $\mathcal{D}^{\ge n} \hookrightarrow \mathcal{D}$.

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Proof of Proposition
Given $X \in \mathcal{D}$, by 3) ⇒ distinguished Δ

$$\underset{\mathcal{D}^{\le 0}}{\underset{\cap}{A}} \to \underset{\mathcal{D}}{\underset{\cap}{X}} \to \underset{\mathcal{D}^{\ge 1}}{\underset{\cap}{B}}$$

For $T \in \mathcal{D}^{\le 0}$, $\mathrm{Hom}(T, A) \xrightarrow{\sim} \mathrm{Hom}(T, X)$:
Consider long exact of $\mathrm{Hom}(T, \text{our } \Delta)$

$$\Rightarrow \underset{\substack{\| \\ 0 \ (1)}}{\mathrm{Hom}(T, B[-1])} \to \mathrm{Hom}(T, A) \xrightarrow{\sim} \mathrm{Hom}(T, X) \to \underset{\substack{\| \\ 0 \ (1)}}{\mathrm{Hom}(T, B)}$$

$$\Rightarrow \mathrm{Hom}_{\mathcal{D}}(T, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(T, X)$$

$\|$

$\mathrm{Hom}_{\mathcal{D}^{\le 0}}(T, A)$

$\|$

$\boxed{\tau_{\le 0} X}$ definition. ⇒ uniquely defined arrow $\tau_{\le 0} X \to X$.

Similarly get $X \to \tau_{\geq 1} X$. □

Claim $\exists!$ $\tau_{\geq 1} X \xrightarrow{\quad d \quad} \tau_{\leq 0} X[1]$ s.t.

$\tau_{\leq 0} X \to X \to \tau_{\geq 1} X \xrightarrow{d} \tau_{\geq 0} X[1]$ is distinguished.

Uniqueness follows from having no morphisms from $\tau_{\leq 0} X[1]$ to $\tau_{\geq 1} X$: suppose $\exists\, d, d'$ as above

$$\begin{tikzcd}
\tau_{\leq 0} X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & \tau_{\geq 1} X \arrow[r, "d"] \arrow[d, dashed] & \tau_{\leq 0} X[1] \\
\tau_{\leq 0} X \arrow[r] & X \arrow[r] & \tau_{\geq 1} X \arrow[r, "d'"] & \tau_{\leq 0} X[1]
\end{tikzcd}$$

$\dashrightarrow$ must be identity since different choices differ by $f \in \mathrm{Hom}(\underset{\substack{\cap \\ \mathcal{D}^{\leq -1}}}{\tau_{\leq 0} X[1]}, \tau_{\geq 1} X) = 0$

Existence follows by construction. □

Facts 1. $a \leq b$ $\quad \mathcal{D}^{\leq a} \subset \mathcal{D}^{\leq b} \subset \mathcal{D}$

$\Rightarrow$ canonical morphism $\tau_{\leq a} X \longrightarrow \tau_{\leq a} \tau_{\leq b} X$ which we claim is iso: $\tau_{\leq a} X \to X$ factors through $\tau_{\leq b} X$ hence through $\tau_{\leq a} \tau_{\leq b} X$ ......

2. $a \leq b$ $\exists$ $\tau_{\leq b} \tau_{\geq a} \overset{\text{canonical}}{\simeq} \tau_{\geq a} \tau_{\leq b}$:

$$\begin{tikzcd}
\tau_{\leq b} X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq a} X \\
\tau_{\geq a} \tau_{\leq b} X \arrow[urr, dashed] & &
\end{tikzcd}$$

$\dashrightarrow$ exists by adjunction of $\tau_{\geq a}$

(*) $\underset{\substack{\| \\ \tau_{\leq a-1} X}}{\tau_{\leq a-1} \tau_{\leq b} X} \longrightarrow \underset{\substack{\cap \\ \mathcal{D}^{\leq b}}}{\tau_{\leq b} X} \to \tau_{\geq a} \tau_{\leq b} X \to \underset{\in \mathcal{D}^{\leq a-2} \subset \mathcal{D}^{\leq}_{b}}{\tau_{\leq a-1} X[1]}$

Lemma $X \in \mathcal{D}^{\leq 0}$ iff $\tau_{\geq 0} X \xrightarrow{\sim} X$ iff $\tau_{\geq 1} X = 0$ iff $\mathrm{Hom}(X, \mathcal{D}^{\geq 1}) = 0$ □

$\Rightarrow$ follows that this condition is ~~set~~ closed under extensions $X' \to X \to X''$ dist (X'', X' have it $\Rightarrow$ so does X)

So $\mathcal{D}^{\leq 0}$ is stable under extensions by long exact of Homs (same for $\mathcal{D}^{\geq a}$, $\mathcal{D}^{\leq b}$ etc.)

