# LECTURE 11

Oct 9

$\mathcal{D}_U \xleftarrow{j^*} \mathcal{D} \xleftarrow{i_*} \mathcal{D}_F$

$j_{!*}B = \mathrm{im}\big(\underbrace{{}^p j_! B}_{[illegible]} \longrightarrow \underbrace{{}^p j_* B}_{[illegible]}\big)$ $\qquad \underline{I \in \mathcal{C}_U}$

$i^* j_{!*} B \in \mathcal{D}^{\leq -1}$

$i^! j_{!*} B \in \mathcal{D}^{\geq 1}$

<u>Exercise</u> i) simple objects of $\mathcal{C}$ are
1) $i_* A \quad A \in \mathcal{C}_F$ simple
2) $j_{!*} B \quad B \in \mathcal{C}_U$ simple

<u>Exercise</u> 1) $X = S^2 \quad F = pt, \quad U = S^2 \setminus pt$

Let $\mathcal{D}_U$ have standard t structure [illegible]

On $\mathcal{D}_F$ use $(\mathcal{D}_F^{\leq p}, \mathcal{D}_F^{\geq p})$

Compute $j_{!*}\mathbb{C}$ (depends on $p$)

2) $X = \mathbb{C} \quad F = \{0\}, \quad U = \mathbb{C}^*$

$\mathcal{D}_U$ t-str. given by $(\mathcal{D}_U^{\leq -1}, \mathcal{D}_U^{\geq 1})$ [illegible]

$\mathcal{D}_F$ t-str given by $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$

$\mathcal{D}$ = complexes $K$ st. $H^i K|_U$ = loc syst of finite rank
and $H^i K|_0$ = finite $\quad \forall i$

Find all simple objects of $\mathcal{C} \subset \mathcal{D}$ and compute $\mathrm{Ext}^1$ between them.

Assume $X$ = loc comp top space

$\mathcal{S}$ = finite strat by loc closed subsets st. if $S \in \mathcal{S} \Rightarrow \bar{S}$ = union of strata

Let $\mathcal{O}$ = sheaf of rings on $X$

$\mathcal{D}(X) = \mathcal{D}(X, \mathcal{O})$ der cat of sheaves of $\mathcal{O}$-modules

Let $p : \mathcal{S} \to \mathbb{Z}$

$i_S : S \hookrightarrow X$ inclusion of $S \in \mathcal{S}$

**Def:** $^p\mathcal{D}^{\leq 0} := \{K \in \mathcal{D}(X) \mid H^i i_S^* K = 0 \text{ for } i > p(S) \ \forall S \in \mathcal{S}\}$

$^p\mathcal{D}^{\geq 0} := \{K \in \mathcal{D}^{+?}(X) \mid H^i i_S^! K = 0 \ \forall i < p(S) \ \forall S \in \mathcal{S}\}$

for $\ast = +, -, b$ define $^p\mathcal{D}^*(X)^{\leq 0} = {}^p\mathcal{D}^{\leq 0} \cap \mathcal{D}^*(X)$

$^p\mathcal{D}^*(X)^{\geq 0} = {}^p\mathcal{D}^{\geq 0} \cap \mathcal{D}^*(X)$

have, since are right term. pres $\mathcal{D}^+_U \xleftarrow{j^*} \mathcal{D}^+ \xleftarrow{i_*} \mathcal{D}^+_F$

**Thm:** $({}^p\mathcal{D}^+(X)^{\leq 0}, {}^p\mathcal{D}^+(X)^{\geq 0})$ is $t$-structure on $\mathcal{D}^+(X)$ don't have $\to \mathcal{D}_U \leftarrow \mathcal{D} \leftarrow \mathcal{D}_F$

easy if $j_*$ has fin. hom. dim. $\Rightarrow$ glueing works

**Proof:** by induction using: $\bar{S}$ = union of strata $\forall S \in \mathcal{S}$ + glueing lemma p.51

$U \subseteq X \supset S$

$\mathcal{D}(U)$ has prev. glued $t$-str. $\qquad (\mathcal{D}(S)^{\leq p(S)}, \mathcal{D}(S)^{\geq p(S)})$

glue these $t$-str to get described $t$-str on $X$

$(S$ = max'l strata among those $\neq X)$

Note if $A \overset{i}{\hookrightarrow} B$ is loc. closed, can define $i^!$ on the level of sheaves as $A \underset{i_1}{\xrightarrow{\text{open}}} \bar{A} \underset{i_2}{\xrightarrow{\text{closed}}} B$ ; $i^! = \underset{\text{t-exact}}{i_1^!}\ \underset{\text{left t-exact}}{i_2^!}$

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Assume $a, b \in \mathbb{Z}$ and $a \le p \le b$ ie $a \le p(S) \le b$ all $S$ ?

Then
$$\left.\begin{array}{l} \mathcal{D}^{\le a} \subseteq \underline{{}^p\mathcal{D}^{\le 0} \subseteq \mathcal{D}^{\le b}} \\ \underline{\mathcal{D}^{\ge a} \supset {}^p\mathcal{D}^{\ge 0}} \supset \mathcal{D}^{\ge b} \end{array}\right\} \text{claim}$$

need only check underlined inclusions: others follow by duality. $\mathcal{D}^{\le b} = \{X \mid \mathcal{H}om(X, \mathcal{D}^{>b}) = 0\}$

If $K \in {}^p\mathcal{D}^{\ge 0}$ want: $K \in \mathcal{D}^{\ge a}$

$\Leftrightarrow$ $H^i i_S^! K = 0$ for $i < p(S)$.

$\wr\|$ (?) if $i<a$ $H^i i_S^! \tau_{<a} K$ trunc. w. resp to usual t-str.

(?):
$$\begin{tikzcd}
 & \tau_{\ge a} i_S^! K \arrow[d, "\tau_{\ge a} i_S^!(\text{can})"'] & i_S^! K \arrow[l] \arrow[d, equal] & \tau_{<a} i_S^! K \arrow[l] \arrow[d, dotted, "TR3"] \\
\mathcal{D}^{\ge a} \ni & i_S^! \tau_{\ge a} K \arrow[d] & i_S^! K \arrow[l] \arrow[d] & i_S^! \tau_{<a} K \arrow[l] \arrow[d] \\
 & \tau_{\ge a} i_S^! \tau_{<a} K[1] & 0 \arrow[l, "TR3"'] & \tau_{\ge a} i_S^! \tau_{<a} K \arrow[l, "TR3"']
\end{tikzcd}$$

l.e.s

$$\to \underset{\substack{\| \, i<a \\ 0}}{H^{i-1} \tau_{\ge a} i_S^! \tau_{<a} K} \to \underset{\substack{\| \, i<a \\ H^i i_S^! K}}{H^i \tau_{<a} i_S^! K} \overset{\cong}{\to} H^i i_S^! \tau_{<a} K \to \underset{\substack{\| \, i<a \\ 0}}{H^i \tau_{\ge a} i_S^! \tau_{<a} K} \to$$

$\cong$ ✓

$\Rightarrow$ $H^i i_S^! \tau_{<a} K = 0$ if $i<a$ so $i_S^! \tau_{<a} K \in \mathcal{D}_S^{\ge a}$ (standard) $\Rightarrow$ $\tau_{<a} K \in \mathcal{D}^{\ge a}$

$\Rightarrow \tau_{<a} K = 0 \quad \Rightarrow K \in \mathcal{D}^{\geq a}$

- quicker: given $K \in {}^p\mathcal{D}^{\geq 0}$

$\tau_{<a} K \to K \to \tau_{\geq a} K$ exact

$\Rightarrow i_S^! \tau_{<a} K \to i_S^! K \to i_S^! \tau_{\geq a} K$ exact

$\Rightarrow H^i i_S^! \tau_{<a} K \to \underset{0}{\overset{}{H^i i_S^! K}} \to H^i i_S^! \tau_{\geq a} K \to$

$\Rightarrow H^i i_S^! \tau_{<a} K \cong H^{i-1}(\underbrace{i_S^! \tau_{\geq a} K}_{\in \mathcal{D}_S^{\geq a}}) = 0$ if $i \leq a$ and $i < p(S)$

$\Rightarrow i_S^! \tau_{<a} K \in \mathcal{D}_S^{\geq a}$ in fact $\in \mathcal{D}_S^{>a}$ if $a < p(S)$

$\approx$ if $a \leq p(S)$

$\Rightarrow \tau_{<a} K \in \mathcal{D}^{\geq a}$ ← standard t-str

$\Rightarrow \tau_{<a} K = 0$, $K \in \mathcal{D}^{\geq a}$

Locally near $S$ Supp $\tau_{<a} K \subset S$ ?

$\Rightarrow i_S^! \tau_{<a} \cong i_S^* \tau_{<a} K$

$\Rightarrow H^i i_S^* \tau_{<a} K = H^i i_S^* K = 0$ for $i < a$

t-str. on S

<u>Prop</u>: $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ is a $t$-structure

<u>Proof</u>: if $K \in {}^p\mathcal{D}^{\leq 0}$, $\left.\begin{array}{l} L \in {}^p\mathcal{D}^{\geq 1} \\ L \in \mathcal{D}^+ \end{array}\right\}$ then want to prove $\mathcal{H}om(K, L) = 0$

$$\tau_{\leq *-1} K \to \underset{\substack{\cap \\ {}^p\mathcal{D}_{\leq 0}}}{K} \to \underset{\substack{\cap \\ {}^p\mathcal{D}_{\leq 0}}}{\tau_{\geq *} K} \to \underset{\substack{\cap \\ {}^p\mathcal{D}_{\leq 0}}}{\tau_{\leq *-1} K[1]}$$

$* << 0 \Rightarrow \tau_{\leq *-1} K \in \mathcal{D}^{\leq a}$ choose $a \leq p$

then $L \in {}^p\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq a+1}$ (see p. 64)

$\Rightarrow \mathcal{H}om(\tau_{\leq *-1} K, L) = 0$

$* << 0 \Rightarrow \mathcal{H}om(\tau_{\geq *} K, L) = 0$ since $L \in \mathcal{D}^+$

$\Rightarrow$ (l.e.s) $\mathcal{H}om(K, L) = 0$

$\left[ \overset{?}{\Rightarrow} \mathcal{H}om(K, L) = 0 \text{ any } L \in {}^p\mathcal{D}^{\geq 1} \left( \overset{\text{since } \forall n}{\mathcal{H}om(K, \tau_{\geq n} L) = 0} \right) \right]$

Axiom 2 is clear

Axiom 3: $\exists ?\ \underset{\cap\, {}^p\mathcal{D}^{\leq 0}}{A} \to K \to \underset{\cap\, {}^p\mathcal{D}^{\geq 1}}{B}$ exact $\triangle$-le $\forall K \in {}^p\mathcal{D}(X)$

If $U \subset X \supset F$ and t-structure of $\mathcal{D}_X$ obtained by gluing t-structures of $\mathcal{D}_U$ and $\mathcal{D}_F$

union of strata

$\mathcal{C}_X$ = perverse sheaves w. perversity $p$, stratification $\mathcal{S}$

Have $j_{!*} : \mathcal{C}_U \to \mathcal{C}_X$

If $U \overset{j}{\hookrightarrow} X$ not open, but loc. closed union of strata

Can define functors $j_!, j_*$

right t-exact   left t-exact

by $U \xrightarrow[\text{open}]{j_1} \bar{U} \xhookrightarrow[\text{closed}]{j_2} X$

$j_! = j_{1!} \circ j_{2!} = \underbrace{j_{1!}}_{\text{right exact}} \circ \underbrace{j_{2*}}_{\text{t exact}}$ = right t-exact

$j_* = \underbrace{j_{1*}}_{\text{left exact}} \circ j_{2*}$ = left t exact

get ${}^p j_! \longrightarrow {}^p j_*$ just as before (p. 61)

Can define Goreski-MacPherson extension $j_{!*} := \mathrm{im}({}^p j_! \to {}^p j_*)$

If $U \overset{j}{\hookrightarrow} X$ closed then $j_* = j_!$

Note: If $V, U$ both loc. closed in $X$ and

$$U \overset{k}{\hookrightarrow} V \overset{j}{\hookrightarrow} X$$

then $(jk)_{!*} = j_{!*} \circ k_{!*}$

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Assume $p$ has property: $S \subset \overline{T} \Rightarrow p(S) \geq p(T)$
$S, T \in \mathfrak{S}$

Take $U_n = \bigcup_{p(S) \leq n} S$ ... assume it is open $\forall n$

$$j_n : U_{n-1} \hookrightarrow U_n$$

Let $p \leq a$ ... $a \in \mathbb{Z}$ (so $U_a = X$)

$$j : U_k \hookrightarrow U_a = X$$

Prop: $A \in \mathcal{C}_{U_k} \Rightarrow$

$$j_{!*} A = \tau_{\leq a-1} j_{a*} \circ \cdots \circ \tau_{\leq k} j_{k+1*} A$$

standard truncations!

Right derived functors, like $j_*$, act naturally on $D^+$ bounded below — but not in general for unbounded categories: need functor to have finite homological dimension

Exercises from last time:

1. $p$ ◯ $0$ $\quad p\in\mathbb{Z}$ perversity at $\bullet$, consider $j_{!*}$
2. pt $\in D$ disc, what are simple objects? (for $p(pt)=0$, $p(D-pt)=-1$) what are ext's between them
3. $X$ is smooth $\underset{\cup}{\supset} U$ open stratum, take $j_{!*}\mathbb{C}_U$ wrt perversity $p(U)=0$, $\forall S\subset X\setminus U$ strata, $0<p(S)<\operatorname{codim} S$ claim is $j_{!*}\mathbb{C}_U=\mathbb{C}_X$.

<u>Theorem</u> Assume $S\subset\bar{T}\Rightarrow p(S)\geq p(T)$, $S,T$ strata. $U_n=\bigcup_{p(S)\leq n} S$,
$j_n: U_{n-1}\hookrightarrow U_n$, $\quad p\leq a$, i.e. $X=U_a$, $\quad U_k \xhookrightarrow{j} X=U_a$.

<u>Then</u> $j_{!*}A \simeq \tau_{\leq a-1}\, j_{a*}\cdots\tau_{\leq k+1}\, j_{k+2*}\,\tau_{\leq k}\, j_{k+1*}A$
($A$ perv sheaf on $U_k$, $\tau$ truncations for standard $t$-structure).

<u>Proof</u> Sufficient to prove one step in $U_k\to U_{k+1}\to\cdots\to U_a=X$; that $j_{k+1!*}A\simeq\tau_{\leq k}\, j_{k+1*}A$

$j_{k+1!*}A = {}^p\tau^F_{\leq -1}\, j_{k+1*}A$ by definition.
$F=\bigcup_{p(S)=k+1} S$, so ${}^p\tau^F_{\leq -1}=\tau^F_{\leq k}$ standard truncation.

While $\tau_{\leq k}\, j_{k+1*}A=\tau^F_{\leq k}\,\tau^{U_k}_{\leq k}\, j_{k+1*}A$ is what we need to get ...

Over $U_k$, $p\leq k \Longrightarrow {}^pD^{\leq 0}\subset {}^{stan}D^{\leq k}$

$A$ perverse $\Rightarrow A\in {}^pD_U^{\leq 0}\subset D^{\leq k}$

so $\tau^{U_k}_{\leq k}\, j_{k+1*}A = j_{k+1*}A$ — don't need any additional truncation ...

