Perverse sheaves on cell-stratified space
(after R. MacPherson)
Notes for Math 278, Fall 1996

1. Extensions between simple objects

Let X be a regular CW-complex. This means that X is a CW-complex such that the closure of each cell C is homeomorphic to the closed ball (in a way compatible with the structural homeomorphism of C with the open ball). Let $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ be a perversity function, such that $p(0) = 0$, $0 \leq p(m) - p(n) \leq n - m$ for $m \leq n$. We consider the category $\mathcal{C}$ of perverse sheaves on X with respect to p, smooth along the stratification by cells. Let C be a cell in X, $i_C : C \to X$ be the inclusion. Let $i_{\overline{C}} : \overline{C} \to X$ be the inlcusion of the closure of C. The simple objects of $\mathcal{C}$ are Goresky-MacPherson extensions $P_C := i_{C,!*}\mathbb{Q}_C[-p(C)]$, where $p(C) = p(\dim C)$. Let us call an integer $i > 0$ *of type* $*$ if $p(i) = p(i-1) - 1$, and *of type* ! otherwise (i.e. if $p(i) = p(i-1)$). It is convenient to assume that 0 is of both types. We say that a cell C is of type $*$ (resp. of type !) if $\dim C$ is.

Lemma 1.1. *For any cell C we have $P_C \simeq i_{C,!}\mathbb{Q}_C[-p(C)]$ if C is of type !, and $P_C \simeq i_{C,*}\mathbb{Q}_C[-p(C)]$ if i is of type $*$. In the latter case $P_C \simeq i_{\overline{C},*}\mathbb{Q}_{\overline{C}}[-p(C)]$.*

Proof. This follows easily from the fact that if $j : B \to \overline{B}$ is an inclusion of the open ball into its closure, then $j_*\mathbb{Q}_B \simeq \mathbb{Q}_{\overline{B}}$. □

In this situation R. MacPherson defines a *perverse dimension* of a cell C as $\delta(C) = \delta(\dim C)$ where $|\delta(d)|$ is the number of i's, $0 < i \leq d$, which are of the same type as d (in particular, $\delta(0) = 0$), the sign of $\delta(d)$ for $d > 0$ is defined by the following rule: $\delta(d) > 0$ if d is of type $*$, and $\delta(d) < 0$ otherwise. Notice that δ maps every interval $[0, d]$ $(d \geq 0)$ to an interval of integers containing zero, and the above definition establishes a bijective correspondence between perversities and maps $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ with this property.

From now on we assume that the cell decomposition of X satisfies the following condition:

for every pair of cells C and C', the intersection $\overline{C} \cap \overline{C'}$ is either empty or homeomorphic to a closed ball.

Below we compute Ext-groups in the derived category of sheaves on X between simple objects of $\mathcal{C}$.

Proposition 1.2. *Let C and C' be cells. Then $\mathrm{Ext}^i(P_C, P_{C'}) = 0$ unless $i \neq \delta(C) - \delta(C')$. Furthermore, $\mathrm{Ext}^*(P_C, P_{C'})$ is non-zero precisely in one of the following cases: 1) both C and C' are of $*$-type (resp. !-type) and $C' \subset \overline{C}$ (resp. $C \subset \overline{C'}$), 2) C is of $*$-type, C' is of !-type and the intersection $\overline{C} \cap \overline{C'}$ is non-empty. In these cases $\mathrm{Ext}^*(P_C, P_{C'})$ is one-dimensional.*

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DF - filtered derived category (graded components have constructible cohomology).

t-structure on D $\rightsquigarrow$ t-structure on DF via

$DF^{\leq 0} = \{X \mid Gr^i \in D^{\leq i}\}$ etc.

<u>Claim:</u> $\mathcal{C}F = DF^{\leq 0} \cap DF^{\geq 0} \xrightarrow[\sim]{H_F} C^b(\mathcal{C})$ complexes in $\cancel{\text{heart}}(D)$.