$\Rightarrow$ $\tau_{\geq a}\tau_{\leq b}X \in \mathcal{D}^{\leq b}$ in above extension $*$

$\Rightarrow$ by adjunction $\tau_{\geq a}\tau_{\leq b}X \longrightarrow \tau_{\leq b}\tau_{\geq a}X$.

This is an isomorphism: use octahedron axiom

$$\begin{tikzcd}
 & \tau_{\leq b}X \arrow[r] \arrow[dd] & \tau_{\geq a}\tau_{\leq b}X \arrow[dd] \arrow[drr, dashed, "\sim"] & & \text{by axiom } \exists! \text{ such making it commutative} \\
\tau_{<a}X \arrow[ur] \arrow[dr] & & & & \tau_{\leq b}\tau_{\geq a}X \arrow[dll] \\
 & X \arrow[r] \arrow[dr] & \tau_{\geq a}X \arrow[d] & & \\
 & & \tau_{>b}X & &
\end{tikzcd}$$

$\square$

$\Longrightarrow$ get functors $\tau_{[a,b]} : \mathcal{D} \rightarrow \mathcal{D}^{[a,b]} = \mathcal{D}^{\leq b} \cap \mathcal{D}^{\geq a}$

Not adjoint to an embedding — composition of right adjoint & left adjoint.

Heart $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, get functor $H^0 : \mathcal{D} \rightarrow \mathcal{C}$ by truncations both way

Theorem 1. $\mathcal{C}$ is an admissible ($\Rightarrow$abelian) category.

2. $H^0 : \mathcal{D} \rightarrow \mathcal{C}$ is a cohomological functor,

[i.e. additive functor taking distinguished Δ to induced exact sequence

$$X \rightarrow Y \rightarrow Z \rightarrow X[1] \quad \Rightarrow \quad H^0 X \rightarrow H^0 Y \rightarrow H^0 Z,$$

so if we denote $H^n X := H^0(X[n])$ get long exact.]

Proof 1. $\mathrm{Hom}^{<0}(\mathcal{C}, \mathcal{C}) = 0$: satisfied

since $\mathrm{Hom}^i(\mathcal{C},\mathcal{C}) = \mathrm{Hom}(\underset{\mathcal{D}^{\leq 0}}{\underset{\cap}{\mathcal{C}}}, \underset{\mathcal{D}^{\geq 0}[i] \subset \mathcal{D}^{\geq -i \geq 1}}{\underset{\cap}{\mathcal{C}[i]}})$

$$\begin{tikzcd}
N[1] \arrow[rr, phantom, ""] & & k \arrow[ll, dashed, "{[1]}"'] \\
 & S \arrow[ur, dashed] \arrow[u, phantom, "*"] \arrow[dl, "{[1]}"'] & \\
X \arrow[rr, "f"] \arrow[ur, phantom, ""] & & Y \arrow[ul, bend right] \arrow[ll, phantom, "*" near end]
\end{tikzcd}$$

$$N[1] \dashrightarrow S$$

2, we need $X \xrightarrow{f} Y$

we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d, phantom, sloped, "\in"] & Y \arrow[r] \arrow[d, phantom, sloped, "\in"] & S \arrow[r] \arrow[d, phantom, sloped, "\in"] & X[1] \arrow[d, phantom, sloped, "\in"] \\
\mathcal{C} & \mathcal{C} & \mathcal{D}^{[-1,0]} & \mathcal{C}[1]
\end{tikzcd}$$

$\Rightarrow$ $K = \tau_{\geq 0} S$, $N[1] = \tau_{\leq -1} S$ get canonical triangle upstairs, & $K \in \mathcal{C}$, $N[1] \in \mathcal{C}[1]$ as desired. (■)

H^0 is cohomological:

Given dist Δ need to show we get long exact..

$\boxed{a.}$ Assume $X \to Y \to Z \to X[1]$, $X, Y, Z \in \mathcal{D}^{\leq 0}$

$\Rightarrow$ $H^0X \to H^0Y \to H^0Z \to 0$ is exact?

Same as to show that for $\forall T \in \mathcal{C}$ get exact seq

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(H^0(Z), T) \arrow[r] \arrow[d, "\wr", "\text{adjointness}"'] & \mathrm{Hom}(H^0Y, T) \arrow[r] \arrow[d, "\wr", "\text{adjoint}"'] & \mathrm{Hom}(H^0X, T) \arrow[d, "\wr", "\text{adjoint}"'] \\
 & \mathrm{Hom}^{-1}(X,T) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}}(Z, T) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(Y, T) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(X, T) \\
 & 0 & & &
\end{tikzcd}$$

since $X \in \mathcal{D}^{\leq 0}$, $T[-1] \in \mathcal{D}^{\geq 1}$ so we get exactness.

$\boxed{b.}$ Assume $X \in \mathcal{D}^{\leq 0}$, Y, Z arbitrary.