∎

Compare with our definition of the IC-sheaf:

$X\supset X_{n-2}\supset X_{n-3}\cdots$

$\cup\, U_2 \quad \cup\, S_{n-2}\ \cdots$

$p(V_2) = p_2 = 0,\ p_3 = \phi(S_{n-2}) \ldots \quad \Rightarrow \quad \underline{IC} \simeq j_{!*}\left(\begin{smallmatrix}\text{orientation}\\ \text{sheaf on } U_2\end{smallmatrix}\right)[?]$

## Review of Verdier Duality

$X \xrightarrow{f} Y$ continuous map between loc. compact fin.dim. topological spaces.

$f_! A|_U$ = sections of $A|_{f^{-1}(U)}$ whose support have compact [image (?)] – sections w/compact support $\Gamma_c$

$R f_!$ has a right adjoint $f^! : \mathcal{D}^+(Y) \to \mathcal{D}^+(X)$

– first place derived category really matters...

To construct it must take flat soft resolution on $X$, ...

$$F^\bullet \in \mathcal{D}^+(X), \quad G^\bullet \in \mathcal{D}^+(Y)$$

$$\Rightarrow \quad \mathrm{Hom}(f_! F^\bullet, G^\bullet) \simeq \mathrm{Hom}(F^\bullet, f^! G^\bullet)$$

in fact $\quad R\mathrm{Hom}(\text{''} \ \text{''}) \simeq R\mathrm{Hom}(\text{''}, \text{''})$

Dualizing complex: $p_X : X \to pt$, Consider sheaves of $k$-vector spaces $\Rightarrow$ take $\mathcal{D}_X := p_X^!(k) \in \mathcal{D}^+(X)$ dualizing complex.

In particular transitivity of $f_! \Rightarrow$

$f^!(\mathcal{D}_Y) \simeq \mathcal{D}_X$.

$X$ smooth, $\dim X = n$ then $D_X \simeq \omega_X[n]$ orientation sheaf

$\Gamma(U, \omega_X) = H_c^n(U, k)^*$

$X$ orientable $\Rightarrow$ this is trivial, or if char $k = 2$.

Adjointness $\Rightarrow$ $R\mathrm{Hom}(R\Gamma_c(F^\bullet), k) \simeq R\mathrm{Hom}(F^\bullet, D_X)$

$\quad = R\Gamma(\underline{R\mathrm{Hom}}(F^\bullet, D_X))$

Denote $D(F^\bullet) := \underline{R\mathrm{Hom}}(F, D_X)$

so $\quad R\mathrm{Hom}(R\Gamma_c(F^\bullet), k) \simeq R\Gamma(D(F^\bullet))$.

$H_c^i(F)^* \simeq H^{-i}(D(F^\bullet))$

$F$ loc con. finite rank $\Rightarrow \mathcal{D}(F) \simeq F^\vee \otimes D_X$.

$X$ smooth, $\dim = n$ & $F$ as above

$$H_c^i(F)^* \simeq H^{n-i}(F^\vee \otimes \omega_X)$$

– Poincaré duality.

General formula for $\mathbb{D}_X$: for $U$ consider complex $C^\bullet(X, X\setminus U, k)$
$\rightsquigarrow$ sheafify get local cohomology complex representing $\mathbb{D}_X$.
Smooth case - local calculation on $\mathbb{R}^n$.

Verdier duality $\mathcal{F}^\bullet \mapsto D(\mathcal{F}^\bullet)$ can be represented as follows:
$\mathcal{F}$ sheaf $\longrightarrow$ $\mathcal{F}^{*\text{naive}}(U) = \Gamma_c(U, \mathcal{F})^*$.
$\mathcal{F}$ soft $\longrightarrow$ this is a sheaf, not just presheaf.
$D$ = right derived functor of $\mathcal{F} \mapsto (\mathcal{F}^{*\text{naive}})^{\text{sheafify}}$
To compute take soft sheaves resolution $S^\bullet \longrightarrow \mathcal{F}$,
$D(\mathcal{F}) = (S^\bullet)^{*\text{naive}}$.

Natural morphism $\mathcal{F} \longrightarrow D^2(\mathcal{F})$ not in general an isomorphism
$\Rightarrow$ introduce better subcategory of sheaves where it is:
<u>Constructible sheaves</u>: $X$ stratified, strata top. varieties of pure dimension & $S \subsetneq \bar{T}$ $S \neq T \Rightarrow \dim S < \dim T$,
& $\exists$ a loc. fin. triangulation s.t. every stratum is a union of open simplices. Also $j_S : S \hookrightarrow X$
we want $j_{S*}$ has finite cohomological dimension,
i.e. only fin. many $R^i j_{S*} \neq 0$

$\Rightarrow$ $\mathcal{D}_c(X,k) = \{ K^\bullet \in \mathcal{D}(X)$ sheaves of $k$-vector spaces $/$
$\forall S\ \forall i:\ H^i K|_S$ is local system of finite rank $\}$

<u>Claim</u>: $\mathbb{D}_X \in \mathcal{D}_c(X,k)$ and $D$ preserves $\mathcal{D}_c(X,k)$, where $D^2 \simeq Id$.
[e.g. Borel's book on $IC$ ... ]
Also $f: X \to Y$ morphism then $f^! D \simeq D f^*$,
$D f_! \simeq f_* D$
To show $D^2 = Id$ check by induction on strata:
easy on single stratum .. use triangulation & embedding of simplices.

$\mathcal{D}_c$ is stable under $j_!\ j_*\ j^!\ j^*$ for locally closed inclusion of strata: Use triangulation to replace with :

$\mathcal{F}$ loc const on $S$, $j_* \mathcal{F} \in \mathcal{D}_c(X,k) \iff R^i j_* \mathcal{F}$ vanishes for all but fin many $i$ where it is loc constant on strata.

Easy that $j_!, j^*$ preserve constructibility.

<u>**Prop**</u> $j : U \hookrightarrow X$ loc closed union of strata.
Then $\mathcal{D}_c(U,k) \overset{j^!}{\underset{j_*}{\rightleftarrows}} \mathcal{D}_c(X,k)$

Proof $j_*$ : single stratum case is easy. $k : S \hookrightarrow U \supset U'$

$K \to k_* k^* K \to L$ with $L$ constructible, support on $U'$ (exact triangle), so reduced to condition on $S$, $U'$ which we know by induction

$j^!$ : write map as composition of open & closed embedding. Open $j^! = j^*$ we're done, so assume closed

$U \overset{j}{\hookrightarrow} X \overset{l}{\hookleftarrow} V$ $K \in \mathcal{D}_c(X)$ $\Rightarrow$ $j_! j^! K \to \underset{\text{constr}}{K} \to \underset{\text{constr}}{l_* l^* K}$

$\Rightarrow j_! j^! K$ constructible, which is extension by zero of $j^! K$ $\Rightarrow$ $j^! K$ constructible ■

Assume $\mathcal{S}, \mathcal{T}$ stratifications on $X$, $\mathcal{T}$ is a refinement of $\mathcal{S}$, ~~and also~~ : $p : \mathcal{S} \to \mathbb{Z}$, $q : \mathcal{T} \to \mathbb{Z}$ perversities

Prop!! (proof next time)

<u>**Prop**</u> Assume $\mathcal{S} \ni S \supset T \in \mathcal{T} \Rightarrow p(S) \le q(T) \le p(S) + \dim S - \dim T$
& stratifications are good, as before.
Then perverse t-structure on $\mathcal{D}_{c,\mathcal{T}}(X,k)$ induces perverse t-structure on $\mathcal{D}_{c,\mathcal{S}}(X,k) \subset \mathcal{D}_{c,\mathcal{T}}(X,k)$

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Example $X \underset{\text{smooth}}{\supseteq} U$ $p(U) = 0$ $\forall S \in X \setminus U$ $0 < p(S) < \operatorname{codim} S$
$j_{!*} \mathbb{C} = \mathbb{C}$

<u>simpler example</u> $\mathbb{C}, pt$ $p(\mathbb{C} \setminus pt) = 0$ $p(pt) = 1$

Simple objects are $\delta_0[-1]$, $\delta_{!*}\mathcal{L}_T$ — local system of monodromy $T \in \mathbb{C}^*$

$T \neq 1 \Rightarrow \quad j_{!*}\mathcal{L}_T = j_!\mathcal{L}_T = j_*\mathcal{L}_T$

$T = 1 \Rightarrow \quad j_{!*}\mathbb{C} = \mathbb{C}$

Why? $\quad (j_*\mathcal{L})_0$ = complex of vector spaces, can replace with cohomology $\Rightarrow$ $H^0(\dot{U}, \mathcal{L})$

$H^1(\dot{U}, \mathcal{L})$ $\quad \dot{U}$ punctured neighborhood.

${}^p\tau^F_{\leq -1}(j_*\mathcal{L})$ : write distinguished triangle

$X \to j_*\mathcal{L} \to i_* H^1(\dot{U}, \mathcal{L})$

- note $p = 1 \Rightarrow {}^p\tau^F_{\leq -1} = \tau^F_{\leq 0}$

$\Rightarrow \quad X_0 = H^0(\dot{U}, \mathcal{L}) = 0$ for $T \neq 1$, $\mathbb{C}$ $T = 1$.

In our previous example: $j_{!*}\mathbb{C}$ from $U \subset X$ smooth. ($\forall$ strata smooth of pure dimension)

$A$ perverse on $U$ $\tilde{A} = j_{!*}A$ is unique extension s.t.

$\forall S \subset X \setminus U$, $i_S : S \to X$, $i_S^*\tilde{A} \in D^{<p(S)}(S)$.

$i_S^! \in D^{>p(S)}(S)$

$\mathbb{C}_X$ is an extension of $\mathbb{C}_U$ — so check these conditions:

$i_S^*\mathbb{C} = \mathbb{C} \in D^{<p(S)}$ fine

$i_S^!\mathbb{C}$ sections with support on $S$:

it is in $D^{>p(S)}$:

$S \xhookrightarrow{i_S} X$, $\mathcal{L}$ loc sys on $X$ then $i_S^!\mathcal{L} = i_S^*\mathcal{L} \otimes or[-c]$

$or$ — rank 1 loc sys of normal orientations,

$c = \operatorname{codim}_X S$

— follows from Verdier duality:

$D\mathcal{L} = \mathcal{L}^\vee \otimes \omega_X[\dim X]$

Then $\quad D i_S^!\mathcal{L} = i_S^* D\mathcal{L}$

i.e. $\quad i_S^!\mathcal{L} = D\, i_S^* D\mathcal{L} = D i_S^*(\mathcal{L}^\vee \otimes \omega_X[\dim X])$

$= D(i_S^*\mathcal{L}^\vee \otimes i_S^*\omega_X[\dim X])$

$= D(i_S^*\mathcal{L}^\vee \otimes i_S^*\omega_X)[-\dim X]$

$= \quad i_S^* \mathcal{L} \otimes i_S^* \omega_X^{-1} \otimes \omega_S \,[\dim S - \dim X]$

– so arrive at our formula, where $Or = i_S^* \omega_X^{-1} \otimes \omega_S$.

In our case $i_S^! \mathbb{C}$ concentrated in degree $-\mathrm{codim}\, C$
$\Rightarrow$ in $\mathcal{D}^{>p(S)}$, so $\mathbb{C}_X$ is indeed $j_{!*} \mathbb{C}_U$.

– Goresky-MacPherson extension as substitute for $\mathbb{C}$ on singular varieties...

Thus if $\mathcal{L}$ loc sys on $U \underset{\subset U' \subset}{\subset} X$ $\quad U'$ smooth still,
$\mathcal{L}$ extends to $\mathcal{L}'$ on $U'$
$\Rightarrow j_{!*}\mathcal{L} = j_{!*}\mathcal{L}'$ – can pass to $U'$ and then extend.

<u>Proof of Proposition from last lecture</u> [Prop !!]

* We need nontrivial condition on stratification of local normal triviality : $\mathcal{L}$ loc sys on $S \overset{i}{\subset} X$ then $j_* \mathcal{L}$ should be constructible wrt our stratification
– then our functors respect constructibility.

[Note: here we want to work for $\mathcal{D}(X, R)$ $R$ noeth. ring – ~~here~~ on this there's no Verdier duality... but our previous example can be solved without using duality).

First show ${}^p\mathcal{D}_S^{\leq 0} \subset \mathcal{D}_{\sigma_T}^{\leq 0}$ : i.e.

$i_S^* K \in \mathcal{D}^{\leq p(S)}$ : cover $S$ with strata in $\mathcal{T}$, this condition holds iff holds for these finer strata
$i_T^* K \in \mathcal{D}^{\leq p(S)} \subset \mathcal{D}^{\leq q(S)}$ so this inclusion holds – ~~in fact ${}^p\mathcal{D}_S^{\leq 0} = {}^p\mathcal{D}_{\sigma_T}^{\leq 0}$~~ ...

Other direction: need ${}^{p}\mathcal{D}_{S}^{\geq 0} \subset {}^{p}\mathcal{D}_{\mathcal{C}}^{\geq 0}(X, R)$

- LHS is $K$ s.t. $i_S^! K \in \mathcal{D}^{\geq p(S)} \overset{?}{\Rightarrow} \forall T < S$,
$i_T^! K \in \mathcal{D}^{\geq p(S) + \mathrm{codim}}$ (codim $c = \dim S - \dim T$) :

To prove this use smooth inclusion $T \subset S$ :
cohomologies of $i_S^! K$ are local systems on $S$
- $i_T^!$ can add only $c$ (codim) to degree
(this proof only works for $\mathcal{D}^b$ but Prop is true in general. . ) ■

In fact, if $U \xhookrightarrow{j} X$ loc closed, union of strata
$\Rightarrow$ ${}^{p}j_!$, ${}^{p}j_*$, ${}^{p}j^!$, ${}^{p}j^*$ (functors on perverse sheaves) on category $\mathcal{M}_S(X)$ (perverse constructible wrt $S$) $\subset \mathcal{M}_{\mathcal{C}}(X)$

$\Rightarrow$ get
$$\begin{array}{ccc} \mathcal{M}_S(U) & \xrightarrow{{}^{p}j_!} & \mathcal{M}_S(X) \\ \cap & & \cap \\ \mathcal{M}_{\mathcal{C}}(U) & \xrightarrow{{}^{p}j_!} & \mathcal{M}_{\mathcal{C}}(X) \end{array}$$
commutes (same for other functors) :

follows from compatibility of $t$-structures due to $S, T$.

Denote $\mathrm{im}({}^{p}j_! \to {}^{p}j_*) = {}^{p}j_{!*}$, diagram will commute also for ${}^{p}j_{!*}$.

[ **Remark** In the situation of first lecture, of stratifications of $\psi$-manifolds $X \supset X_{n-2} \supset X_{n-3} \supset$ which are good from our POV, and perversity function satisfies $p(U_2) = p_2 = 0$, $p(S_{n-2}) = p_3$ etc. and $p_n \leq p_{n+1} \leq p_{n+1}$ $\Rightarrow$ conditions of proposition are satisfied automatically
$\Rightarrow$ IC independent of stratification
(can always find common refinement) ]

Consider now only sheaves of $k$-vectorspaces, & we have PL structure.