$K \in \mathcal{C}F \iff Gr^i K \in \mathcal{C}[-i]$
$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{K^i[-i]}$

$K^i \overset{?}{\to} K^{i+1}$: we have truncation-of-filtration functors $\sigma_{\geq n}$, $\sigma_{\leq n}$ etc, as adjoints of inclusions

$\Rightarrow \sigma_{[a,b]} = \sigma_{\geq a}\sigma_{\leq b} \simeq \sigma_{\leq b}\sigma_{\geq a}$

We have exact triangle $Gr^{i+1}K \to \sigma_{[i,i+1]}K \to Gr^i K \to$

$\Rightarrow$ $H^i(Gr^i K) \longrightarrow H^{i+1}(Gr^{i+1}K)$ boundary map
$\qquad\quad \overset{\shortparallel}{K^i} \longrightarrow \overset{\shortparallel}{K^{i+1}}$

gives our differential.

H_F is full & faithful: by spectral sequence from last time: E_1^{pq} is $\neq 0$ only in $p, q \geq 0$,

$\prod Hom(K^i, L^i) \xrightarrow{d_1} \prod Hom(K^i, L^{i+1})$

d_1 is $[d,]$ d map above $\Rightarrow$

$E_2^{00} = Hom_{C^b(\mathcal{C})}(K^\cdot, L^\cdot)$ & SS degenerates at E_2.

Map is surjective on objects... get composition

$$\begin{tikzcd} & DF \arrow[r, "\omega"] & D \\ \mathcal{C}F \arrow[ur, hook] \arrow[r, "\sim"] & C^b(\mathcal{C}) \arrow[r] & D^b(\mathcal{C}) \arrow[u, "\text{real}"'] \end{tikzcd}$$

Why is H_F essentially surjective?

$a \leq p < b$ $\qquad 0 \to K^a \to K^{a+1} \to \dots \to K^b \to 0$

write as $\qquad 0 \to K^a \to \dots \to K^p$
$\qquad\qquad\qquad\quad \uparrow \qquad\qquad\qquad \downarrow f \qquad\qquad \downarrow$ deg b
$\qquad\qquad\qquad$ deg $a+1$ $\qquad\qquad K^{p+1} \to \dots \to K^b \to 0$

By induction on $b-a$, f comes from map of filtered complexes $f = H_F(\tilde{f} : (A,F) \to (B,F))$

$$\begin{tikzcd} (A,F^{\circ+1}) \arrow[r] \arrow[rr, bend right, "g"] & (A,F^{\circ}) \arrow[r] & (B,F) \end{tikzcd}$$

$\Rightarrow H_F(\text{cone } g) = K^\bullet$

Remark: we had $A = \oplus \operatorname{Ext}^*(P_c, P_{c'})$

S = ordered set of cells, Claim: $A \simeq A_S$

(before we only claimed A Koszul $\Leftrightarrow A_S$ Koszul)

c ! type $P_c = i_! \mathbb{Q}_c [\ldots]$

$*$ $P_{c'} = i_* \mathbb{Q}_c [\ldots]$

- don't see this isomorphism.. but can rewrite in ! case as $i_{c!}$ or $_c[\ldots]$: selfdual

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Semi-small stratified map:

$T \subset f^{-1}(S)$ $T \to S$ loc triv $f : X \to Y$

$f_* : \mathcal{D}_T(X) \to \mathcal{D}_S(Y)$

f is semismall if $\forall S \subset f(\bar{T})$, $x \in S$. $\dim(f^{-1}(x) \cap \bar{T}) \leq \frac{1}{2}(\dim f(\bar{T}) - \dim S)$

& $f|_{\bar{T}}$ proper

Claim f semismall $\Rightarrow$ P perverse on $X \to f_*P$ perverse on Y.