Claim $\tau_{\geq 1} Y \xrightarrow{\sim} \tau_{\geq 1} Z$: take $T \in \mathcal{D}^{\geq 1}$, show $\mathrm{Hom}(\tau_{\geq 1} Z, T) \xrightarrow{\sim} \mathrm{Hom}(\tau_{\geq 1} Y, T)$ — use adjointness to write as Hom in $\mathcal{D}$ & long exact of Homs – but $\mathrm{Hom}^{-1}(X, T) = \mathrm{Hom}(X, T) = 0$ since $T \in \mathcal{D}^{\geq 1}$. □

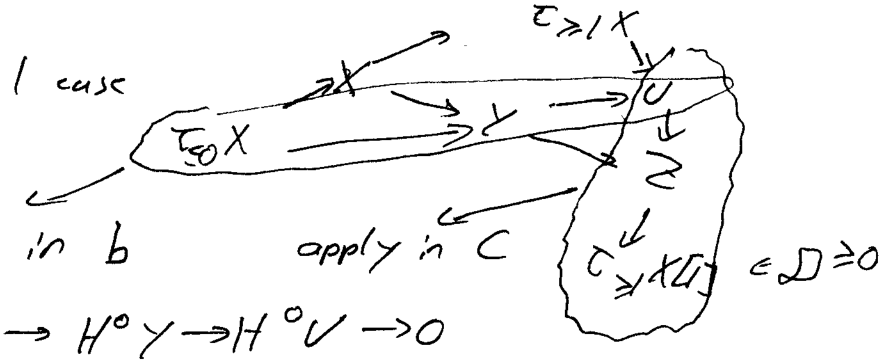
Now use octahedron:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, bend right] & \tau_{\leq 0} Y \arrow[r] \arrow[d] & \tau_{\leq 0} Z \arrow[d] \\
 & Y \arrow[r] \arrow[d] & Z \\
 & \tau_{\geq 1} Y \arrow[r, phantom, "\simeq"] & \tau_{\geq 1} Z
\end{tikzcd}$$

$\to$ can be substituted in part a. to reach our desired conclusion.

$\boxed{c.}$ Dually if $Z \in \mathcal{D}^{\geq 0}$ get $0 \to H^0X \to H^0Y \to H^0Z$ exact.

$\boxed{d}$ General case



$H^0X \to H^0Y \to H^0U \to 0$

$0 \to H^0U \to H^0Z$

$H^0X \to H^0Y \longrightarrow H^0Z$. ☐

Proposition-Def A t-structure is called non-degenerate if $\bigcap_n \mathcal{D}^{\leq n} = \bigcap_n \mathcal{D}^{\geq n} = 0$.

[e.g. $\mathcal{D}$ with $(\mathcal{D}, 0)$ is degenerate t-structure]

This implies that the system of functors $H^i : \mathcal{D} \to \mathcal{C}$ is conservative:

1) $X = 0 \iff H^iX = 0 \;\forall i$

2) $f : X \to Y$ is iso $\iff H^if = 0 \;\forall i$

(can't check a morphism is zero by taking H^i ...)

3) $\mathcal{D}^{\leq 0} = \{X \mid H^iX = 0, \forall i > 0\}$, same for $\mathcal{D}^{\geq 0}$.

Proof 1). $H^iX = 0 \;\forall i$, $X \in \mathcal{D}^{\leq 0}$: $H^0X = 0 \Rightarrow X \simeq \tau_{\leq -1}X \in \mathcal{D}^{\leq -1}$ and so on $X \in \mathcal{D}^{\leq -n} \;\forall n \Longrightarrow X = 0$.

For general $X \Rightarrow \tau_{\leq 0}X \to X \to \tau_{\geq -1}X$ above applies to first term, exactness gives it for X...

2) $f : X \to Y$ iso $\Rightarrow Z = 0$ in $X \xrightarrow{f} Y \to Z$.
so reduce to first statement : $H^iZ = 0 \;\forall i$

3) similar ☐

Gluing construction of t-structures

X - topological space $\qquad \overset{\text{open}}{U} \underset{j}{\subsetneq} X \overset{i}{\supset} F = X \setminus U.$

$$\begin{tikzcd} Sh\,U & Sh_X \arrow[l, "j^*"'] & Sh\,F \arrow[l, "i_*"'] \end{tikzcd}$$

get adjoints

$$\begin{tikzcd} Sh_U \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & Sh_X \arrow[l, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & Sh\,F \arrow[l, "i_*"'] \end{tikzcd}$$

Get triples $(j_!, j^! = j^*, j_*)$ $\qquad (i^*, i_* = i_!, i^!)$

where any two nearby functors are adjoint.