$D: D_S(X,k) \circlearrowleft$

<u>Claim</u>: for any perversity $p: S \to \mathbb{Z}$, take dual perversity $p^*(S) = -p(S) - \dim S$. Then

$D: {}^pD^{\geq 0} \longleftrightarrow {}^{p^*}D^{\leq 0}$ (denote $D^{\geq p} \longleftrightarrow D^{\leq p^*}$)

<u>Proof</u>: $D$ interchanges $i_S^!$, $i_S^*$. On each strata have local systems, we know $D: \mathcal{L} \to \mathcal{L}^\vee \otimes \mathrm{or}[\dim S]$ $\square$

$D$ interchanges ${}^{p^*}H^{i}$ and ${}^pH^{-i}$, ${}^{p^*}j_!$ & ${}^pj_*$ ($j: U \hookrightarrow X$ loc closed)

Since $p^{**} = p \Rightarrow D$ interchanges ${}^{p^*}j_*$ & ${}^pj_!$, ${}^{p^*}j_{!*}$ & ${}^pj_{!*}$.

Assume all strata are even dimensional $\Rightarrow$ middle perversity $p_{\frac{1}{2}} S = -\frac{1}{2}\dim S$, $p_{\frac{1}{2}}^* = p_{\frac{1}{2}}$

so $D: \mathcal{M}^{p_{\frac{1}{2}}}(X) \circlearrowleft$

<u>Proposition</u> $U \hookrightarrow X$ open union of strata, $A \in \mathcal{M}^{p_{\frac{1}{2}}}(U)$ autodual $(DA \simeq A) \Rightarrow j_{!*}A$ is the unique autodual extension $\tilde{A}$ to $\mathcal{M}^{p_{\frac{1}{2}}}(X)$ s.t. $\forall\ S \subset X \setminus U$ $H^i\tilde{A}|_S = 0$ $i \geq -\frac{1}{2}\dim S$

<u>Proof</u> straightforward from duality of $j_{!*}$ & our criterion

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Goresky-MacPherson on Morse theory Astérisque 101-102

Borho-MacPherson on Springer "

Let $X$ be a (separated) scheme of finite type $/\mathbb{C}$.

$S$ - algebraic stratification of $X(\mathbb{C})$ - all strata loc. closed in Zariski topology

<u>Theorem (Verdier)</u> Any such $S$ can be refined to a <u>good</u> stratification.

$\mathcal{D}_{\mathcal{S}}(X(\mathbb{C}), R)$ constructible wrt $\mathcal{S}$ (for $\mathcal{S}$ good) - sheaves in classical topology

$\mathcal{D}_c(X(\mathbb{C}), R) = \{ K^\cdot \mid H^i K$ constructible wrt some alg. stratification$\}$

$= \varinjlim_{\mathcal{S}\text{ good}} \mathcal{D}_{\mathcal{S}}(X(\mathbb{C}), R)$

$p : \mathcal{S} \to \mathbb{Z}$ perversity of form $p(S) = p(\dim_{top} S)$

($\dim_{top} S = 2\dim_{alg} S$) i.e. $p : 2\mathbb{Z}_+ \to \mathbb{Z}$

+ with $0 \le p(n) - p(m) \le m - n$ for $n \le m$

$\Rightarrow$ $t$-structure on $\mathcal{D}^b_c(X(\mathbb{C}), R)$

**Prop** Assume $K \in \mathcal{D}^b_c(X(\mathbb{C}), R)$. Then t.f.a.e.:

i). $K \in {}^p\mathcal{D}^{\le 0}$

ii). $\forall S' \subset X$ irred subvariety, $\exists S \subset S'$ Zariski open s.t. $i : S \hookrightarrow X$, $H^i i^* K = 0$ $i > p(S)$

iii). Assume $\mathcal{T}$ finite stratification s.t. all strata are smooth & $K^\cdot$ $\mathcal{T}$-constructible & equidimensional. $\forall S \in \mathcal{T}$ $i : S \hookrightarrow X$ $H^i i^* K = 0$ $i > p(S)$ & $H^i i^* K$ loc constant on $S$

[Similarly characterize ${}^p\mathcal{D}^{\ge 0}$ by $i^* \Rightarrow i^!$]

**Proof** i, ⇒ iii immediate. i ⇒ ii: $K \in {}^p\mathcal{D}^{\ge 0}$ means for sufficiently fine stratifications $K \in {}^p\mathcal{D}^{\le 0}_{\mathcal{S}}$ - write $S'$ as union of strata

Take $S \subset S'$ open stratum → ii follows from definition

iii ⇒ ... ? ? □

**Prop** $f : X \to Y$ quasifinite (finite fibers), fix $p : 2\mathbb{Z}_+ \to \mathbb{Z}$ as above

Then $f_!$, $f^*$ are right $t$-exact, $f^!$, $f_*$ left $t$-exact

Proof

$K \in {}^p\mathcal{D}^{\le 0} \iff p(2\dim \operatorname{Supp} H^i K) \geq i$

Condition ii in above prop can be written as

$i^*_{\eta_{S'}} H^i K \ne 0$ $i \le p(2\dim S')$ for $\eta_{S'}$ the generic point.

$K \in {}^p\mathcal{D}^{\le 0} \Rightarrow f_! K \in {}^p\mathcal{D}^{\le 0}$: $f_!$ is exact,

so $H^i f_! K = {}^0 f_! H^i K$,

$\dim \operatorname{supp} H^i f_! K = \dim \operatorname{supp} H^i K$ so condition ii satisfied.

Supp $H^i f^* K \subset f^{-1}$ supp $H^i K$, fibers are finite $\Rightarrow$
dim supp $H^i f^* K \leq$ dim Supp $H^i K$.
$\Rightarrow$ we have the proposition for $f_!, f^*$,
follows for other two by adjointness. ∎

<u>Corollary</u> $f$ finite morphism $\Rightarrow$ $f_* = f_!$ t-exact.
$f$ étale $\Rightarrow$ $f^! = f^*$ t-exact

<u>Étale topology</u> Recall definition of constructible sheaves:
$X$ scheme of fin <u>type / $k$</u>, consider $\mathbb{Z}/\ell^n$ sheaves,
$\varprojlim$ $\mathbb{Z}/\ell^n$ sheaves $=$ $\mathbb{Z}_\ell$ sheaves.
($\mathbb{Z}_\ell$-sheaves) $\otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ $=$ $\mathbb{Q}_\ell$-sheaves

$\varprojlim \mathcal{D}^b(X, \mathbb{Z}/\ell^n)$ won't be triangulated in general..
will be if $k$ is algebraically closed, or if $k$ is <u>finite</u> -
need $H^i(Gal(k'/k), M)$ finite for
$M$ fin $\mathbb{Z}/\ell$-module - then all Hom sets will be
finite. $\Rightarrow$ $\mathcal{D}^b(X, \mathbb{Z}_\ell)$

$K \in \mathcal{D}^b(X, \mathbb{Z}_\ell) \Leftrightarrow K_n \in \mathcal{D}^b(X, \mathbb{Z}/\ell^n)$ ;
$K_{n+1} \overset{L}{\otimes}_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \simeq K_n$
$H^i K = \varprojlim H^i K_n$, $\mathbb{Z}_\ell$-sheaf.

Problems with $\ell$-adic constructibility: want $j_* L$ constructible
for $L$ loc sys on strata (Zariski-loc-closed strata)
- but there are always bad loc sys where this fails...

<u>Def</u> $F$ finite sheaf on $X_{ét}$ is <u>constructible</u> if one of following (equivalent) holds:
1. $\forall$ irred closed subsch. $Z \subset X$ $\exists U \subset Z$ nonempty open
s.t. $F|_U$ is ét.-loc-const.
2. $\forall U \subset \cancel{X}$ quasi-compact open $\Rightarrow \exists U = \bigcup Z_i$ $Z_i$ loc closed
subschemes, s.t. $F|_{\tilde{Z}_i} \xrightarrow[\text{étale}]{\text{fin}} Z_i$ is constant.

- in our case $F$ defines space $\underset{X}{\overset{\tilde{F}}{\downarrow}}$ étale scheme, which is finite: étale constructible schemes are representable thus.

Problem - can't define stratification good for all loc systems.. construct more complicated data:

$\mathcal{S}$ finite algebraic stratification of $X$.

$\mathcal{L}$: $\forall S \in \mathcal{S} \Rightarrow \mathcal{L}(S)$ finite set of isom classes of irred loc constant $\mathbb{Z}/l$ sheaves on $S$, s.t.

a) Strata $S$ smooth of pure dimension, reduced (geometrically).

b.) $S \in \mathcal{S}$, $F \in \mathcal{L}(S)$, $j: S \hookrightarrow X \Rightarrow j_* F$ is $(\mathcal{S}, \mathcal{L})$ constructible (on strata restriction is loc system constructed as consecutive extension of sheaves in $\mathcal{L}(S)$)

(Theorem - starting from such set $\mathcal{S}, \mathcal{L}$ can always refine the data (refine $\mathcal{S}$, add stuff to $\mathcal{L}$) then - - -)

We say $(\mathcal{S}', \mathcal{L}')$ is a refinement of $(\mathcal{S}, \mathcal{L})$ if $\forall S \in \mathcal{S}, F \in \mathcal{L}(S)$ $\Rightarrow$ $S = \bigcup S'$, $S' \in \mathcal{S}'$, and $F$ is $(\mathcal{S}'|_S, \mathcal{L}'|_S)$-constructible.

Theorem - can always refine set $(\mathcal{S}, \mathcal{L})$ so that b. is satisfied. [follows from constructibility]

$$\mathcal{D}_c(X, \mathbb{Z}/l) = \bigcup \mathcal{D}_{(\mathcal{S},\mathcal{L})}(X, \mathbb{Z}/l)$$

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$\mathbb{Z}/l$ sheaf / $X$ scheme $^{\text{fin type}}/k$ finite $\Rightarrow$ $\underset{X}{\overset{F}{\downarrow}}$ étale but not separable..

e.g. $Z \subset X$ closed subscheme, $i_* (\mathbb{Z}/l)_Z$ is represented by ~~group~~ taking $l$ copies of $X$ and identifying all of them over $X \setminus Z$ ... non-separated..

$\mathbb{Z}_l$ (or $\mathbb{Q}_l$) sheaf over spec $k$ - continuous rep of $\mathrm{Gal}(\bar{k}/k)$ in $\mathbb{Z}_l$ (resp $\mathbb{Q}_l$).

Classical topology $X(\mathbb{C})$ :

Proposition Assume $U \subset X \supset F$ $U$ Zariski open, $F$ smooth of dim $d$.
$A$ perverse on $U$ $j_{!*}A = {}^p\tau^F_{\leq -1} j_*A$
Assume $H^i j_* A$ loc. constant over $F$ for $i \geq p(2d) = t$
Then $j_{!*}A = \tau^F_{\leq t+1} j_* A$

Proof
1) $\tau_{<t}\, i^* j_* A \in {}^p\mathcal{D}_c^{<0}$ since $p \geq t$ over $F$
2) $\tau_{\geq t}\, i^* j_* A \in {}^p\mathcal{D}_c^{\geq 0}$ since cohomologies in these degrees are loc. constant, we showed this holds for loc systems..
$\Rightarrow$ ${}^p\tau_{\leq -1}\, i^* j_* A = \tau_{<t}\, i^* j_* A$ so statement holds. ■

Note - in étale topology we don't have truncation functors - $\mathcal{D}^b(X)$ not defined by gluing... can still define $j_{!*} = \mathrm{im}(j_! \to j_*)$

- actually do have gluing t-structure on each $\mathcal{D}^b_{S,\mathcal{L}}$ pass to the limit .. use conditions on perversity
$Y \subset X$ smooth embedding of smooth varieties, $\mathcal{L}$ over $X$
$c$ = codimension $i^! \mathcal{L} \cong i^* \mathcal{L}[2c](c)$ (Tate twist)
- follows from Verdier duality, $\Rightarrow$ t-structures compatible
pass to limit. .. Still have problems with $\tau_\leq$ :
don't preserve condition of finite Tor-dimensions...
define t-structure using $H^i K = \varprojlim H^i K_n$ (Weil II)

Middle Perversity $p_{\frac{1}{2}}(S) = -\frac{1}{2}\dim_{top} S$
${}^{p_{\frac{1}{2}}}\mathcal{D}^{\leq 0} = \{K \in \mathcal{D}^b_c(X, \mathbb{Q}_\ell) \mid \forall x \in X$ (not nec. closed) $i_x : x \to X$ , $H^i i_x^* K = 0$ $i > -\dim x\}$
where $\dim x := \dim \overline{x}$.

${}^{\frac{1}{2}}\mathcal{D}^{\geq 0} = \{$ same but with $H^i i_x^! K = 0$ $i < \dim x\}$

Equivalently ${}^{\frac{1}{2}}\mathcal{D}^{\leq 0} = \{\dim \operatorname{supp} H^i K \leq -i\}$

Denote $Perv(X) = {}^{p}\mathcal{D}^{\leq 0} \cap {}^{p}\mathcal{D}^{\geq 0}$

eg $\mathcal{L}$ loc. sys. $/X$, $X$ smooth of pure dimension $n$

$\mathcal{L}[n] \in Perv(X)$.

$D: \mathcal{D}_c^b(X)^{op} \longrightarrow \mathcal{D}_c^b(X)$, $D^2 = Id$. (case of field coefficients)

$X$ smooth of dim $n$ $\quad D_X \simeq (\mathbb{Q}_\ell)_X[2n](n)$ twisted.

- we have a canonical rank 1 local system

$\mathbb{Q}_\ell(1) := \mathbb{Z}_{\ell,X}(1) \otimes \mathbb{Q}_\ell$

$\mathbb{Z}_\ell(1) = \varprojlim \mathbb{Z}/\ell^n(1)$, $\quad \mathbb{Z}/\ell^n(1) = \ker\left(\mathbb{G}_m \xrightarrow{(\ )^{\ell^n}} \mathbb{G}_m\right)$

$D$ preserves $Perv(X)$.

<u>Theorem</u> Assume $f: X \to Y$ affine morphism of schemes.

$f_*: \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathcal{D}_c^b(Y, \overline{\mathbb{Q}}_\ell)$ is right $t$-exact.

[ <u>Remark</u> $\overline{\mathbb{Q}}_\ell$ sheaves: take $\mathbb{Q}_\ell \overset{\text{finite ext}}{\subset} K$ $\Rightarrow$

$\mathbb{Z}_\ell \subset R$ - ring of integers

$R$-sheaves $= \varprojlim R/\ell^n$ sheaves,

$K$ sheaves $= R$-sheaves $\otimes K$, $\quad \overline{\mathbb{Q}}_\ell$-sheaves are

filtered unions of $K$ sheaves for $K$ finite extension of $\mathbb{Q}_\ell$. ]

<u>Proof</u> <u>Artin's theorem</u>: $\mathcal{F}$ constructible on $X$,

$d(\mathcal{F}) := \dim \operatorname{supp} \mathcal{F}$. Then $d(R^i f_* \mathcal{F}) \leq d(\mathcal{F}) - i$

In particular if $X/k$ $k$ alg. closed, $\dim X = n$ and

$X$ affine $\quad H^i(X, \mathcal{F}) = 0 \quad i > \dim X$.