Pf Suffices to consider & show $f_* {}^p\mathcal{D}^{\leq 0} \subset {}^p\mathcal{D}^{\leq 0}$

- properness $\Rightarrow$ $f_* = f_!$, commutes with duality.

~~use local~~

Condition $P \in {}^p\mathcal{D}^{\leq 0}$ is pointwise condition: sufficient to consider restriction to every stratum

$\Leftrightarrow P|_T \in \mathcal{D}^{\leq -\dim T}$

Thus we have to show $f_* (j_! P|_T)|_S \in \mathcal{D}^{\leq -\dim S}$:

$\mathcal{L}$ loc sys on $T \Rightarrow R^i f_* \mathcal{L} = 0$ $i > \dim f(\bar{T}) - \dim S$

$\Rightarrow f_*(P|_T) \in D^{\le -\dim T + \dim f(T) - \dim S} \subset D^{\le -\dim S}$

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X - triangulated, or more generally cell stratified:
X stratified, every stratum $S \simeq$ open ball, $\bar{S} \simeq$ closed ball
$\mathcal{C}$ category of perverse sheaves on X wrt stratification, some perversity p
is B-mod, B quadratic algebra ←
Ext^1 occurs only when difference in perverse dimension is 1 $\Rightarrow$ get filtrations, morphism compatible with filtrations. $B = \mathrm{Aut}$ (fibre functor $\oplus$ Gr (filtration))
This functor has a geometric construction as in loop Grassmannian case..

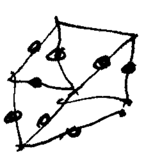
$\boxed{\text{Perverse cells}}$ [Note: in cell stratified case can construct barycentric subdivision, by induction on dimension of strata — cone from center of ball to subdivision of boundary

- get simplicial triangulation of X. X is $\simeq$ geometric realization of poset of cells in subdivision.]

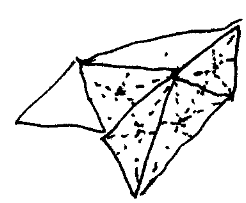
k-simplices $\longleftrightarrow$ $C_0 \subset \dots \subset C_k$

Perverse filtration: $F_d X = \bigcup_{\delta(C_i) \le d} \langle C_0 \subset \dots \subset C_k \rangle$
union of simplices in barycentric subdivision..
Perverse cells are connected components of $F_d X \setminus F_{d-1} X$.

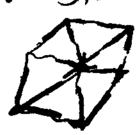
Example: $\delta: 0 \to 0,\ 1 \mapsto -1,\ 2 \to 1$
$F_{-1} X$ = centers of cells of dim 1



F_{-1}



F_0 : ~~come~~ from cells of dim 0 - union of all stars of vertices



connected components $\longleftrightarrow$ vertices of original triang

F_2 = union of open cells..
Perverse cells are in bijective corresp with usual cells:

$\partial^\delta C$: need to add condition $C^* \neq \tilde{C}$.

$\Rightarrow \quad {}^\delta\bar{C} \cap \bar{C}' = |\{\tilde{C} : \tilde{C} \subset \bar{C}', \delta(\tilde{C}) \leq \delta(C') \}|$

which has a maximal element C' $\Rightarrow$ contractible.

$\partial^\delta C \cap \bar{C}'$

= same set with condition

$\tilde{C} \neq C$, but C' still gives maximal element $\Rightarrow$ contractible

$\rightarrow C'$ of type ! so $\delta(C') \leq \delta(C)$ ~~but~~ because of $C \subset \bar{C}'$ ...

b. $C \not\subset \bar{C}'$ (C' of type !) . Then ${}^\delta C \cap \bar{C}' = \emptyset$

c. $C = C'$! $\Rightarrow$ ${}^\delta\bar{C} \cap \bar{C}' = pt$, $\partial^\delta C \cap \bar{C}' = \emptyset$

${}^\delta C'' \cap \bar{C}'$.