Formal consequences $j^* i_* = 0 \iff i^* j_! = 0,\ i^! j_* = 0$

Also have exact sequences $0 \longrightarrow j_! j^* F \longrightarrow i_* i^* F \longrightarrow 0$

$0 \to i_* i^! F \to F \to j_* j^* F$

Pass to derived categories: just need to worry about j_* $i^!$ which are just left exact (resp exact).

$\Rightarrow$ look in $\mathcal{D}^+$ take derived functors

Adjunctions $\Rightarrow$ $\quad i^* i_* F \xrightarrow{\sim} F \to i^! i_* F$

$j^* j_* F \xrightarrow{\sim} F \xrightarrow{\sim} j^* j_! F$, same for derived functors, above exacts become distinguished triangles

$$\begin{tikzcd} \mathcal{D}_U \arrow[r, bend left] \arrow[r, bend right] & \mathcal{D}_X \arrow[l] \arrow[r, bend left] \arrow[r, bend right] & \mathcal{D}_F \arrow[l] \end{tikzcd}$$

From t-structures on $\mathcal{D}_U, \mathcal{D}_F \Rightarrow$ t-structure on $\mathcal{D}_X$ by gluing.

$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ t-structure $\Rightarrow$ shifted t-structure $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$

Have stratified space glued step by step out of strata, add larger & larger open sets. Each stratum has perversity p - use p-shifted t-structure $\Rightarrow$ t-structure on $\mathcal{D}_X$ with heart the category of perverse sheaves.

Gluing Construction

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Given three categories (triangulated) $$\begin{tikzcd} \mathcal{D}_U \arrow[r, bend left] \arrow[r, hook] & \mathcal{D}_X \arrow[l, "j^*"'] \arrow[r, bend left] & \mathcal{D}_F \arrow[l, "i^*"'] \arrow[l, hook'] \end{tikzcd}$$
exact functors with pairs of adjoints

$j^* i_* = 0$, $j_! j^* X \longrightarrow X \to i_* i^* X \to$ exact Δ

$i^* j_! = 0$, $i^! j_* = 0$ $\quad i_* i^! X \to X \to j_* j^* X \to$ " "

There are no morphisms

$$\mathrm{Hom}(j_! A, i_* B) = 0 = \mathrm{Hom}(i_* B, j_* A) \quad \forall A \in \mathcal{D}_U, B \in \mathcal{D}_F$$

$\Rightarrow$ above exact Δ's are canonically defined by first two arrows.

$$\mathcal{D}_F \xrightarrow{i_*} \overset{\mathcal{D}}{\overset{\shortparallel}{\mathcal{D}_X}} \xrightarrow{j^*} \mathcal{D}_U \quad \text{exact triple of categories:}$$

i.e. i_* fully faithful, with image $\mathcal{D}_F = \{A \in \mathcal{D}_X \mid j^* A = 0\}$

- follows from above exact Δ's.

Assume we have t-structures $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ on $\mathcal{D}_U$, $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ on $\mathcal{D}_F$

Definition $\mathcal{D}_X^{\leq 0} = \mathcal{D}^{\leq 0} = \{A \in \mathcal{D} \mid j^* A \in \mathcal{D}_U^{\leq 0}, i^* A \in \mathcal{D}_F^{\leq 0}\}$

$\mathcal{D}^{\geq 0} = \{A \in \mathcal{D} \mid j^* A \in \mathcal{D}_U^{\geq 0}, i^! A \in \mathcal{D}_F^{\geq 0}\}$

- sheaf structure

Theorem $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.

Proof.

1) $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$: take $A \in \mathcal{D}^{\leq 0}$, $B \in \mathcal{D}^{\geq 1}$.

exact Δ ~~$i_* i^*$~~ $j_! j^* A \to A \to i_* i^* A$

Take $\mathrm{Hom}(\cdot, B)$

$$\begin{tikzcd} \mathrm{Hom}(i_* i^* A, B) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Hom}(A,B) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}(j_! j^* A, B) \arrow[d, phantom, "\wr"] \\ \mathrm{Hom}(\underset{\mathcal{D}_F^{\leq 0}}{\underset{\cap}{i^* A}}, \underset{\mathcal{D}_F^{\geq 1}}{\underset{\cup}{i^! B}}) = 0 & 0 & \mathrm{Hom}(\underset{\mathcal{D}_U^{\leq 0}}{\underset{\cap}{j^* A}}, \underset{\mathcal{D}_U^{\geq 1}}{\underset{\cup}{j^* B}}) = 0 \end{tikzcd}$$

2) Need $\mathcal{D}^{\geq n} \supset \mathcal{D}^{\geq n+1}$ etc — obvious

3) Given $X \in \mathcal{D}$ need Δ $\quad \overset{\in \mathcal{D}^{\leq 0}}{A} \to X \to \overset{\in \mathcal{D}^{\geq 1}}{B} \to ..$

Use Δs in $\mathcal{D}_U, \mathcal{D}_F$: $\quad$ morphism $j^*X \to \tau_{>0} j^*X$

by adjunction $\Rightarrow$ $X \to j_* \tau_{>0} j^* X$

Complete to Δ : $\quad Y \to X \to j_* \tau_{>0} j^* X$

$i^* Y \simeq i^* X$: same over F,

$j^* Y \simeq \tau_{\leq 0} j^* X$. $\quad$ – from exactness of j^*

$(\tau_{\leq} j^* X \longrightarrow j^* X \to \tau_{>0} j^* X)$.