(Milne, Etale cohomology)

$$K \in {}^{p}\mathcal{D}^{\leq 0} \iff \forall i \;\; H^i K[-i] \in {}^{p}\mathcal{D}^{\leq 0}$$

<u>Claim</u> $d(\mathcal{F}) =$ smallest $d \in \mathbb{Z}$ s.t. $\mathcal{F} \in {}^{p}\mathcal{D}^{\leq d}$

similarly if $d(K) := \sup(i + d(H^i K)) =$ smallest $d$ s.t. $K \in \mathcal{D}_c^{\leq d}$

$\Rightarrow d(Rf_* K) \leq d(K)$

this follows since it's the same as saying

$\forall i: d(R^i f_* K) \leq d(K) - i$

<u>Corollary</u> $f$ quasifinite & affine $\Rightarrow f_*$ $t$-exact.

$f$ affine $\Rightarrow$ $f_*$ right t-exact, $f_!$ left t-exact

$f$ affine, quasi-finite $\Rightarrow$ $f_*, f_!$ t-exact

Consider $j: U \hookrightarrow X \overset{i}{\hookleftarrow} D$ open affine (e.g. if $D$ is Cartier divisor).

**Prop** i. $G$ perverse $/U$ $\Rightarrow$ $j_! G$, $j_* G$ perverse

ii. $F$ perverse $/X$ $\Rightarrow$ $i^* F \in {}^pD^{[-1,0]}$, $i^! F \in {}^pD^{[0,1]}$

**Proof** i. follows from above

ii: dist. $\Delta$ $\quad \underset{\in {}^pD^{[0]}}{j_! j^* F} \to \underset{\in {}^pD^{[0]}}{F} \to \underset{\in {}^pD^{[-1,0]}}{i_* i^* F}$

dual triangle $\Rightarrow$ other statement. $\square$

$\Rightarrow$ long exact ${}^pH$-sequence

$$0 \to i_* {}^pH^{-1} i^* F \to j_! j^* F \to F \to i_* {}^pH^0 i^* F \to 0$$

$i_* i^! F \to F \to j_* j^* F \Rightarrow$

$$0 \to i_* {}^pH^0 i^! F \to F \to j_* j^* F \to i_* {}^pH^0 i^! F \to 0$$

Ex. $i^* j_{!*} G[-1]$ perverse for $G$ perv. $/U$

$'' \; {}^pH^{-1} i^* j_* G$,

$i^! j_{!*} G[1] = {}^pH^0 i^* j_* G$ also perverse.

k alg closed $X$ affine, $F$ constructible $\Rightarrow$ $H^i(X, F) = 0$ $\quad i > \dim X$

since $F[\dim X] \in {}^pD^{\le 0}$ (Artin's theorem for $f$ a projection $f: X \to pt$)

**Corollary** $H^i(X, K) = 0$ $\quad i > 0$ for $K \in {}^pD^{\le 0}$ (for $X$ affine / $k$ alg closed)

$X$ arbitrary, $V \to X$ quasi-finite $\Rightarrow$ $\forall K \in {}^pD^{\le 0}$, $H^i(V, K) = 0$ $\quad i > \dim X$

**Theorem** $k$ alg. closed, $K \in D^b_c(X)$ $\quad \forall$ Zar open affine $V \hookrightarrow X$ $\quad H^i(V, K) = 0$ $\quad i > 0$ $\Longrightarrow$ $K \in {}^pD^{\le 0}$.

**Lemma** $X$ affine, smooth, irred $\dim = d \geq 1$, $\mathcal{F}$ lc. const. $/X$, nonzero

$\Rightarrow \exists\ U = X[\frac{1}{f}]$ $f \in \Gamma(\mathcal{O}, X)$ s.t.

$H^d(U, \mathcal{F}) \neq 0$

**Pf** $f_1 \ldots f_d$ s.t. $S = (\underset{1 \leq i \leq d}{f_i = 0}) \neq \emptyset$ zero-dim

$H^{2d}_S(X, \mathcal{F}) = H^{2d}(S, i^! \mathcal{F}^\cdot) \neq 0$

But $i^! \mathcal{F} \simeq i^* \mathcal{F}[-2d](d)$

$\Rightarrow H^{2d}(S, i^! \mathcal{F}^\cdot) = H^0(S, i^* \mathcal{F})(-d) \neq 0$

$$H^{2d-1}(X \setminus S, \mathcal{F}) \to H^{2d}_S(X, \mathcal{F}) \to \underset{\substack{\| \\ 0}}{H^{2d}(X, \mathcal{F})} \text{ by Artin: } 2d > d$$

So first term is $\neq 0$ since middle is $\neq 0$.

Take $U_i = X[\frac{1}{f_i}]$ $X \setminus S = \bigcup U_i$, write Čech resolution

$$E_1^{p,q} = \bigoplus_{\substack{|I| = p+1 \\ I \subset \{1, \ldots d\}}} H^q(\underset{= (\bigcap_I U_i)}{U_I}, \mathcal{F}) \Longrightarrow H^{p+q}(X \setminus S, \mathcal{F})$$

Apply Artin to $U_I$ $\Rightarrow$ $E_1^{pq} = 0$ for $q > d$ or $p > d-1$ ($\cap$ empty)

$p+q = 2d-1$ contributing to $H^{2d-1}(X \setminus S, \mathcal{F})$ : only $E_1^{d-1,d}$ contributes so $\neq 0$ $\Rightarrow H^d(\bigcap U_i, \mathcal{F}) \neq 0$

$\bigcap U_i = X[\frac{1}{\prod f_i}]$ ∎

**Proof of Theorem** $K$ is constructible $\Rightarrow$ find a filtration

$F_{-1} = \emptyset \subset F_0 \subset \ldots \subset F_n = X$ $n = \dim X$ $F_i$ closed.

$W_i = F_i \setminus F_{i-1}$ smooth of pure dimension $i$,

$H^i K|_{W_i}$ locally constant.

Prove by descending induction in $i$ that $K|_{U_i = X - F_{i-1}} \in D^{\leq -i}$

Standard structure.

$\Rightarrow \operatorname{supp} H^0 K \subset F_{-j}$

$\Rightarrow \dim \operatorname{supp} H^0 K \leq j$

$\Leftrightarrow K \in {}^p D^{\leq 0}$

Assume we know $K|_{U_j} \in \mathcal{D}^{\leq -j}$ $j > i$. Consider everything $/U_i$ -

1. $\tau_{\leq -i} K \in {}^p\mathcal{D}^{\leq 0}$ : $U_{i+1} \hookrightarrow U_i \hookleftarrow W_i$

we know statement on $U_{i+1}$, but on $W_i \Rightarrow H^j$ loc constants

- ~~#~~ truncate $\Rightarrow$ loc sys in deg $-i$ & less $\Rightarrow$ we're set.

2. $j \geq -i$. $H^j K$ loc. constant, supported on $W_i$ : since $K|_{U_{i+1}} \in \mathcal{D}^{\leq -i-1}$

$$\underset{\in {}^p\mathcal{D}^{\leq 0}}{\tau_{\leq -i} K} \longrightarrow K \longrightarrow \tau_{> -i} K$$

$V \subset X$ Zariski open & affine. $H^j(V, \tau_{\leq -i} K) \to H^j(V, K) \to H^j(V, \tau_{> -i} K) \to$.

$\overset{\shortparallel}{0}$ $j>0$ (Artin) $\quad \overset{\shortparallel}{0}$ $j>0$ by assumption

$\Rightarrow H^{>0}(V, \tau_{> -i} K) = 0$

$\shortparallel$

$H^{>0}(V \cap W_i, \tau_{> -i} K)$

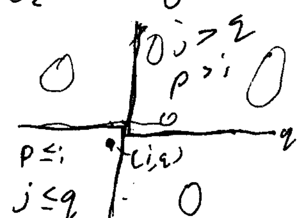
$$E_2^{pq} = H^p(\underset{\text{affine of dim } i}{V \cap W_i}, H^q \tau_{> -i} K) \Rightarrow H^{p+q}(V \cap W_i, \tau_{> -i} K)$$

$E_2^{pq} = 0$ for $p > i$ (Artin)

Assume $H^j_{\tau > -i} K = 0$ for $j > q$, $q > -i$

$\Rightarrow$ want to prove $H^q_{\tau > -i} K$

$d_2^{(i,q)} = 0$




$\Rightarrow H^{i+q}(V \cap W_i, \tau_{> -i} K) = H^i(V \cap W_i, H^q_{\tau > -i} K)$

$\overset{\shortparallel}{0}$

denote $\mathcal{L} = H^q \tau_{> -i} K$ loc sys on $W_i$

$\forall V \subset X$ affine open $H^i(V \cap W_i, \mathcal{L}) = 0$ :

so pass from $W_i$ to $V_1 \subset W_i$ affine open in

$\Longrightarrow$ by lemma $\mathcal{L} = 0$ (apply to irred components of $V \cap V_1$).

- actually need $V \subset X$ affine open, $\Rightarrow \exists f_1 \in \Gamma(V_1 \cap W_i, \mathcal{O})$

s.t. $H^i(V_1 \cap W_i[\frac{1}{f_1}], \mathcal{L}) \neq 0$

- $V_1$ affine, $W_i$ closed $\Rightarrow$ lift $f_1$ to a function $f$ on $V_1$,

take $V = V_1[\frac{1}{f}]$ $\Rightarrow V_1 \cap W_i[\frac{1}{f_1}] = V \cap W_i$ . . . .

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$T: \mathcal{D}_1 \to \mathcal{D}_2$ t categories.

$T$ has cohomological amplitude $[a,b]$ $(a \le b)$

$\iff$ $T\mathcal{D}_1^{\le 0} \subset \mathcal{D}_2^{\le b}$, $T\mathcal{D}_1^{\ge 0} \subset \mathcal{D}_2^{\ge a}$

e.g. $T$ has coh. amplitude $\le b$ if has $[-\infty, b]$ ..

Prop $f: X \to Y$ morphism of schemes, dim of fibers $\le d$.

Then coh. amps are: $f^* f_! : \le d$ $f^! f_* : \ge -d$

Proof $f^*, f_*$ adjoint - so statement for $f^* \Rightarrow$ for $f_*$:

$f^*[d]$ right t-exact $\Rightarrow$ $f_*[-d]$ left t-exact

Using Verdier duality $Df^*D = f^!$ $\Rightarrow$ $f^*$ statement gives $f^!$ statement, & finally adjoint back to get $f_!$.

So check just for $f^*$: $F \in {}^p\mathcal{D}^{\le 0}(Y) \overset{?}{\Longrightarrow} f^*F \in {}^p\mathcal{D}^{\le d}(X)$

$x \in X$, $H^i i_x^*(f^*F) = 0$ $i > \dim(x) + d$

$y = f(x)$. We know that $H^i i_y^* F = 0$ $i > \dim(y)$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & X \arrow[d] \\ y \arrow[r] & Y \end{tikzcd}$$

Know $H^i i_x^*(f^*F) = H^i f^* i_y^* F$

but $\dim x \le \dim y + d$ so we're done. ■

Cor. 1. If $f$ is proper, $f_! = f_*$ has coh amplitude $[-d, d]$

2. If $f$ is smooth, rel dim $d$, $f^! = f^*[2d](d)$

$\Rightarrow$ $f^*[d](d) = f^![-d]$ is exact (wrt p-structure)

(f still smooth) & 3 ( ${}^pH^d f_!(d)$, ${}^pH^d f^*$, ${}^pH^{-d} f_*$ ) nearby pairs are adjoint

proof in 3: use adjointness of $(f^*[d], f_*[-d])$ for right, & ${}^pH^d f^* = {}^pH^{-d} f^![-d]$ .. ■

Proposition $f$ smooth of rel. dim $d$, with geometrically connected ($\neq \emptyset$) fibers. Then $f^*[d]: \mathrm{Perv}\, Y \to \mathrm{Perv}\, X$ is a full faithful embedding.

Lemma $K, L$ perverse $\Rightarrow$ $\underline{R\mathrm{Hom}}(K, L) \in \mathcal{D}^{\ge 0}$ (standard)

Pf We know there's no $\mathrm{Hom}(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 1})$.

So $\mathrm{Hom}^{<0}(X,Y)=0$ for $X,Y\in D^{[0]}$ — our statement is sheafification of this ... more generally we'll show:

* $p,q$ perversities $K\in D^{\leq p}$, $L\in D^{\geq q}$
$\Rightarrow \underline{R\mathrm{Hom}}(K,L)\in D^{\geq q-p}$.

– The statement is Verdier dual to:
$A\in D^{\leq p}$, $B\in D^{\leq q}$ $\Rightarrow$ $A \overset{L}{\otimes} B\in D^{\leq p+q}$
which is straightforward for constructible sheaves: $\otimes$ commutes with restriction to strata.

Now $D\,\underline{R\mathrm{Hom}}(K,L)\simeq K\overset{L}{\otimes}DL$

since $D(K\otimes DL)=\underline{R\mathrm{Hom}}(K\otimes DL, D_X)$
$=\underline{R\mathrm{Hom}}(K,\underline{R\mathrm{Hom}}(L,D_X))$ ~~adjoint to~~.

For any perversity we have dual perversity $q^*=q_0-q$, $q_0(S)=-\dim_{top}S$.

$L\in D^{\geq q} \Leftrightarrow DL\in D^{\leq q^*}$ $\quad$ $\mathcal{D}^{p}_{S}$ $K\otimes DL\in D^{\leq q_0-q}$

$D\,R\mathrm{Hom}(K,L)\in D^{\leq q_0-q+p}$ since

In our case $q-p=0=$ standard perversity / t-structure ∎

<u>Proof of Proposition</u> $\mathrm{Hom}(K,L)\to \mathrm{Hom}(f^*K,f^*L)$
we want to show it's an isomorphism for $K,L\in\mathrm{Perv}(Y)$.
Locally, since $f$ is smooth,
$f^*\underline{R\mathrm{Hom}}(K,L)\simeq \underline{R\mathrm{Hom}}(f^*K,f^*L)$

Apply perverse ${}^0f_*H^0$ $\Rightarrow$ $\quad$ ${}^0f_*$ standard pushforward of sheaves, $H^0$ – $H^0$ of complexes of sheaves

$${}^0f_*\,f^*H^0\underline{R\mathrm{Hom}}(K,L)\simeq {}^0f_*H^0\underline{R\mathrm{Hom}}(f^*K,f^*L)$$

[ <u>Claim</u> For any $\mathcal{H}$ constructible sheaf on $Y$,
${}^0f_*f^*\mathcal{H}\simeq\mathcal{H}$

Apply global sections $\Gamma$:
$F=\underline{R\mathrm{Hom}}(K,L)\in D^{\geq 0}$.