$\boxed{\text{Perversity \& Purity}}$

$X_0 / \mathbb{F}_q$ scheme of finite type $\rightsquigarrow X / \bar{\mathbb{F}}_q$.

$\mathcal{F}_0 / X_0$ - $\mathbb{Q}_\ell$ sheaf $\longrightarrow$ $\mathcal{F}/X$ $\bar{\mathbb{Q}}_\ell$ sheaf

$Fr_q : X \to X$. $F_q : Fr_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ from $\begin{array}{ccc} & Fr_q : X \to X & \\ \uparrow & & \uparrow \\ \mathcal{F} & \xrightarrow{Fr} & \mathcal{F} \end{array}$

K_0, L_0 elements of (derived category of ~~f~~ $\mathbb{Q}_\ell$-sheaves

$\in D_c^b(X_0, \bar{\mathbb{Q}}_\ell)$

$$R\mathrm{Hom}(K_0, L_0) = R\Gamma \underbrace{R a_* \, \underline{R\mathcal{H}om}(K_0, L_0)}_{M_0 \in D_c^b(\mathrm{Spec}\, \mathbb{F}_q, \bar{\mathbb{Q}}_\ell)}$$

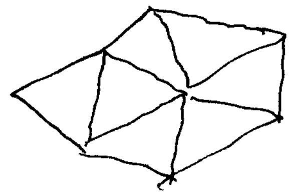
$a : X_0 \to \mathrm{Spec}\, \mathbb{F}_q$.

Compute via s.s. , $E_2^{pq} = H^p(\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q), H^q M)$

* (objects without $(\ )_0$ are extended by scalars to alg closure)

$\Longrightarrow H^{p+q} R\Gamma M_0$

This is <u>continuous</u> cohomology - comes from inverse limit of such objects for $\mathbb{Z}/\ell$-sheaves...



C cell $\rightsquigarrow$ perverse cell $\quad {}^\delta C = \bigcup_{\substack{\delta(C_i)\le\delta(C) \\ \text{\& one of } C_i \text{ is } C}} \langle C_0 < \dots < C_k \rangle \quad \dots$

$\Rightarrow$ get another stratification.

<u>Theorem</u> $\quad H^*_{\delta_C}(X, P_{C'}) = \begin{cases} 0, & C \neq C' \\ \text{1-dim of degree } -\delta(C), & C = C' \end{cases}$

(If we replaced $\delta \le \delta(C)$ by $\ge$ would get H with compact support ...)

Thus $F \longmapsto \bigoplus_C H^*_{\delta_C}(X, F)$ will be our fibre functor – this is decomposition into isotypic Components, graded piece of filtration corresponding to simple object P_C ...

<u>Proof</u> 1. C' of type $*$: then ${}^\delta C \cap C' = \emptyset$ unless $C' = C$

$C = C' \Rightarrow C \subset {}^\delta C$, so taking $H_{\delta C} \Rightarrow$

$P_C = j_{C*}\mathbb{Q}_C(-\underbrace{p(C)}_{\delta(C)}) \quad \dots$

2. C' of type $!$: $\quad P_{C'}$ is extension by zero – so by Verdier duality we need to compute

$H^*_c({}^\delta C, i_{\bar{C}'*}\mathbb{Q}_{\bar{C}'})$ $\quad$ get constant sheaf on closure...

<u>case a.</u> $\quad C \subset \bar{C'}$. $\quad$ if $C \neq C'$, $\quad \left.\begin{array}{l} {}^\delta C \cap \bar{C}' \\ \partial^\delta C \cap \bar{C}' \end{array}\right\}$ contractible

But H^*_c of difference of compact contractible spaces vanishes (by long exact sequence of H^*_c).