By adjunction have morphism $i^* Y \to \tau_{>0} i^* Y$

$\Rightarrow$ $\quad A \to Y \to i_* \tau_{>0} i^* Y$ $\quad$ (adj. again).

$$\begin{tikzcd}
 & & & i_*\tau_{>0}i^*Y \arrow[dd] \\
 & Y \arrow[urr] \arrow[dr] & & \\
A \arrow[ur] \arrow[rr] & & X \arrow[r] \arrow[dr] & \boxed{B} \arrow[d] \\
 & & & j_*\tau_{>0}j^*X
\end{tikzcd}$$

– from octahedron.

Now $j^* A \simeq j^* Y$ since $i_* \tau_{>0} i^* Y$ is concentrated on F.

$\simeq \tau_{\leq 0} j^* X \in \mathcal{D}^{\leq 0}$.

$i^* A \simeq \tau_{\leq 0} i^* Y \in \mathcal{D}_F^{\leq 0}$

$\Longrightarrow \quad A \in \mathcal{D}^{\leq 0}$

Need $B \in \mathcal{D}^{\geq 1}$:

$j^* B \simeq \tau_{>0} j^* X \in \mathcal{D}^{\geq 1}$

$i^! B \xleftarrow{\sim} i^! i_* \tau_{>0} i^* Y \simeq \tau_{>0} i^* Y \in \mathcal{D}^{\geq 1}$

■

Exercise If t-structures on $\mathcal{D}_U, \mathcal{D}_F$ are nondegenerate then same holds for glued structure

Glued $\tau_{\leq 0}$ is composition of truncations on U, F :

We can glue $(\mathcal{D}_U, 0)$ & $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$
(degenerate t-structure on $\mathcal{D}_U$) $\Rightarrow$ canonical truncation $\tau_{\leq 0}^F$: to take $\tau_{\leq 0}$ for glued structure only need to truncate over F.

$$\Rightarrow \quad \tau_{\leq 0}^F X \to X \to i_* \tau_{>0} i^* X$$

Taking $(0, \mathcal{D}_U)$ on $\mathcal{D}_U$ same on F $\Rightarrow$ $\tau_{\geq 0}^F$

$i_* \tau_{\leq 0} i^! X \to X \to \tau_{\geq 0}^F X$

similarly for $\tau_{\leq 0}^U$, $\tau_{\geq 0}^U$.

<u>Claim</u> $\tau_{\leq 0} = \tau_{\leq 0}^F \circ \tau_{\leq 0}^U$. [follows from construction]
(order does matter..)

Take $\mathcal{D}_U, \mathcal{D}_F, \mathcal{D}_X$ derived categories (bounded below) of sheaves (because we're deriving left exact functors)
glue standard t-structures on $\mathcal{D}_U, \mathcal{D}_F$ $\Rightarrow$ standard on $\mathcal{D}_X$

$$\tau_{\leq 0}^F K^\bullet \hookrightarrow K^\bullet = (K^{-1} \to K^0 \to K^1 \to \dots)$$

$$\tau_{\leq 0}^F K^\bullet = K^{-1} \to \{s \mid s|_F = 0\} \to j_* j^* K^1 \to \dots$$

" $\{s \mid s|_F = 0\}$

- because $\mathcal{D}_{\text{glued}}^{\leq 0}$, $\mathcal{D}_{\text{standard}}^{\leq 0}$ can both be defined via stalks, & agree on U, F.

<u>General setup again</u> How does heart of glued structure $\mathcal{C}$ relate to hearts $\mathcal{C}_U, \mathcal{C}_F$ of $\mathcal{D}_U, \mathcal{D}_F$?

<u>Definition</u> given $Y \in \mathcal{D}_U$ An extension of Y to $\mathcal{D}_X$ is $X \in \mathcal{D}$ s.t. $j^* X = Y$

<u>Proposition</u> 1) $\forall\, Y \in \mathcal{D}_U$ $\exists!$ extension of Y, X, (up to ! isomorphism) s.t. $i^* X \in \mathcal{D}^{\leq -1}$, $i^! X \in \mathcal{D}^{\geq 1}$.