$\mathcal{F} \in \mathcal{D}^{\geq 0}$, $R\Gamma$ left exact $\Rightarrow H^0 R\Gamma \mathcal{F} \simeq \Gamma H^0 \mathcal{F}$:

since $\underset{0}{\overset{\shortparallel}{H^{-1}()}} \to \Gamma H^0 \mathcal{F} \xrightarrow{\sim} H^0 R\Gamma \mathcal{F} \to \underbrace{H^0(R\Gamma \tau_{>0} \mathcal{F})}_{\overset{\shortparallel}{0}}$ $\in \mathcal{D}^{>0}$

So we get

$$\mathrm{Hom}(K, L) \simeq \Gamma(X, H^0 \underline{R\mathcal{H}om}\, f^* K[d], f^* L[d])$$
$$= \mathrm{Hom}(f^* K[d], f^* L[d])$$

(RHoms don't change when we shift.) $\Rightarrow$ and we're done! ■

k-perfect **Theorem** i. $\mathrm{Perv}(X)$ is artinian and noetherian.

ii. $j : V \hookrightarrow X$ irred subvariety, $\mathcal{L}$ irred. loc const/$V$
$\Rightarrow j_{!*}(\mathcal{L}[d])$ is a simple object of $\mathrm{Perv}(X)$
+ these are all simple objects in $\mathrm{Perv}(X)$.

**Lemma** $X$ smooth, irred of dim $d$, $\mathcal{L}$ loc system $/X$,
$j : U \hookrightarrow X$ nonempty open. Then $\mathcal{L}[d] \cong j_{!*}(j^* \mathcal{L}[d])$

*Proof* $j_{!*}\mathcal{F}$ is characterized as unique extension of $\mathcal{F}$
$\hat{\mathcal{F}}$ s.t. $\forall x \notin U$, $H^i i_x^* \hat{\mathcal{F}} = 0$ $\quad i \geq -\dim x$
$\qquad H^i i_x^! \hat{\mathcal{F}} = 0 \quad i \leq \dim x$

- use that $i^!$ on local system for smooth embedding just shifts by codimension... ■

**Lemma** $X, \mathcal{L}$ as above & $\mathcal{L}$ irred $\Rightarrow \mathcal{L}[d]$ is simple.

*Proof* Assume we have a perverse subobject
$\mathcal{F} \subset \mathcal{L}[d] \Rightarrow \exists U \subset X$ nonempty s.t.
$\mathcal{F}|_U = \mathcal{M}[d]$, $\mathcal{M}$ local system on $U$.
$\mathcal{M} \hookrightarrow \mathcal{L}|_U$, $\pi_1(U) \twoheadrightarrow \pi_1(X)$ ($X$ smooth)
so $\mathcal{L}|_U$ is irred $\Rightarrow$ either $\mathcal{M} = 0$ or $\mathcal{M} = \mathcal{L}|_U$.
But $\mathcal{L}[d] = j_{!*}\mathcal{L}|_U[d]$ has no subquotients supported off $U$ $\Rightarrow$ can't have $\mathcal{M}$ supported off $U$.
and in second case get $\mathcal{F}$ = everything since $\mathcal{L}[d]/\mathcal{F}$ would be a quotient supported off $U$. ■

Proof of Theorem  $j_{!*}\mathcal{L}$ ($\leftarrow$ irred) is simple - couldn't have subquotients off $U$. $F \in Perv(X) \Rightarrow$
$\exists\, \emptyset \neq U \subset X$ / $F|_U = M[d]$.
Have exact triple $0 \to i_* G \to {}^p j_! M[d] \to j_{!*} M[d] \to 0$
$G$ supported on complement
Induce on codimension of supp $F$

$$\begin{tikzcd}
0 \arrow[d] & \\
i_* G \arrow[d] \arrow[dr, "\text{on } U"] & \\
{}^p j_! M[d] \arrow[r] \arrow[d] & F \arrow[d] \\
j_{!*} M[d] \arrow[r] \arrow[d] & F' \arrow[d] \\
0 & 0
\end{tikzcd}$$
isom/$U$

We know how to represent $j_{!*}M$ as succ. extension, and we know statement outside $U \Rightarrow$ we get Artinian structure in general. ■

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<u>Localization</u>
$F$ sheaf $\longleftrightarrow$ $F|_{U_i} = F_i$, $F_i|_{U_i \cap U_j} \simeq F_j|_{U_i \cap U_j}$
compatible.
$f: F \to G \longleftrightarrow f_i = f|_{U_i}$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$

<u>Not</u> true for complexes: $f: K^\cdot \to L^\cdot$ morphism in $D(Sh)$
$(f|_{U_i} = 0 \ \forall i) \not\Rightarrow f = 0$ !

<u>Claim</u> For <u>perverse</u> sheaves, gluing from $\{U_i\}$ is fine.

<u>General nonsense</u> Let $F_\alpha = F|_{U_\alpha} = j_\alpha^* F$,
$j_\alpha^*: Sh\, X \to Sh\, U_i$.
$j_{\alpha *}: Sh\, U_i \to Sh\, X$ right adjoint

$\alpha, \beta \Rightarrow j_\beta^* j_{\alpha *}: Sh\, U_\alpha \to Sh\, U_\beta$
$\quad\quad\quad \shortparallel$
$\quad\quad\quad \psi_{\alpha\beta}$

$$\psi_{\alpha\gamma} \xrightarrow{\mathrm{can}_{\alpha\beta\gamma}} \psi_{\beta\gamma} \circ \psi_{\alpha\beta}, \quad \psi_{\alpha\alpha} = \mathrm{Id}, + \text{coassociativity}$$

morphism of functors

Consider collection $F_\alpha \in Sh\, U_\alpha$, morphisms $\varphi_{\alpha\beta} : F_\beta \to \psi_{\alpha\beta} F_\alpha$

s.t.

$$\begin{tikzcd} F_\gamma \arrow[r, "\varphi_{\alpha\gamma}"] \arrow[d, "\varphi_{\beta\gamma}"] & \psi_{\alpha\beta} F_\alpha \arrow[d, "\mathrm{can}_{\alpha\beta\gamma}"] \\ \psi_{\beta\gamma} F_\beta \arrow[r, "\psi_{\beta\gamma}(\varphi_{\alpha\beta})"'] & \psi_{\beta\gamma}\psi_{\alpha\beta} F_\alpha \end{tikzcd}$$

These data form an abelian category, equivalent to $Sh\, X$

$$\begin{tikzcd} X & U_\alpha \arrow[l, hook', "j_\alpha"'] \\ U_\beta \arrow[u, hook, "j_\beta"] & U_{\alpha\beta} \arrow[l, hook', "j'_\alpha"] \arrow[u, hook, "j'_\beta"'] \end{tikzcd}$$

$j_\beta^* j_{\alpha *} F_\alpha \simeq j'_{\alpha *} j_\beta'^* F_\alpha$

$\to$ same as morphism $j_\alpha'^* F_\beta \to j_\beta'^* F_\alpha$.

<u>Gluing data</u> $\{\mathcal{A}_\alpha\}$ collection of abelian categories, $\psi_{\alpha\beta} : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ left exact, $\psi_{\alpha\alpha} = \mathrm{Id}$ & have morphism $\mathrm{can}_{\alpha\beta\gamma} : \psi_{\alpha\gamma} \to \psi_{\beta\gamma}\psi_{\alpha\beta}$ with coassociativity

$\Rightarrow$ $\mathrm{Glue}(\{\mathcal{A}_\alpha\}, \psi_{\alpha\beta})$ – category of collections $A_\alpha \in \mathcal{A}_\alpha$ + $\varphi_{\alpha\beta} : A_\beta \to \psi_{\alpha\beta} A_\alpha$ satisfying compatibility as before, + $\varphi_{\alpha\alpha} = \mathrm{Id}$

– THEN this category is <u>abelian</u> (uses left exact).

<u>Localization data</u> $\{\mathcal{A}_\alpha\}$ collection of abelian categories, $\mathcal{A}$ ab. cat, $j_\alpha^* : \mathcal{A} \to \mathcal{A}_\alpha$ exact with $j_{\alpha *} : \mathcal{A}_\alpha \to \mathcal{A}$ its right adjoint (left exact) & $j_\alpha^* j_{\alpha *} \xrightarrow{\sim} \mathrm{Id}$ natural adjunction morphism

$\Rightarrow$ associated gluing data $\{\mathcal{A}_\alpha\}$, $\psi_{\alpha\beta} = j_\beta^* j_{\alpha *}$ left exact,

Canonical functor $Loc : \mathcal{A} \to \mathrm{Glue}(\mathcal{A}_\alpha, \psi_{\alpha\beta})$

$$A \mapsto (A_\alpha = j_\alpha^* A,\ A_\beta \to \psi_{\alpha\beta} A_\alpha)$$

$$\underset{j_\beta^* A}{\overset{A_\beta}{\|}} \to \underbrace{\underset{j_\beta^* j_{\alpha *} j_\alpha^* A}{\overset{\psi_{\alpha\beta} A_\alpha}{\|}}}_{\text{adjunction morphism}}$$

Loc is exact.

Is Loc an equivalence?

**Theorem** Assume Loc is faithful (equivalently, $A=0 \Leftrightarrow j_\alpha^* A = 0\ \forall \alpha$)
Then Loc is an equivalence.

**Example** Perverse sheaves, with $j^*$, ${}^p j_*$ $\Rightarrow$ localization data. Suppose $K, L$ perverse. $f: K \to L$
$f|_{U_\alpha} = 0 \Rightarrow f = 0$ : in fact only need not perversity but $\mathcal{R}\mathcal{H}om\,(K,L) \in D^{\geq 0}$ :
$$\mathrm{Hom}\,(K,L) = H^0 R\mathrm{Hom}\,(K,L) = H^0 R\Gamma\, \underline{R\mathcal{H}om}\,(K,L)$$
$\Rightarrow$ get spectral sequence
$$H^p(\underline{\mathcal{H}}^q R\mathcal{H}om\,(K,L)) \Rightarrow \mathrm{Hom}^{p+q}(K,L)$$
$$\searrow 0 \quad q<0 \text{ or } p<0$$
$$\Rightarrow \mathrm{Hom}\,(K,L) \simeq H^0(\underline{\mathcal{H}}^0 R\mathcal{H}om\,(K,L))$$
- sections over $X$ of a <u>sheaf</u> ! uniquely determined by its restrictions! [redacted] (perverse sheaves glue)

<u>Proof of Theorem</u> Have to check 1.) $X \in \mathrm{Glue}$, $\exists A \in \mathcal{A}$ and embedding $X \hookrightarrow \mathrm{Loc}\, A$.
2). Loc is full, i.e. $\mathrm{Hom}_{\mathcal{A}}(A, A') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Glue}}(\mathrm{Loc}\, A, \mathrm{Loc}\, A')$

Then we take
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \mathrm{Loc}\, A \arrow[d] \arrow[dr, "\mathrm{Loc}\, f)"] & \\ \mathrm{Loc}\,(\ker f) & X' \arrow[r, hook] & \mathrm{Loc}\, A' \end{tikzcd}$$
& we're done.

1). $X = (X_\alpha \in \mathcal{A}_\alpha,\ \varphi_{\alpha\beta} : X_\beta : \psi_{\alpha\beta} X_\alpha)$.
Take $A = \oplus_\alpha j_{\alpha *} X_\alpha \Rightarrow X \hookrightarrow \mathrm{Loc}\, A$
via $X_\alpha \longrightarrow \oplus_\beta \underbrace{j_\alpha^* j_{\beta *}}_{= \psi_{\beta\alpha}} X_\beta$.
$\qquad \downarrow \varphi_{\beta\alpha}$

2). $f : \mathrm{Loc}\, A \to \mathrm{Loc}\, A'$
$$\rightsquigarrow \tilde{f} : \oplus j_{\alpha *} j_\alpha^* A \longrightarrow \oplus j_{\alpha *} j_\alpha^* A'$$
$$\begin{tikzcd} \oplus j_{\alpha *} j_\alpha^* A \arrow[r] & \oplus j_{\alpha *} j_\alpha^* A' \\ A \arrow[u, hook] \arrow[r, dashed, "?"] & A' \arrow[u, hook, "\tau"'] \end{tikzcd}$$

suffices to find map $\overset{?}{\dashrightarrow}$, i.e. check $\tilde{f}(A) \overset{?}{\subset} \mathrm{im}\, \tilde{C}$

But $\mathcal{L}oc$ is faithful $\Rightarrow$ suffices to check this on every open: $S_1 \subset A \supset S_2$ then $j_\alpha^* S_1 \subset j_\alpha^* S_2$ $\forall \alpha$ $\Rightarrow S_1 \subset S_2$

– take $Z = S_1 / S_1 \cap S_2$, $j_\alpha^* Z = 0$ $\forall \alpha \Rightarrow Z = 0$. ∎

## Back to topology — cell complexes

Suppose $X$ CW complex (finite), $c \subset X$ cell with

open ball $\simeq c \subset \bar{c} \simeq$ closed $d$-dimensional ball (e.g. not allowed)

For example a triangulation satisfies this.

Set perversity $p(c) = p(\dim c)$,

$p(0) = 0$ $\quad 0 \le p(m) - p(n) \le m - n$ for $m \ge n$.

Simple objects of Perv $X$ are $i_{c!*} \mathbb{Q}_c[-p(c)]$

Exercise Calculate $i_{c!*} \mathbb{Q}_c[-p(c)]$

Hint: Consider two cases: ($\dim c = d$): 1. $p(d-1) = p(d)$

2. $p(d-1) = p(d) + 1$.

2nd case $\Rightarrow$ $i_{c*} \mathbb{Q}_c[-p(c)]$, first gives $i_{c!} \mathbb{Q}_c$.
– follows from $j_* \mathbb{Q}_B = \mathbb{Q}_{\bar{B}}$.

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$X$ cell stratified space, $\bar{c} \simeq$ closed ball (cells)

$p : \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ perversity $\quad p(0) = 0$, $p(i) = \begin{cases} p(i-1) \\ p(i-1) - 1 \end{cases}$

$\mathcal{C}$ perverse sheaves on $X$ wrt cell stratification.

Cell of $\dim c = d$ is of $\begin{cases} * \text{ type} & p(d) = p(d-1) - 1 \\ ! \text{ type} & p(d) = p(d-1) \end{cases}$

$P_c$ simple object associated with $c$

$c$ of type $!$ $\Rightarrow P_c \simeq i_{c!} \mathbb{Q}_c[-p(c)]$

" " " $*$ $\Rightarrow P_c \simeq i_{c*} \mathbb{Q}_c[-p(c)] \simeq i_{\bar{c}*} \mathbb{Q}_{\bar{c}}[-p(c)]$

(from last time).

$*$ case, need to check for $c' \neq c$,

$i_{c'}^! P = 0$ since if $i : \partial B \hookrightarrow \bar{B}$, $i^! j_* = 0$

$i_{c'}^* P = \begin{cases} 0 & c' \not\subset \bar{c} \\ \mathbb{Q}_{c'}[-p(c)] & c' \subset \bar{c} \end{cases}$

$!$ - case : $i_{c!}\, \mathcal{P}_c[-p(d)]$

$i^*_{c'}\, P_c = 0 \quad c' \neq c$

$i^!_{c'}\, i_{c!}$ : apply duality : interchanges $p$ & $p^*$

interchanges the two types of cells, & $! \longleftrightarrow *$ so reduces to $*$-case.