Why are these contractible? both of them are actually geometric realization of sets of cells :

$$\bar{C}' = \bigcup_{C_i \subset \bar{C}'} \langle C_0 \subset \dots \subset C_k \rangle = |\{\tilde{C} : \tilde{C} \subset \bar{C}'\}|$$

$${}^\delta C^* = \bigcup_{\delta(C_i) \le \delta(C^*)} \langle C_0 \subset \dots \subset C_k \rangle = |\{\tilde{C} \mid \delta(\tilde{C}) \le \delta(C^*),\ \tilde{C} \sim C^*\}|$$

$C_i \sim C^*$: one is in closure of the other

This ss degenerates - only H^0, H^1 nonzero $\Rightarrow$

$$* \quad 0 \to \underbrace{(Hom^{i-1}(K, L))_F}_{\text{coinvariants}} \to Hom^i(K_0, L_0) \to \overbrace{Hom^i(K, L)^F}^{\text{invariants}} \to 0$$

F geometric Frob $\in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, inverse to arithmetic Frobenius

<u>Definition</u> $\mathcal{F}_0 / X_0$ is called <u>pure</u> of weight $w \in \mathbb{Z}$ iff for $\forall x \in X_0(\mathbb{F}_{q^n})$ (i.e. $Fr_{q^n} : X \to X$ fixes x) $F_{q^n}^* : \mathcal{F}_x \circlearrowleft$ has all eigenvalues algebraic numbers, all their conjugates have absolute value $q^{\frac{nw}{2}}$ (complex absolute value, via some isomorphism $\overline{\mathbb{Q}_\ell} \to \mathbb{C}$)

<u>Definition</u> $\mathcal{F}_0 / X_0$ is <u>mixed</u> if it has a filtration s.t. all graded factors are pure.

Conjecturally all $\mathcal{F}_0$ are mixed...

Denote $D^b_m(X_0, \overline{\mathbb{Q}_\ell}) = \{K : \forall i \ H^iK \text{ mixed}\} \subset D^b_c(X, \overline{\mathbb{Q}_\ell})$

Stable under $f_*, f_!, f^!, f^*, \otimes, R\mathcal{H}om$, Verdier duality, ${}^p\tau_{\leq i}, {}^p\tau_{\geq i}, \tau_{\geq i}, \tau_{\leq i}$

$\rightsquigarrow$ perverse t-structure, can talk about mixed perverse sheaves.

Any subquotient of a mixed perverse sheaf is mixed

D^b_m is triangulated & (mixed perverse sheaves) is abelian.

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$K_0 \in D^b_m(X_0, \overline{\mathbb{Q}_\ell})$

<u>Def</u> 1) K_0 has weights $\leq w$ iff $\forall i \ H^iK_0$ has weights $\leq w+i$.
- will write $K_0 \in D_{\leq w}(X_0, \overline{\mathbb{Q}_\ell})$

Thus $D_{\leq w}[1] = D_{\leq w+1}$ & $K_0 \in D_{\leq w} \Leftrightarrow \forall x, \ i_x^* K \in D_{\leq w}$: pointwise definition.

2) $K_0 \in D_{\geq w}$ iff $DK_0 \in D_{\leq -w}$

[Hard] <u>Theorem</u> (Deligne) i) $f_!, f^*$ respect $\mathcal{D}_{\leq w}$

i*) $f^!, f_*$ respect $\mathcal{D}_{\geq w}$

ii) $\mathcal{D}_{\leq w} \otimes \mathcal{D}_{\leq w'} \subset \mathcal{D}_{\leq w+w'}$

ii*) $\underline{R\mathcal{H}om}(\mathcal{D}_{\leq w}, \mathcal{D}_{\geq w'}) \subset \mathcal{D}_{\geq -w+w'}$

iii) $D : \mathcal{D}_{\leq w} \longleftrightarrow \mathcal{D}_{\geq w}$

Hard part (Weil II) : Show $f_!$ statement...