2)
$$X \simeq \tau_{\geq 1}^F j_! Y \simeq \tau_{\leq -1}^F j_* Y$$

If $Y \in \mathcal{C}_U$ can ask for extensions of Y in $\mathcal{C}$: i.e. $i^* X \in \mathcal{D}_F^{\leq 0}$, $i^! X \in \mathcal{D}_F^{\geq 0}$: so condition in proposition is stronger

---

Proof

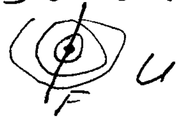
$$j_! X \to j_* X \to i_*(\text{something}) \quad \xleftarrow{\text{canonically defined}} \quad \longrightarrow$$

$(j^* j_* \simeq \mathrm{id})$ since $j_!, j_*$ agree pulled back ... so call this "something" $j_*/j_! : \mathcal{D}_U \to \mathcal{D}_F$

$$\Rightarrow \Delta : \quad \underset{\substack{\text{is} \\ j_! X}}{j_! j^*(j_* X)} \to j_* X \to i_* i^*(j_* X)$$

$$\Rightarrow \quad i^* j_* \xrightarrow{\sim} j_*/j_! \xrightarrow{\sim} i^! j_! [1]$$

Sheaf situation : $i^* j_*$ is cohomologies of sheaf extended from U along F - limit over



$$\begin{tikzcd}
j_! j^* X \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d] & i_* i^* X \arrow[d] \\
 & j_* j^* X \arrow[r] \arrow[dr] & i_* j_*/j_! (j^* X) \arrow[d] \\
 & & i_* i^! X[1]
\end{tikzcd}$$

Canonical octahedron (no morphisms where there shouldn't be)

$$\Rightarrow \text{canonical } \Delta \qquad i^! X \to i^* X \to j_*/j_! \, j^* X \to i^! X[1]$$

We want X with $i^* X \in \mathcal{D}_F^{\leq -1}$, $i^! X[1] \in \mathcal{D}_F^{\geq 0}$

$\Rightarrow$ above Δ is unique one around

$\mathcal{D}^{\leq -1} \to j_*/j_! \, j^* X \to \mathcal{D}^{\geq 0}$ as in definition.

$$\Rightarrow \quad i^* X \simeq \tau_{\leq -1} \, i^! j_!$$

$$i^! X[1] \simeq \tau_{\geq 0}\left(\frac{j_*}{j_!} Y\right) \Leftrightarrow i^* X \simeq \tau_{\leq -1}\left(\frac{j_*}{j_!} Y\right) \Leftrightarrow \begin{cases} i^* X \in \mathcal{D}_F^{\leq -1} \\ i^! X[1] \in \mathcal{D}_F^{\geq 0} \end{cases}$$

We also have

$$\begin{tikzcd}
i_* i^! X \arrow[r] & X \arrow[r] & j_* j^* X \arrow[r] \arrow[d, equal] & i_* i^! X[1] \arrow[d, phantom, "\wr"] \\
 & \underbrace{\tau^F_{\le -1} j_* Y}_{\text{by definition of } \tau^F} \arrow[r] & j_* Y \arrow[r] & i_* \tau_{\ge 0}(i^* j_* Y)
\end{tikzcd}$$

$\Rightarrow$ equivalent to above conditions get $X \simeq \tau^F_{\le -1} j_* Y$, $X \simeq \tau^F_{\ge 1} j_! Y$ $\Rightarrow$ uniqueness. ■

10/7

Definition $F : \mathcal{D}_1 \to \mathcal{D}_2$ additive functor between t-categories (triangulated w/ t-structures). F is called right t-exact iff $F(\mathcal{D}_1^{\le 0}) \subset \mathcal{D}_2^{\le 0}$. & F exact
left t-exact iff $F(\mathcal{D}_1^{\ge 0}) \subset \mathcal{D}_2^{\ge 0}$. & F exact

Example Assume F is right derived functor between $\mathcal{D}_1 = \mathcal{D}(\mathcal{A}_1)$, $\mathcal{D}_2 = \mathcal{D}(\mathcal{A}_2)$
${}^0F : \mathcal{A}_1 \to \mathcal{A}_2$ left exact, $F = R\,{}^0F$ $\Rightarrow$
F left t-exact Cat complexes concentrated $0 \to 1 \to \dots$!

Proposition F right t-exact, $\mathcal{C}_i \subset \mathcal{D}_i$ hearts $\Rightarrow$
${}^pF : \mathcal{C}_1 \to \mathcal{C}_2$ defined by ${}^pF(A) = {}^pH^0 F(A)$
("0th cohomology" wrt $\mathcal{D}_2$ ${}^pH^0 = \tau_{\le 0}\tau_{\ge 0} : \mathcal{D}_2 \to \mathcal{C}_2$)

[Same for left exact]

1. Then ${}^pF$ is right exact.
2. Also $K \in \mathcal{D}_1^{\le 0} \Rightarrow H^0 F K \xrightarrow{\sim} {}^pF\, H^0 K$.
3. $F_1 : \mathcal{D}_1 \to \mathcal{D}_2$, $F_2 : \mathcal{D}_2 \to \mathcal{D}_3$ right t-exact functors $\Rightarrow$ $F_2 \circ F_1$ right t-exact & ${}^p(F_2 \circ F_1) \simeq {}^pF_2 \circ {}^pF_1$
4. (F^*, F_*) adjoint ~~with~~: F^* right t-exact $\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "F^*"] & \mathcal{D}_2 \arrow[l, bend left, "F_*"] \end{tikzcd}$ is equivalent to F_* left t-exact.
If this is so then $({}^pF^*, {}^pF_*)$ also adjoint.