Perverse dimension $\quad c \rightsquigarrow \delta(c) = \delta(\underset{d}{\dim\, c}) \in \mathbb{Z}$

1. $\delta(0) = 0$
2. $\delta(d) > 0 \iff d$ of type $*$ $\quad (p(d) = p(d-1) - 1)$

   $\delta(d) < 0 \iff d$ of type $!$
3. $|\delta(d)| = \#\{ i \mid 0 < i \leq d,\ i$ of same type as $d\}$

$\not{3}$ (equivalently) $j \geq 0$ $\delta : [0, j] \to$ interval $\subset \mathbb{Z}$ containing $0$.

There is a bijection between perversities $\{p\}$ and such functions $\{\delta\}$

**Proposition** $\quad \underset{D^b(S)}{\mathrm{Ext}^i}(P_c, P_{c'}) \quad (= \mathrm{Hom}(P_c, P_{c'}[i]))$

$= 0$ unless $\quad i = \delta(c) - \delta(c')$

- (Under assumption $\quad \bar{c} \cap \bar{c}' \simeq$ closed ball).

$\mathrm{Ext}^*(P_c, P_{c'}) \neq 0 \Rightarrow$ one of following hold

1. $c, c'$ type $*$, $\quad c' \subset \bar{c}$

1'. $c, c'$ type $!$, $\quad c \subset \bar{c}'$

2. $c$ of type $*$, $c'$ type $!$ & $\bar{c} \cap \bar{c}' \neq \emptyset$

In all these cases $\dim \mathrm{Ext}^*(P_c, P_{c'}) = \begin{cases} 0 \\ 1 \end{cases}$

**Proof**. Claim : $c\ !,\ c'\ * \Rightarrow \mathrm{Ext} = 0$ :

$\mathrm{Ext}^*(i_{c!}\,\mathcal{P}_c,\ i_{c'*}\,\mathcal{P}_{c'}) = \mathrm{Ext}^*(\mathcal{P}_c, \underbrace{i_c^!\, i_{c'*}}_{0}\, \mathcal{P}_{c'})$.

$c, c'$ of type $*$ :

$\mathrm{Ext}^*(i_{\bar{c}*}\,\mathcal{P}_c,\ i_{c'*}\,\mathcal{P}_{c'}) = \mathrm{Ext}^*(\mathcal{P}_c, \underbrace{i_{\bar{c}}^!\, i_{c'*}}\, \mathcal{P}_{c'})$

$0$ if $\bar{c} \not\subset \bar{c}$

$1$ if $c' \subset \bar{c}$

$$c' \subset \bar{c} \Rightarrow \mathrm{Ext}^i(P_c, P_{c'}) = \begin{cases} 0 & i \neq p(c')-p(c) \\ 1. & i = p(c')-p(c) \end{cases} = \delta(c')-\delta(c)$$

- $d' \leq d \Rightarrow \delta(c')-\delta(c) = \#$ of $i$ of type $*$, $d' < i \leq d$
- each such is responsible for increase in perversity, so $p(c')-p(c) = \delta(c')-\delta(c)$.

<u>Both!</u>: obtained by Verdier duality: $\delta^* = -\delta$ (<u>exercise</u>)

$$\mathrm{Ext}(P_c, P_{c'}) \Longrightarrow \mathrm{Ext}(\underset{\substack{\shortparallel \\ P^{p*}_{c'}}}{DP_{c'}}, \underset{P^{p*}_c}{DP_c})$$

<u>$c\ *$, $c'\ !$</u>: $\mathrm{Ext}^i(i_{\bar{c},*}\mathbb{Q}_{\bar{c}}, i_{c'!}\mathbb{Q}_{c'}) =$

$$= H^i(\bar{c}, \underbrace{i^!_{\bar{c}}\, i_{c'!}\mathbb{Q}_{c'}}) \qquad \text{Apply duality}$$

$$i^!_{\bar{c}}\, i_{c'!}\mathbb{Q}_{c'} = D\, \underbrace{i^*_{\bar{c}}\, \underbrace{i_{c'*}\mathbb{Q}_{c'}}_{\substack{\shortparallel \\ i_{\bar{c}'*}\mathbb{Q}_{\bar{c}'}}}}_{\text{constant sheaf on } \bar{c}\cap\bar{c}'} [d']$$

But $\bar{c}\cap\bar{c}' \simeq$ ball. $\Rightarrow \mathrm{Ext}^i(\cdot,\cdot) = H^i(\bar{c}, \bar{c}' D(\mathbb{Q}_{\bar{c}\cap\bar{c}'}[d']))$

$(\bar{c}\cap\bar{c}'$ compact$)$ $\simeq H^{-i}(\bar{c}\cap\bar{c}', \mathbb{Q}_{\bar{c}\cap\bar{c}'}[d'])^*$

$$= H^{d'-i}(\mathbb{Q}_{\bar{c}\cap\bar{c}'})^* = \begin{cases} 0 & i \neq d' \\ 1 & i = d \end{cases}$$

$$\Rightarrow \mathrm{Ext}^i(P_c, P_{c'}) = \begin{cases} 0 & i \neq d' + p(d') - p(d) \overset{?}{=} \delta(d)-\delta(d') \\ 1 & i = \quad '' \end{cases}$$

Now $\delta(d) > 0$, $\delta(d') < 0$

$\#\{i \mid 0 < i \leq d',\ i \text{ of type } *\} = d' - |\delta(d')|$

$\delta(d) = (d' + \delta(d')) + p(d') - p(d)$, so we're set ∎

Let's write $c' \leq c \Longleftrightarrow \mathrm{Ext}^*(P_c, P_{c'}) \neq 0$

$\Longrightarrow$ partial order on cells:

$c, c'$ — $(*,*)$ then $c' \leq c \Leftrightarrow c' \subset \bar{c}$

$c, c'$ — $(!,!)$ then $c' \leq c \Leftrightarrow \bar{c}' \supset c$

$c, c'$ — $(*,!)$ then $c' \leq c \Leftrightarrow \bar{c}\cap\bar{c}' \neq \emptyset$.

$$c_1 \leq c_2,\ c_2 \leq c_3 \overset{?}{\Longrightarrow} c_1 \leq c_3$$

Interesting cases: $(!,*,*) \Rightarrow \bar{C}_1 \cap \bar{C}_2 \neq \emptyset \quad C_2 < C_3 \Rightarrow \bar{C}_1 \cap \bar{C}_3 \neq \emptyset$

$(!,!,*)$ same holds . . so indeed set partial order.

Proposition $C_1 \leq C_2 \leq C_3 \Rightarrow Ext^*(C_3, C_2) \times Ext^*(C_2, C_1) \to Ext^*(C_3, C_1)$ is nonzero.

Proof Sufficient to consider 1. $(*,*,*)$ 2. $C_2 = x$ a point, $C_1$ type $!$, $C_3$ - $*$. [ $0$ is of type $*$ and $!$ ]

- e.g. suppose we have $!**$ : we know $\bar{C}_1 \cap \bar{C}_2$ intersect, pick $x$ in there (0-dim cell)


break up into cases 1, 2.

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Proof continued

Case 2: $C''$ $*$, $C$ $!$, $C'$ point-cell.

write $C_1 = C''$, $C_2 = C$. $x$ point $\in \bar{C}_1 \cap \bar{C}_2$

$Ext^*(P_{C_1}, P_{C_2})$ — nonzero element comes from canonical $\neq 0$

$\quad = i_{\bar{C}_1 *} \cdots \quad = i_{C_2 !} \cdots$ morphism $i_{\bar{C}_1 *} \Phi_{\bar{C}_1} \to i_{C_2 !} \Phi_{C_2}[d_2]$

This morphism was obtained by adjointness from $\Phi_{\bar{C}_1} \to i^!_{\bar{C}_1} i_{C_2 !} \Phi_{C_2}[d_2]$

- by duality $\rightsquigarrow D\, i_{\bar{C}_1 \cap \bar{C}_2 *} \Phi_{\bar{C}_1 \cap \bar{C}_2}[0]$

Intersection has canonical generator $[x]$ for its $H^0$:

$$i_{\bar{C}_1 *} \Phi_{\bar{C}_1} \longrightarrow i_{x *} \Phi \longrightarrow i_{C_2 !} \Phi_{C_2}[d_2] \qquad \neq 0\ ?$$

Denote $B = \bar{C}_1 \cap \bar{C}_2 \underset{i_B}{\subset} \bar{C}_1$

— comes from $\Phi \to i_x^! i_{C_2 !} \Phi_{C_2}[d_2] = D(\underbrace{i_x^* i_{\bar{C}_2 *} \Phi_{\bar{C}_2}}_{=\,\Phi})$ - has canonical element.

$$\Rightarrow \Phi_{\bar{C}_1} \longrightarrow i_{x*} \Phi \longrightarrow i^!_{\bar{C}_1} i_{C_2 !} \Phi_{C_2}[d_2]$$

$$\cong$$

$$i_{B *} D \Phi_{\bar{B}} \qquad \cong i_{B *} j_! \Phi_B[d]$$

- dualizing sheaf on closed ball. $j_* \Phi_B = \Phi_{\bar{B}} \Rightarrow$

$D \Phi_{\bar{B}} = j_! \Phi_B [d]$

This reduces the statement to a question purely on the closed ball $\bar{B}$ : $x \in \partial\bar{B} = \bar{B} \setminus B$

$\Rightarrow$ nonzero composition $\varphi_{\bar{B}} \to i_{x*}\varphi \to j_! \varphi_B[1]$

assume the composition were zero. st $k : \underbrace{\bar{B} \setminus x}_{U} \hookrightarrow \bar{B}$

$$\begin{tikzcd} k_! \varphi_U \arrow[r] & \varphi_{\bar{B}} \arrow[r] \arrow[dr, "0?"'] & i_{x*}\varphi \arrow[r] \arrow[d] & k_! \varphi_U[1] \arrow[dl, dotted] \\ & & j_! \varphi_B[1] & \end{tikzcd}$$

But $\mathrm{Hom}^{d+1}(k_! \varphi_U, j_! \varphi_B) = H^{d-1}(U, j_! \varphi_B)$

<u>Exercise</u> $H^{d-1}(U, j_! \varphi_B) = 0$

- Use $i : \partial\bar{B} \hookrightarrow \bar{B}$, $j_! \varphi_B \to \varphi_{\bar{B}} \to i_* \varphi_{\partial\bar{B}}$

consider separate cases $d = 1, 2$. ∎

$\Rightarrow C_1, C_2$ of $*$ type, $C_1 \leq C_2$, $\dim C_1 < \dim C_2$, $\delta(C_1) < \delta(C_2)$ with nonzero Ext then can choose $C_1 \leq \tilde{C} \leq C_2$ so that $\delta(\tilde{C}) = \delta(C_1 + 1)$ & Exts will factor.

<u>Corollary</u> Algebra $A = \bigoplus_{c,c'} \mathrm{Ext}^*(P_c, P_{c'})$ is 1-generated (generated by $A_0, A_1$).

<u>Lemma</u> $\mathcal{D}$ t-category, $\mathcal{C}$ heart of $\mathcal{D}$, $A, B \in \mathcal{C}$.

$\Rightarrow \mathrm{Ext}^1_{\mathcal{C}}(A,B) \simeq \mathrm{Ext}^1_{\mathcal{D}}(A,B)$

& $\mathrm{Ext}^2_{\mathcal{C}}(A,B) \hookrightarrow \mathrm{Ext}^2_{\mathcal{D}}(A,B)$

<u>Cor</u> $\mathrm{Ext}^1_{\mathrm{Perv}(X)}(P_c, P_{c'}) = 0$ unless $\delta(c) - \delta(c') = 1$

<u>Proof of Lemma</u> Given $0 \to B \to E \to A \to 0$ in $\mathcal{C}$ $\Rightarrow$ dist $\Delta$ $B \to E \to A \xrightarrow{d} B[1]$ unique

If $d$ is $0$ then the sequence splits:

$$\begin{tikzcd} B \arrow[r] & E \arrow[r] & A \arrow[r] & B[1] \\ & A \arrow[r] \arrow[u, dashed] & A \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

similarly given such $\Delta \Rightarrow E$ is in heart ......

$\mathrm{Ext}^2$ : $0 \to B \to E_2 \to E_1 \to A \to 0$

$$\begin{tikzcd}[column sep=small] & & & & 0 \arrow[dr] & & 0 & & \\ 0 \arrow[r] & B \arrow[r] & E_2 \arrow[rr] \arrow[dr] & & E_1 \arrow[r] & A \arrow[r] & 0 \\ & & & C \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

- - - - ?

<u>Theorem</u> $\mathcal{C} = \mathrm{Perv}\, X$ is equivalent to the category of finite dimensional representations of a quadratic algebra (i.e. $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \mathcal{B}^{\otimes} \ldots$ s.t. $\mathcal{B}$ is generated by $\mathcal{B}_1$ over $\mathcal{B}_0$ + quadratic defining relations.)

[$\mathcal{B}$ here is in some sense dual to algebra of Ext's above]

<u>Pf</u> 1. Every $K \in \mathcal{C}$ has a canonical filtration $F_{\leq t} K \subset F_{\leq t+1} K \subset \ldots$

s.t. $F_{\leq t} \in \mathcal{C}_{\leq t} = \langle P_c,\ \delta(c) \leq t \rangle$

2. Every morphism $f: K \to L$ in $\mathcal{C}$ is strictly compatible with filtrations, i.e. $F_{\leq t} L \cap f(K) = f(F_{\leq t} K)$

1. Given an extension $S \subset K \twoheadrightarrow S'$ by simple objects and $\delta(S) \not< \delta(S')$ then the extension splits — we're using $\mathrm{Ext}^1(P_c, P_{c'}) = 0$ unless $\delta(c) - \delta(c') > 0$

$\Rightarrow$ existence of filtration

2. $\Leftrightarrow Gr: K \longrightarrow \bigoplus_t F_{\leq t} K / F_{\leq t-1} K$ is exact, conservative (something has $Gr\, K = 0 \Rightarrow K = 0$).

Given abelian category with conservative exact functor to fin. dim. vector spaces $\Rightarrow$ the category is equivalent to fin dim reps of an algebra (Tannakian formalism..)

Here $Gr\, K = \bigoplus_t \left( \bigoplus_{\delta(c) = t} V_c \otimes P_c \right)$ with $V_c$ vector space

$\longmapsto \bigoplus V_c$.