$$\begin{tikzcd} X/\overline{\mathbb{F}}_q \arrow[loop left, "\mathrm{Frob}"] \arrow[d] \\ X_0/\mathbb{F}_q \end{tikzcd}$$

<u>Proposition</u> i) $K_0 \in \mathcal{D}_{\leq w}$, $L_0 \in \mathcal{D}_{> w}$. Then $\mathrm{Hom}^i(K_0, L_0) = 0$, $i > 0$.

ii) $K_0 \in \mathcal{D}_{\leq w}$, $L_0 \in \mathcal{D}_{\geq w}$. Then (base change to closure of field)
$\mathrm{Hom}^i(K, L)^{\mathrm{Frob}} = 0$, $i > 0$.

<u>Proof</u> Denote $a : X_0 \longrightarrow \mathrm{Spec}\, \mathbb{F}_q$.
$Ra_* \underline{R\mathcal{H}om}(K_0, L_0) \in \mathcal{D}_{> 0}$ by Theorem
$\mathrm{Hom}^i(K, L)$ has weights $> i$, $i \geq 0 \Rightarrow$ weights > 0

Recall $\star$: $0 \longrightarrow (\mathrm{Hom}^{i-1}(K, L))_F \longrightarrow \mathrm{Hom}^i(K_0, L_0) \longrightarrow \mathrm{Hom}^i(K, L)^F \to 0$

weights $> 0 \Rightarrow$ no invariants or coinvariants for F
$\Rightarrow$ 1st & 3rd terms vanish $\Rightarrow$ i.

ii $\Rightarrow$ in particular $\mathrm{Hom}^i(K_0, L_0) \longrightarrow \mathrm{Hom}^i(K, L)$ is zero...

[Hard] <u>Theorem</u> ($\overset{\text{B-Be}}{\text{Deligne}}$) $\mathcal{F}_0$ mixed perverse sheaf $/X_0$. Then $\mathcal{F}_0 \in \mathcal{D}_{\geq w}$
$\Longleftrightarrow$ $\forall\, U_0 \longrightarrow X_0$ affine étale, $H^0(U, \mathcal{F})$ has weights $\geq w$ (over alg. closure).

Note ($\Rightarrow$) easy from previous thm : for étale map $f^! = f^*$..
($\Leftarrow$) hard : induction on dimension, vanishing cycles...

<u>Theorem</u> $\mathcal{F}_0$ perverse $\in \mathcal{D}_{\geq w} \Rightarrow$ same true for any subquotient.
(same for $\leq w$ by duality.)

<u>Proof</u> Quotient case $\mathcal{F}_0 \twoheadrightarrow \mathcal{G}_0$: $\forall\, U_0 \to X_0$ affine étale
$H^0(U_0, \mathcal{F}_0) \twoheadrightarrow H^0(U_0, \mathcal{G}_0)$ (cohomologies of perverse sheaf on affine étale vanish in deg > 0)
so $\mathcal{F}_0 \in \mathcal{D}_{\geq w} \Rightarrow \mathcal{G}_0 \in \mathcal{D}_{\geq w}$.

Sub case $\mathcal{G}_0 \hookrightarrow \mathcal{F}_0$:

$H^{-1}(U_0, \mathcal{F}_0/\mathcal{G}_0) \to H^0(U_0, \mathcal{G}_0) \to H^0(U_0, \mathcal{F}_0)$

can assume $w=0$ by shifting by constant systems on X, nonconstant on X_0 with given Frob action.

By quotient case $H^{-1}(U_0, \mathcal{F}_0/\mathcal{G}_0)$ has weights ≥ -1. ~~=~~ $\Rightarrow$ H^0 ~~$(U_0, \mathcal{G}_0) \to$~~ $\mathcal{G}_0$ has weights ≥ -1.