Proof 1. $0 \to A \to B \to C \to 0$ exact triple
we need ${}^pFA \to {}^pFB \to {}^pFC \to 0$ .. ?

By admissibility of $\mathcal{C}_1$ ⇒ exact △ $A \to B \to C \to A[1]$

⇒ triangle $FA \to FB \to FC \to \cdots$

~ $\boxed{\in \mathcal{D}_2^{\leq 0}}$ ⇒ ${}^pFA \to {}^pFB \to {}^pFC \to \underset{\overset{\|}{0}}{H^1FA}$

2. $K \in \mathcal{D}_1^{\leq 0}$: $\tau_{\leq -1}K \to K \to H^0K$

$\underset{\in \mathcal{D}^{\leq -1}}{F\tau_{\leq -1}K} \to FK \to FH^0K$ another exact △

⇒ on cohomology get

$H^0(FK) \simeq H^0(FH^0K) \simeq {}^pFH^0K$.

3. ${}^pF_2\,{}^pF_1A = H^0F_2H^0\underset{\in \mathcal{D}^{\leq 0}}{F_1A}$ ⇒ (using 2)

$\simeq H^0F_2F_1A = {}^p(F_2 \circ F_1)A$.

4. Assume F^* right t-exact ⇒ we need F_* left t-exact,

i.e. $F_*(\mathcal{D}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$?

sufficient to show that $Hom(\mathcal{D}^{\leq -1}, F_*(\mathcal{D}^{\geq 0})) = 0$

(since $\mathcal{D}^{\leq 0} = {}^{\perp}\mathcal{D}^{\geq 1}$, $\mathcal{D}^{\geq 1} = \mathcal{D}^{\leq 0 \perp}$).

⇔ $Hom(F^*\mathcal{D}^{\leq -1}, \mathcal{D}^{\geq 0})$ by adjointness

$= 0$ by exactness of F^*.

To show ${}^pF^*$, ${}^pF_*$ adjoint!

remark that if $A \in \mathcal{C}_1$, ${}^pF^*A = H^0\overbrace{F^*A}^{\in \mathcal{D}^{\leq 0}}$

$= \tau_{\geq 0}F^*A$. similarly if $B \in \mathcal{C}_2$,

${}^pF_*B = \tau_{\leq 0}F_*B$

$Hom({}^pF^*A, B) = Hom(\tau_{\geq 0}F^*A, B)$ (use adj. of τ, embedding)

$= Hom_{\mathcal{D}_2}(F^*A, B) = Hom(A, F_*B)$

$= Hom(A, \tau_{\leq 0}F_*B)$ ⇒ done. ■

Return to our situation $$\begin{tikzcd} \mathcal{D}_U \arrow[r, bend left] \arrow[r, bend right] & \mathcal{D} \arrow[l] \arrow[r, bend left] \arrow[r, bend right] & \mathcal{D}_F \arrow[l] \end{tikzcd}$$

Prop 1. $j_!$, i^* right t-exact, j^*, i_* t-exact,

j_*, $i^!$ left t-exact

2. $$\begin{tikzcd} \mathcal{C}_U \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \mathcal{C} \arrow[l, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathcal{C}_F \arrow[l, "i_*"'] \end{tikzcd}$$ (call with ^{p})

⇒ $({}^pj_!, {}^pj^*, {}^pj_*)$, $(i \ldots)$ adjoint pairs

[Note j^*, i_* t-exact $\Rightarrow$ don't need to truncate to get functors on heart $\mathcal{C}$].

3. Same compositions vanish as before.

4. There are canonical exact sequences

$${}^pj_!\, j^*A \longrightarrow A \longrightarrow i_*\, {}^pi^*A \longrightarrow 0 \qquad \forall A \in \mathcal{C}.$$

$$0 \to i_*\,{}^pi^!A \to A \to {}^pj_*\, j^*A$$

5. $i_* : \mathcal{C}_F \hookrightarrow \mathcal{C}$ full, faithful & identifies $\mathcal{C}_F$ with a subcategory in $\mathcal{C}$ closed under quotients, subobjects & extensions (Serre subcategory.)