$Gr = \bigoplus Gr_t$. $\quad B = \bigoplus_{t, t'} \mathrm{Hom}(Gr_t, Gr_{t'})$

$B_0 = \bigoplus_c \mathbb{Q}$

Denote $Gr\, K = \bigoplus_c Gr_c K \otimes P_c$

$\Rightarrow$ extension $\quad 0 \to \bigoplus_{\delta(c)=t} Gr_c \otimes P_c \to F_{\leq t+1}K / F_{\leq t-1}K \to \bigoplus_{\delta(c)=t+1} Gr_c \otimes P_c \to 0$

$\Rightarrow$ $c$ dim $t+1$, $c'$ $t$ $\Rightarrow$ canonical element in
$\mathrm{Ext}^1(Gr_c \otimes P_c, Gr_{c'} \otimes P_{c'})$

$\Rightarrow$ canonical embedding $\quad \mathrm{Ext}^1(P_c, P_{c'})^* \hookrightarrow \mathrm{Hom}(Gr_{t+1}, Gr_t) = B_1$
$\to$ gives some generators in degree one.
$B_0 = \bigoplus_c Q \Rightarrow M = \bigoplus_c M_c$ for any $B$-module

[ Assume we have a graded algebra $B = B_0 \oplus B_1 \oplus \ldots$
where $B_0$ is semisimple $B_0 = \bigoplus_{s \in S} k_s$ field

$\Rightarrow$ 1. $B$ is generated by $B_1$ over $B_0$ $\Leftrightarrow$
$\mathrm{Ext}^1(k_s, k_{s'}) = 0$ unless $\delta(s) - \delta(s') = 1$ :

$B_i = \bigoplus_t \mathrm{Hom}(Gr_{t+i}, Gr_t)$ $\quad M = \bigoplus_t M_t$ graded module
$B_1 \cdot M_t \subset M_{t+1}$. The simple object $k_s$ has
gradings $(k_s)_t = \begin{cases} 0 & t \neq \delta(s) \\ 1 & t = \delta(s) \end{cases}$
$\Rightarrow$ defines degree function $\delta$.

2. Assume this satisfied. Then $B$ is quadratic $\Leftrightarrow$
$\bigoplus_{s,s'} \mathrm{Ext}^2(k_s, k_{s'})$ is generated by $\oplus \mathrm{Ext}^1(k_s, k_{s'})$

<u>Example</u> $\quad A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$
$\Rightarrow$ $k_i$ which is 1 in component $i$, 0 in all others

generators $\Rightarrow$ elements of $\mathrm{Ext}^1(k_i, k_j)$.
$\mathrm{Ext}^1(k_i, k_j) = 0$ unless $j = i+1$ $\Leftrightarrow$ $A$ generated
by $A_1$ : $\quad \to A_1 \otimes A[-i-1] \twoheadrightarrow A[-i] \to k_i \to 0$.

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$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots$ graded $k$-algebra, $A_0 = k$

Consider graded left $A$-modules, $A$-mod

$A$ is generated by $A_1/A_0 \iff \mathrm{Ext}^1_{A\text{-mod}}(k_i, k_j) = 0 \quad j \neq i+1$

$(k_i$ - $k$ in degree $i$). $\quad i = 0$: $\quad (P_0 = A_0 \; \underset{\cup}{A_+ = A_1} \; A_2 \; \dots$
$k_0$ )

$\mathrm{Ext}^1(k_0, k_j)$ = generators of $~~A~~ A_+$ as $A$-mod, of degree $j$

$A_+ = A_{>0}$

Denote $P_i = A[-i]$

$\exists$ resolution $A_1 \otimes P_1 \to P_0 \to k_0 \iff$ all $\mathrm{Ext}^1 = 0$ as above...

$A$ is quadratic: $A$ gen. by $A_1$ & determined by $I < A_1 \otimes A_1 \to A_2$

$\left( A_n = A_1^{\otimes n} \big/ (I \otimes A_1^{\otimes n-2} + \dots + A_1^{\otimes n-2} \otimes I) \right)$ )

$\iff \mathrm{Ext}^1(k_i, k_j) = 0 \quad j \neq i+1, \quad \mathrm{Ext}^2(k_i, k_j) = 0 \quad j \neq i+2$

$\iff \exists$ resolution $V_2 \otimes P_2 \to V_1 \otimes P_1 \to V_0 \otimes P_0 \to k_0 \to 0$

of $k_0$ ($V_i$ vector spaces - multiplicities)

since then can compute Ext's. Since $\mathrm{Hom}(P_i, k_j) = \delta_{ij}$

Assume $A$ quadratic $\Rightarrow$ $I \otimes P_2 \to A_1 \otimes P_1 \to P_0 \to k_0$ : in degree 2 get $I \hookrightarrow A_1 \otimes A_1 \to A_2$

exact, in degree $n$ $I \otimes A_n \to A_1 \otimes A_{n+1} \to A_{n+2} \to 0$

exact $\rightarrow$ get resolution as desired

& similarly conversely ... ?

---

**Definition** $A$ is a Koszul algebra if $\mathrm{Ext}^n_{grA}(k_i, k_j) = 0, \; j \neq i+n$

for all $n$.

**Example** $S^\bullet V$, $\Lambda^\bullet V$. $x_1, \dots x_n$ (noncomm) gens of deg 1 w/ $x_i x_j = 0$, $(i,j) \in S$ some set.

Koszul condition $\iff \exists$ resolution $\dots \to V_2 \otimes P_2 \to V_1 \otimes P_1 \to V_0 \otimes P_0 \to k_0 \to 0$

(necessarily unique: $V_i$ come as kernels of previous maps)

$A = S^\bullet V \Rightarrow$ Koszul complex $\quad \dots \to \Lambda^3 V \otimes S^\bullet V \to \Lambda^2 V \otimes S^\bullet V \to$
$\to V \otimes S^\bullet V \to S^\bullet V \to k_0 \to 0$

Quadratic duality

$V$ vector space, $I \subset V \otimes V \Rightarrow A_1 = V,\ A_2 = V \otimes V / I$ etc.

$\longleftrightarrow\ V^*,\ I^\perp \subset V^* \otimes V^*$ annihilator of $I$

$\Rightarrow A^!$, dual quadratic algebra.

$A^{!!} = A$.

Then $\operatorname{Ext}^1(k_0, k_1) = V^*$ : it is

$\operatorname{Hom}(I \otimes P_2 \to V \otimes P_1 \to P_0 \to k_0,\ k)$

- in deg 2 get $V^*$.

$\operatorname{Ext}^2(k_0, k_2) = I^* \longleftarrow V^* \otimes V^* = \operatorname{Ext}^1 \otimes \operatorname{Ext}^1$

with kernel $I^\perp$

So $A^!$ determined by algebra of Ext's: $A^!$ is quadratic part of $\operatorname{Ext}_A^*(k, k)$

Theorem $A$ Koszul $\Rightarrow A^!_\bullet \simeq \operatorname{Ext}^*(k, k)$
$(A^!_n = \operatorname{Ext}^n_A(k_0, k_n))$ and $A^!$ is Koszul.

[In some cases get equivalence of derived categories of modules].

[S. Priddy, Beilinson-Ginzburg-Schechtman]

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$\mathcal{C}$ a category, $F : \mathcal{C} \to \mathrm{Vect}$ exact conservative

$\Rightarrow \mathcal{C} \simeq A$-mod for graded assoc. $A$.

Warning: Start from $A = A_0 \oplus A_1 \oplus \dots$, $\mathcal{C}$ = graded findim $A$-mod

$\Rightarrow$ get actually $\tilde{A} = \bigoplus_{i \le j} A_{j-i}$, not $A$.

$\mathcal{C}$ perverse sheaves on $X$ in cell stratification

$F : K \mapsto \bigoplus_c Gr_c V$ - multiplicity of $P_c$.

$c \mapsto \delta(c)$ gives grading on $B$ & on $B$-modules

$B = \operatorname{End}(\bigoplus Gr_c)$

$\mathcal{C} \sim B$-mod.

$B = \bigoplus_{c, c'} B_{c,c'}$ with degree $\delta(c') - \delta(c)$ $\quad B_{c,c'} = \operatorname{Hom}(Gr_{c'}, Gr_c)$

$\operatorname{Ext}^1(P_c, P_{c'}) = 0$ unless $\delta(c) = \delta(c') + 1$

$B_0 = \bigoplus_c B_{c,c} \rightsquigarrow$ modules decompose $M = \bigoplus_c M_c$ in deg $f(c)$

Simple modules $S_c \longleftrightarrow \{c\}$

$Ext^1_{\mathcal{C}} = Ext^1_{B\text{mod}} \Rightarrow B$ generated by degree 1

S-algebras

Consider more general algebras: $S$ set, $A = \bigoplus_{s,s' \in S} A_{s,s'}$

$A_{s,s'} \otimes A_{s',s''} \longrightarrow A_{s,s''}$ gives multiplication,

$A_{s,s} = k$ field all $s$.

$w : S \longrightarrow \mathbb{Z}$ degree function : $\deg A_{s,s'} = w(s') - w(s)$

Ex. graded algebras : $w : S = \mathbb{Z} \to \mathbb{Z}$

$A_{ij} = A_{j-i}$ .. everything generalizes

— ~~mod~~

$M$ left $A$-mod $\Rightarrow M = \bigoplus_{s \in S} M_s$. ($M_s = A_{ss} M$ : idempotent)

$A_{ss'} : M_{s'} \longrightarrow M_s$

simple modules $(S_s)_{s'} = \delta_{s,s'}$.

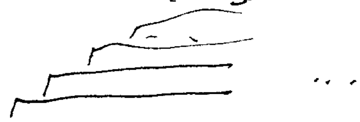
Projective covers $\bigoplus_{s'} A_{s',s} = P_s \longrightarrow S_s$ ...

An $S$-algebra is Koszul iff $Ext^n(S_s, S_{s'}) = 0$ unless $n = w(s) - w(s')$

<u>Remark</u> $Ext^n(S_s, S_{s'}) = 0$ $n > w(s) - w(s')$

- same holds for $A = A_0 \oplus A_1 \oplus \dots$, $Ext^n(k_i, k_j) = 0$ $n > j - i$

- since have resolution $\longrightarrow \bigoplus_{m_j \geq i+2} P_{m_j} \longrightarrow \bigoplus_{n_i \geq i+1} P_{n_i} \longrightarrow P_i \longrightarrow k_i$



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1. Finish proof : $\mathcal{C} = Perv(X)$, $\mathcal{C} \simeq B$-mod, $B$ quadratic
2. $X$ has simplicial triangulation. $A = \bigoplus_{c,c'} Ext^\bullet_{D(X)}(P_c, P_{c'})$
 $\Rightarrow$ <u>Theorem</u> $A$ is Koszul
3. $X$ simplicial triang. $Ext^*_{\mathcal{C}}(P_c, P_{c'}) \simeq A$,
 & $D^b(\mathcal{C}) \simeq D^b_c(X)$ smooth along our trivializations.

1. Need $\mathrm{Ext}^i_{B\text{-mod}}(k_x, k_{c'})$ $i=1,2$ appear in right degrees
but these are same as $\mathrm{Ext}_{\mathcal{C}}$ of corresponding objects. $\mathrm{Ext}^1_{\mathcal{C}} \simeq \mathrm{Ext}^1_{D(X)}$ for our simple objects,
and $\mathrm{Ext}^2_{\mathcal{C}}(P_c, P_{c'}) \hookrightarrow \mathrm{Ext}^2_{D(X)}(P_c, P_{c'})$
— in fact the latter is iso

$f: \tilde{A} = \mathrm{Ext}^*_{\mathcal{C}}(P_c, P_{c'}) \to A$. $A$ is gen. by $A_1$,
$f_1$ iso, $f_2$ mono $\Rightarrow$ $f_2$ isomorphism!

$$\begin{tikzcd} \tilde{A}_1 \otimes \tilde{A}_1 \arrow[r, "\sim"] \arrow[d] & A_1 \otimes A_1 \arrow[d] \\ \tilde{A}_2 \arrow[r, hook] & A_2 \end{tikzcd}$$

∎

2. Let $A = \bigoplus_{x \le y} A_{x,y}$ $x,y \in$ ordered set $S$,
$w: S \to \mathbb{Z}$ weight function, $w(x) \le w(y)$ $x \le y$.
$\dim A_{x,y} = 1$ $\forall x,y$, $A_{xy} \otimes A_{yz} \xrightarrow{\neq 0} A_{xz}$ $x \le y \le z$

Koszulity of such $A$ depends only on $S, \le, w$:
$k$ field $\Rightarrow$ define $A_S = \bigoplus_{x \le y} k_{x,y}$ with identity (nontrivial) multiplications.

Prop T.f.a.e.:
1. $A$ is Koszul
2. $A_S$ is Koszul
3. $x,y \in S$ $x \geq y$ consider $S(x,y) = \{z \in S : x \geq z \geq y\}$
Then $A_{S(x,y)}$ is Koszul $\forall x \geq y$.

Proof
Bar resolution of $k_x$: $\cdots \to \bigoplus_{z_0 < z_1 < x} A_{z_0 z_1} \otimes A_{z_1 x} \otimes P_{z_0} \to \bigoplus_{z_0 < x} A_{z_0 x} \otimes P_{z_0} \to P_x \to k_x$

comes from taking components of $\cdots \to A_+ \otimes A_+ \otimes A \to A_+ \otimes A \to A$.

$(P_x)_y = \begin{cases} A_{yx} & y \le x \\ 0 & \text{otherwise} \end{cases}$ projective

To compute $\mathrm{Ext}(k_x, k_y)$ take Hom from this complex to $k_y$ and see only pieces between $x, y$ matter

- $\mathrm{Hom}(P_z, k_y) = 0$ unless $z = y$

Thus $Ext^*_A(k_x, k_y) = Ext_{A(x,y)}(k_x, k_y)$  $A(x,y) = \bigoplus_{y \le y' \le x' \le x} A_{y'x'}$

$\Rightarrow$ 2 & 3 are equivalent.

To prove $1 \Leftrightarrow 2$ suffices to consider case when $S$ has maximal element (replace $S$ by $S_{(x,y)}$).

Claim $S$ has max elt $\Rightarrow A \simeq A_S$.

- choose bas $e_{xy} \in A_{xy}$, get $c_{x,y,z} \in k^*$ for $x \le y \le z$

Associativity of multiplication $\Rightarrow$ 2-cocycle on geometric realization of $S$, changing base gives equivalent cocycle .. but if $S$ has max. element, $|S|$ is contractible $\Rightarrow H^2(|S|, k) = 0$ ! ■

[ **Prop** $x, y \in S$, $x > y$ consider $S'(x,y) = \{ z \in S \mid x > z > y \}$ poset $|S'(x,y)|$ geometric realization. Then $A_S$ is Koszul $\Leftrightarrow \forall x > y$ $\tilde{H}^i(|S'(x,y)|, k) = 0$ $i \ne w(x) - w(y) - 2$.
($\tilde{H}$ reduced cohomology)

[General result: given $\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}$ adjoint functors between categories $\Rightarrow |\mathcal{C}| \simeq |\mathcal{D}|$ homotopy equivalence. (G right adj to F).

Also $S$ poset, $\exists u \in S$ $\forall x \in S$ either $x \le u$ or $x \ge u$ $\Rightarrow |S|$ contractible : follows from above result.

adjointness means $x \le f_x(y) \Rightarrow f^*(x) \le y$ . .. ]

<u>Proof of Proposition</u> Bar resolution of $A_S = \bigoplus_{x \in Y} k_{xy}$ is

$P^\bullet$ : . . $\to \bigoplus_{z_0 < x} P_{z_0} \to P_x$

$Hom(P^\bullet, k_y) = \ldots \leftarrow \bigoplus_{y < z_1 < z_2 < x} k \leftarrow \bigoplus_{y < z_1 < x} k \leftarrow \underset{z_0 = y}{k} \leftarrow 0$

which calculates <u>reduced</u> cohomology of $|S(x,y)|$, get shift by 2. ■

$A$ is Koszul $\Leftrightarrow A_{S(x,y)}$ Koszul $\forall x > y$

<u>Return to case</u>: $S$ = set of simplices of our triangulation, $\le$ our order

САША ПОЛИЩУК

<u>Lemma</u> $C$ of type $*$, $C'$ type $! \Rightarrow T = |S'(C,C')|$ is contractible

<u>Proof</u> $C, C'$ closed simplices. $C \cap C' \ni x$ vertex,

Consider $\{S'(C,C')\} = T \supset T_x = \{\tilde{C} \in T / \tilde{C} \ni x\}$

$|T| = \bigcup_{\substack{x \in C \cap C' \\ \text{vertex}}} |T_x|$ Sufficient to prove: $\forall I$ set of vertices of $C \cap C'$, $\bigcap_{x \in I} |T_x| = |\bigcap_{x \in I} T_x|$ is contractible and nonempty.