<u>Trick</u>: now take $\mathcal{G}_0 \otimes \mathcal{G}_0 \hookrightarrow \mathcal{F}_0 \otimes \mathcal{F}_0$

$\Rightarrow \mathcal{G}_0 \otimes \mathcal{G}_0$ has weights $\geq -1 \Rightarrow \mathcal{G}_0$ has weights $\geq -\frac{1}{2}$

$\Rightarrow \geq 0$!

■

<u>Cor</u> $j: U_0 \hookrightarrow X_0$ affine, $\mathcal{F}_0$ perverse$/U_0$, weights $\leq w$ ($\geq w$)

$\Rightarrow j_{!*}\mathcal{F}_0$ has same $\leq w$ ($\geq w$).

<u>Proof</u>. $j_! \mathcal{F} \twoheadrightarrow j_{!*}\mathcal{F}_0 \hookrightarrow j_* \mathcal{F}_0$

$j_!$ preserves $\leq w$, j_* preserves $\geq w$ ■

<u>Cor</u> simple mixed perverse sheaves are pure

<u>Proof</u> $\mathcal{F}_0 = j_{!*}\mathcal{L}_0[d]$, $\mathcal{L}_0/U_0$ affine subspace of subvariety

$\mathcal{L}_0$ irred, mixed $\Rightarrow \mathcal{L}_0$ pure. ■

<u>Theorem</u> Every mixed perverse sheaf $\mathcal{F}_0$ has a unique increasing filtration W s.t. $Gr_i^W \mathcal{F}_0$ is pure of weight i & every morphism is strictly compatible with this filtration

<u>Proof</u>: use <u>Fact</u> In abelian category $\mathcal{A}$ s.t. every object has finite length, & we have a partition $\left\{\begin{matrix}\text{isom classes}\\ \text{of simple objects}\end{matrix}\right\} = S^+ \sqcup S^-$ s.t. $Ext^1(S^-, S^+) = 0$

$\Rightarrow$ subcategories $\mathcal{A}^-, \mathcal{A}^+ \subset \mathcal{A}$, get unique filtration on every $A \in \mathcal{A}$ $A_- \subset A \twoheadrightarrow A_+$

& every morphism is strictly compatible.

So use S^- = simple objects weight $\leq w$ $S^+ > w$

etc. ■

<u>Theorem</u> $\mathcal{F}_0/X_0$ pure perverse $\Rightarrow$ $\mathcal{F} \simeq \bigoplus_{\mathcal{L}_0/U_0} j_{!*}(\mathcal{L}[d])$ irred loc sys

Proof Take $\left(\sum \text{all perv. simple subsheaf in } \mathcal{F}\right) = \mathcal{F}' \subset \mathcal{F}$

$\mathcal{F}'$ defined over $\mathbb{F}_q$ – closed under Frob.

$0 \to \mathcal{F}_0' \to \mathcal{F}_0 \to \mathcal{F}_0/\mathcal{F}_0' \to 0$ Both pure of same weight – this extension thus dies over algebraic closure

~ since $\mathrm{Hom}^1(\mathcal{F}_0/\mathcal{F}_0', \mathcal{F}_0') \xrightarrow{0} \mathrm{Hom}^1(\mathcal{F}/\mathcal{F}', \mathcal{F}')$

$\Rightarrow \mathcal{F} = \mathcal{F}' \oplus \underline{\quad}$, $\underline{\quad}$ must have a simple subobject $\Rightarrow$ contradiction. ■

* **Theorem** $K \in \mathcal{D}_{\leq w} \iff {}^pH^iK_0$ has weights $\leq w+i$ (same for $\geq w$).

~~Theorem~~

K is *pure* of weight w iff H^iK pure of weight $w+i$ iff ${}^pH^iK$ pure of weight $w+i$.

Theorem K_0 pure $\Rightarrow K = \bigoplus {}^pH^iK[-i]$ (Decomposition).