<u>Proof</u> 1. j^*, i_* t-exact : $\mathcal{D}^{\leq 0} = \{X \,/\, j^*X \in \mathcal{D}_U^{\leq 0},\ i^*X \in \mathcal{D}_F^{\leq 0}\}$ etc. so exactness follows from definition for j^*.

$X = i_* Y \Rightarrow$ same inclusions hold.

Two others follow from previous proposition by adjointness.

2. Use previous prop to prove adjointness

\# [<u>Remark</u> $j_!$ t-exact wrt standard t-structures in the sheaf setting — not true for perverse t-structures . . .]

3. Again follows from ${}^p(F \circ G) \simeq {}^pF \circ {}^pG$

4. Take cohomologies for same sequence in $\mathcal{D}$

$H^1({}^pj_!\, j^*A \in \mathcal{D}^{\leq 0}) = 0$, same for other sequence

5. i_* full faithful by ${}^pi^*\, i_* \simeq id$.

$\mathcal{C}_F = \{A \,|\, j^*A = 0\}$ by 4. , $j^*i_* = 0$

j^* exact $\Rightarrow$ closed under sub, quot, ext :

$A' \subset A$ and $j^*A = 0 \Rightarrow j^*A' = 0$ . . .

$\boxed{6}$. $i_*\,{}^pi^*A$ is maximal quot. of $A \in \mathcal{C}_F$

$i_*\,{}^pi^!A$ — maximal sub of $A \in \mathcal{C}_F$.

– from sequences of 4 : suppose A has quotient $A \to i_*B \to 0$

$\Rightarrow$ (adjointness) factor

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & i_*B \\ & i_*\,{}^pi^*A \arrow[u] \end{tikzcd}$$

$\blacksquare$

$Y \in \mathcal{D}_U$, we showed $\exists!\ X \in \mathcal{D}$, $j^*X \simeq Y$ s.t. $i^*X \in \mathcal{D}_F^{\le -1}$, $i^!X \in \mathcal{D}_F^{\ge 1}$.

If $Y \in \mathcal{C}_U$ then this unique $X \in \mathcal{C}$

Prop X is the unique extension of Y s.t. $X \in \mathcal{C}$ and X has no non-zero quotients or subobjects in $\mathcal{C}_F$

Proof Let's show X has no quotients: assume $X \twoheadrightarrow i_*B$ $\Rightarrow$ $i^*X \to B$ nonzero, by adjunction — but $i^*X \in \mathcal{D}_F^{\le -1}$, has no morphisms to $B \Rightarrow$ contradiction. For subs use $i^!$ ..

Conversely assume X has no $\neq 0$ quotients or subs in $\mathcal{C}_F$: $i_*{}^p i^* X$ is quotient of X in $\mathcal{C}_F$ $\Rightarrow$ must vanish $\Rightarrow$ ${}^p i^* X = 0$ but this is $H^0 i^* X$, $i^*X \in \mathcal{D}_F^{\le 0}$ $\Longrightarrow$ get $i^*X \in \mathcal{D}_F^{\le -1}$. similarly for subs w/ $i^!$. ■

Prop $Y \in \mathcal{C}_U$, $\exists!$ extension of Y, $X \in \mathcal{C}$, $j^*X \simeq Y$ st. X has no quotients or subs in $\mathcal{C}_F$ & X is constructed as follows: There's a canonical morphism $j_! \to j_*$ $\Longrightarrow$ ${}^p j_! \to {}^p j_*$: ~~def~~ $A \in \mathcal{C}_U$, ${}^p j_! A = \tau_{\ge 0} j_! A$, ${}^p j_* A = \tau_{\le 0} j_* A$

$X =: j_{!*}Y$

$$\begin{tikzcd}
\mathcal{D}^{\le 0} \ni j_! A \arrow[rr] \arrow[dr] & & j_* A \in \mathcal{D}^{\ge 0} & \\
& \tau_{\ge 0} j_! A \arrow[ur] \arrow[rr] & & \tau_{\le 0} j_* A \arrow[ul]
\end{tikzcd}$$

. Set $X \simeq \operatorname{im}({}^p j_! Y \to {}^p j_* Y)$

Proof We know $\exists!$ extension ... assume X has no quot, subs in $\mathcal{C}_F$ $\Rightarrow$ canonical pair of morphisms ${}^p j_! Y \to X \to {}^p j_* Y$ by adjunction from $j^*X \leftrightarrow Y$

This pair of morphisms become isos under j^*: $j^*(\cdot \to \cdot \to \cdot) = (Y = Y = Y)$

$\Rightarrow$ coker of first morphism & ker of second vanish on $U \Rightarrow$ ${}^p j_! Y \twoheadrightarrow X \hookrightarrow {}^p j_* Y$ ■

Example/Claim All simple objects of $\mathcal{C}$ are

1. $i_* A$, A simple in $\mathcal{C}_F$
2. $j_{!*} B$, B simple in $\mathcal{C}_U$.