In $T_I = \bigcap_{x \in I} T_x$ set simplex $C_0 = \langle x \in I \rangle$.

$T_I = \{\tilde{C} \subset C$ or $\tilde{C} \subset C' : \tilde{C} \supset C_0\}$ – so everything is comparable to $C_0 \Rightarrow$ contractible.

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<u>Regular</u> C-W complex as before ($\bar{C}$ = ball, $\bar{C}_1 \cap \bar{C}_2$ = ball)

$\mathcal{C} \simeq B$-mod, $B$ quadratic algebra.

$A = \bigoplus_{c,c'} \mathrm{Ext}_X(P_c, P_{c'})$ 1-generated.

Simplicial case $X = \bigcup_S \Delta$ : 1. $A$ is Koszul

2. $A \simeq \bigoplus \mathrm{Ext}_{\mathcal{C}}(P_c, P_{c'})$ $\Rightarrow \mathcal{D}^b(\mathcal{C}) \simeq \mathcal{D}^b_{\mathcal{C}}(X)$.

1. Need to check $A(x,y)$ is Koszul (cells between $x,y$)

We did this for $x,y$ of different types.

$x,y$ of same type $*$ : cells between them have certain perscribed dimensions .. faces in simplex $\longleftrightarrow$ subsets of vertices of $\Delta_x$ ! so we're looking at set $[1, \ldots, n]$ of vertices, & cardinalities

$0 = d_0 < d_1 < \ldots < d_k = n$

$S$ = set of subsets $I \subset [1, \ldots n]$ s.t.

$\# I \in \{d_0, d_1, \ldots, d_k\}$ – ordered by inclusions

$\delta : d_i \longmapsto i$

$A_S = \bigoplus_{x \leq y} k_{x,y}$ $\Rightarrow \delta : S \longrightarrow \mathbb{Z}$

$\deg k_{x,y} = \delta(y) - \delta(x)$ grading

<u>$A_S$ is quadratic</u> :

generators $e_{IJ}$ $\# I = d_i$, $\# J = d_{i+1}$.

Quadratic relations $\underset{I}{\overset{d_i}{}} \subset \overset{d_{i+2}}{K}$ $\quad e_{IJ}e_{JK} = e_{IJ'}e_{J'K}$

where $\# J = \# J' = d_{i+1}$ $\quad I \subset J \subset K$, $I \subset J' \subset K$

$\Rightarrow \quad A' \twoheadrightarrow A_S$, $A'$ quadratic algebra generated by these.

$A' = \bigoplus_{x \le y} A'_{x,y}$. So S.T.P. $\dim A'_{x,y} = 1 \quad x \le y$

Prove by induction in $r(y) - r(x) = i$ :

holds $i \le 2$. $\quad x < y$, $x < \tilde{x} <$ $\Rightarrow \dim A'_{\tilde{x},y} = 1$ by induction,

$A'_{x,y} = \sum_{r(\tilde{x}) = r(x)+1} A'_{x\tilde{x}} A'_{\tilde{x}y}$ - need all summands to be colinear - but quadratic relations imply this for nearby dimensions .......

reduce to checking graph of subset poset is connected ...

To show quadratic algebras are (sometimes) Koszul:

$A = \underset{k}{A_0} \oplus \underset{V}{A_1} \oplus \ldots$ quadratic

$e_1, \ldots e_n$ basis of $V = A_1$ (ordered)

$S = \{(i,j) \in [1,\ldots n]^2 \mid e_i e_j \in A_2$ is not a linear combination of $e_k e_l$ with $(k,l) < (i,j)$

in lexicographical order.

$\Rightarrow \quad e_i e_j, \; (i,j) \in S$ normal words of length 2 ...

get basis for $A$ in normal words.

Definition $\{e_1, \ldots e_n\}$ are PBW generators iff

$e_{i_1} e_{i_2} \ldots e_{i_k}$ (where any consecutive pair $e_{i_j} e_{i_{j+1}}$ etc are normal words) is a base in $A_2 \oplus A_3 \oplus \ldots$

Eg. $A = S^{\bullet}V$ $\quad e_1, \ldots e_n$ base of $V$ $\Rightarrow$ ordered monomials give base.

[Diamond Lemma] Theorem $\{e_1, \ldots e_n\}$ PBW generators iff all elements $\{e_{i_1} e_{i_2} e_{i_3}, \; (i_1, i_2) \in S, \; (i_2, i_3) \in S\}$ are linearly independent

Def $A$ is a quadratic PBW alg if $\exists$ PBW basis

Theorem (Priddy) $A$ quadratic PBW algebra $\Rightarrow A$ is Koszul.

Proof: get filtration on $A$, with values in free group, s.t. $\mathrm{gr} A$ is monomial algebra $k\langle e_1, \dots e_n\rangle / e_i e_j = 0 \quad (i,j) \notin S$

The latter is Koszul, use s.s. of filtration see Koszulity of $\mathrm{gr} A \Rightarrow$ Koszulity of $A$. ∎

$$A_1 = \bigoplus_{\substack{x \in y \\ \delta(y) - \delta(x) = 1}} A_{xy} = \bigoplus_i \Big( \bigoplus_{\substack{x \in y \\ \delta(x) = i \\ \delta(y) = i+1}} A_{xy} \Big)$$

take ordered base for each $i$.

Prop $S$ = subsets in $[1, \dots n]$, $\# S \in \{0 = d_0 < d_1 < \dots < d_k = n\}$
$\Rightarrow A_S$ is a quadratic PBW algebra ($\Rightarrow A$ Koszul)

Proof Take lexicog. order on subsets $I \subset [1, \dots n]$ of given cardinality $\Rightarrow$ order on $e_{IJ}$

Claim – these are PBW generators: Words of length 2 are $I \subset J \subset K$, card. $d_i, d_{i+1}, d_{i+2}$, $J$ is minimal among all such: $e_{IJ}\, e_{JK}$ normal iff
$J = I \cup [\,(d_{i+1} - d_i)$ first elts of $K - I\,]$

Now need to check there's a unique such word:
PBW condition: given $\underset{d_i}{I} \subset \underset{d_{i+3}}{L}$ $\exists!\ I \subset \underset{d_{i+1}}{J} \subset \underset{d_{i+2}}{K} \subset L$ s.t. each pair is normal.

STP: $J$ is unique, in fact $J = I \cup (d_{i+1} - d_i)$ first elts of $L - I$ (instead of just $K - I$).

They have same number of elts, so show RHS $\subset J$.
Induction on $j = 0, 1, \dots, d_{i+1} - d_i$ that $j$ first elts of $L - I$ belong to $J$.

Assume $j^{th}$ does, $x = (j+1)$st doesn't belong to $J$
$\Rightarrow$ $x$ is minimal element of $L - J \Rightarrow x \in K$
$\Rightarrow$ $x$ is one of first $d_{i+1} - d_i$ elts of $K - I \Rightarrow x \in J$.

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$A = \bigoplus \mathrm{Ext}^*_X(P_c, P_{c'})$, simplicial triangulation $\Rightarrow$ $A$ is Koszul.

$\mathcal{C} \simeq B$-mod with $B$ quadratic

Proposition, $A = B^!$, quadratic dual.

$B^!$ = quadratic part of $\bigoplus_{s,s'} \mathrm{Ext}^*_B(S, S') = \bigoplus_{c,c'} \mathrm{Ext}^*_{\mathcal{C}}(P_c, P_{c'})$.

But this has same quadratic part as $A$.

$(\mathrm{Ext}^1, \mathrm{Ext}^2) \Rightarrow = A$ so $A = B^!$ $\square$

Cor. $B$ is Koszul.

$\boxed{\text{Theorem}}$ $X$ triangulated $\Rightarrow$ $D^b(\mathcal{C}) \simeq D^b_c(X)$.

Proof 1. (General part) Realization functor

real : $D^b(\mathcal{C}) \longrightarrow D^b_c(X)$ $t$-exact, extending usual embedding on $\mathcal{C}$. (wrt standard on LHS, perverse on RHS)

2. $K, L \in \mathcal{C} \Rightarrow \mathrm{Ext}^i_{\mathcal{C}}(K,L) \simeq \mathrm{Ext}^i_X(K,L)$

1, 2 are sufficient to imply theorem:

2 gives full & faithful. Next any object on RHS has finite number of perverse cohomologies $\Rightarrow$ cone of finite number of perverse sheaves, which are in the essential image, so essential image is everything.

To check 2: STP, by five lemma, for $K, L$ simple

$$\begin{array}{c} \tilde{A} = \bigoplus \mathrm{Ext}^*_{\mathcal{C}}(P_c, P_{c'}) \xrightarrow[?]{\sim} A = \bigoplus \mathrm{Ext}_X(P_c, P_{c'}) \\ \wr\wr \\ B^! \xrightarrow{\quad\sim\quad} \end{array}$$

Realization

Difficulty - nonfunctoriality of cone: objects in $D^b(\mathcal{C})$ are successive cones of objects in $\mathcal{C}$, but in $D^b_c(X)$ $\mathcal{C}$ isn't sitting in one-degree.. Can do this on the level of isomorphism classes of objects, but not functorially.

$A$ abelian category $\Rightarrow$ $DF(A)$ category of complexes/$A$ with finite decreasing filtration, localized by filtered quasi-isomorphism:

$C_1^\bullet \xrightarrow{f} C_2^\bullet$, $f(F^i C_1) \subset F^i C_2 \Rightarrow \mathrm{Gr}^i_F f : \mathrm{Gr}_F C_1 \to \mathrm{Gr}_F C_2$

$f$ is filtered quism iff $Gr_F f$ is quism ...

– triangulated category $DF(A) \supset DF(A)(\geq n) =$
$\{ C^\cdot \mid Gr_F^i C^\cdot = 0 \quad i < n \}$ triangulated subcategory (closed under cones)

$D(A) \subset DF(A)$ in degree 0, $Gr_F^i : DF(A) \to D(A)$

shift of filtration $s(C^\cdot, F^\cdot) = (C^\cdot, F^{\cdot - 1})$

$s$ is exact autoequivalence.

$\alpha : Id_{DF(A)} \to s$, $F^i C \to F^{i-1} C$.

$\omega : DF(A) \to D(A)$ forget-filtration functor.

Assume $A$ has enough injectives

$K \in DF(A)$, $L \in D^+F(A) \rightsquigarrow RHom(K, L) \in DF(Ab)$

- can assume $Gr\, L$ injective (via resolution).

$Hom^n(K, L) = \prod_{j-i=n} Hom(K^i, L^j)$

via $f = (f_i) \in F^k Hom^n(K, L)$ iff $f_i(F^m K^i) \subset F^{m+k}(L^{i+n})$

1. $Gr_F^k RHom(K, L) = \prod_{m-n=k} RHom(Gr_F^n K, Gr_F^m L)$
2. $Hom_{FD(A)}(K, L) = H^0 F^0 (RHom(K, L))$
 $Hom_{D(A)}(\omega(K), \omega(L)) = H^0 RHom(K, L)$

$\Rightarrow$ s.s. with $E_1^{pq} = \begin{cases} \prod_{j-i=p} Hom^{p+q}(Gr_F^i K, Gr_F^j L) & p \geq 0 \\ 0 & p < 0 \end{cases} \Longrightarrow Hom^{p+q}_{DF(A)}(K, L)$

$E_1'^{pq} = \left\{ \begin{array}{c} \text{above term} \\ \forall p \end{array} \right\} \Longrightarrow Hom_{D(A)}(\omega(K), \omega(L))$

$\Rightarrow K \in DF(A)(\geq 1)$, $L \in DF(A)(\leq 0) \Rightarrow Hom_{DF(A)}(K, L) = 0$

- need $i \geq 1$, $j \leq 0$, $p = j - i \leq -1$ all terms in s.s. vanish.

$K \in DF(A)(\leq 0) \quad L \in DF(A)(\geq 1) \Rightarrow Hom_{DF(A)}(K, L) \simeq Hom_{D(A)}(\omega(K), \omega(L))$

<u>Definition</u> $DF$ is a filtered triangulated category (f-category) if
given $DF(\leq 0) \subset D \supset DF(\geq 0)$
strictly full triangulated subcategories,

$s: DF \longrightarrow DF$ exact automorphism, $\alpha: Id_{DF} \to s$

[set $DF(\le n) = s^n DF(\le 0)$ etc.)

<u>Axioms</u> i. $DF(\le 1) \supset DF(\le 0)$, $DF(\ge 1) \subset DF(\ge 0)$

$\bigcup_n DF(\le n) = \bigcup_n DF(\ge n) = DF$.

ii. $X \in DF \Rightarrow \alpha_X = s(\alpha_{s^{-1}X})$

iii. $\left.\begin{array}{l}\mathrm{Hom}(X,Y) = 0 \\ \mathrm{Hom}(Y,X) \xrightarrow{\sim} \mathrm{Hom}(s^{-1}Y, X)\end{array}\right\}$ if $X \in DF(\ge 1)$ $Y \in DF(\le 0)$

iv. $\forall X \in DF$ $\exists$ dist $\Delta$ $\underset{DF(\ge 1)}{A} \to X \to \underset{DF(\le 0)}{B} \to \dots$

<u>Example</u> $DF(A)$ f-category over $DA$. [$DF$ f-category over $D$.
$D = DF(\le 0) \cap DF(\ge 0)$. ]

Given t-structure on $D \Rightarrow$ t structure on $DF$:
$A \in DF^{\le 0}$ if $Gr_F^i \in D^{\le i}$

[We have truncation functors $\sigma_{\ge n}: DF \to DF(\ge n)$,
$\sigma_{\le n}$, $Gr_F^n = s^{-n}\sigma_{\le n}\sigma_{\ge n} : DF \longrightarrow D$ ]

If $\mathcal{C}$ is heart of $D$, then the heart $\mathcal{C}F$ of $DF$ is canonically equivalent to the category of complexes $C^b(\mathcal{C})$ (abelian category)
To construct this, send $A \longmapsto \{Gr_F^i A\}$
with canonical complex structure...

Case $\mathcal{A} = Sh\,X$, apply to perverse t-structure on $\mathcal{D}$

$$\begin{array}{ccc} C^b(\mathcal{C}) = \mathcal{C}F \subset DF(X) & \xrightarrow{\omega} & D(X) \\ \cup \\ Perv(X) = \mathcal{C} \subset D(X) \end{array}$$

get functor $\mathcal{C}^b(\mathcal{C}) \longrightarrow D(X)$, verify
it descends to $D^b(\mathcal{C}) \to D(X)$.