Proof (using previous thm): consider exact Δ

$${}^p\tau_{<i} K_0 \to {}^p\tau_{\leq i} K_0 \to {}^pHK_0[-i] \Rightarrow {}^p\tau_{<i}K_0[1]$$

Assume $w=0$. We wish to show middle term splits

All are pure of weight 0 by cohomology description

$\mathrm{Hom}^1(\text{wt } 0, \text{wt } 0)$ goes to 0 over alg. closure

So thick arrow goes to 0 & sequence splits ■

Proof of * : $\Leftarrow$ is exercise.

K. Rietsch – Springer Correspondence 12/4

Setting: G reductive linear alg grp $/k = \bar{k}$ ($k = \mathbb{C}$ usually)

Originally (1979, Invent.) Char $k = p > 0$

$u \in G_{uni}$ unipotent $\Rightarrow$

$$\mathcal{B}_u = \{B \in \mathcal{B} \mid u \in B\} \text{ Springer fiber}$$

$$A_G(u) = Z_G(u)/Z_G^\circ(u)$$

Ref: Shoji. X 168

$\mathcal{B} = G/B$

Springer resolution: $\tilde{G}_{uni} = \{(u, B) \in G_{uni} \times \mathcal{B} \mid u \in B\}$
$$\begin{array}{c} \tilde{G}_{uni} \\ \pi_1 \downarrow \\ G_{uni} \end{array}$$
resolution: small map from smooth variety (birational map). $\tilde{G}_{uni} \simeq T^* \mathcal{B} = G/B \times_B \mathfrak{b}^*$.

$H^i(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$ gets $A_G(u)$ action. Springer constructs rep of $A_G(u) \times W$

Isotypic decomp: $H^{top}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\rho \in A_G(u)^\wedge} V_{u,\rho} \otimes \rho$

$V_{u,\rho}$ multiplicities of reps.

<u>Theorem</u> The $V_{u,\rho}$ are irreps (possibly 0) of W and $\{(u,\rho) \mid V_{u,\rho} \neq 0,\ u \text{ up to conjugacy}\} \xrightarrow{\sim} W^\wedge$.

We'll denote the set of such pairs mod conjugacy $\mathcal{N}_G$.

Lusztig
$$\begin{array}{ccc} \tilde{G}_{reg} & \hookrightarrow & \tilde{G} = \{(g, B) \in G \times \mathcal{B} \mid g \in B\} \\ \downarrow & & \pi \downarrow \\ G_{reg} & \hookrightarrow & G \end{array}$$
Grothendieck-Springer map

G_{reg}: reg s.s. elements.

$\tilde{G}_{reg} = \{(g, xT) \in G_{reg} \times G/T \mid x^{-1}gx \in T_{reg}\}$

- determined for s.s. by pairs (g, xB): g determining torus.

W acts on $\tilde{G}_{reg}$ by $(g, xT) \cdot w = (g, xwT)$ with orbits the fibers of the map.

$\mathcal{L} := (\pi_0)_* \overline{\mathbb{Q}}_\ell$ pushforward of constant sheaf $\Rightarrow$ loc sys on G_{reg} with left W action rank $= |W|$

$\Rightarrow$ take IC extension, $IC(G, \mathcal{L})[\dim G] = j_{!*}\mathcal{L}$ is perverse sheaf on G with action

<u>Prop (Lusztig)</u> $j_{!*}\mathcal{L} \simeq \pi_* \overline{\mathbb{Q}}_\ell[n]$

<u>Proof</u> π is small. (later..?) ∎

Note : G connected

$\Rightarrow$ get W action on stalks $\mathcal{H}^i_x(\pi_* \bar{\mathbb{Q}}_\ell) = H^i(\mathcal{B}_x, \bar{\mathbb{Q}}_\ell)$
$\mathcal{B}_x = \pi^{-1}(x)$.

<u>Special case</u> $x = e$ identity $\Rightarrow \mathcal{B}_e = \mathcal{B}$, get W action on $H^*(\mathcal{B}) \overset{\sim}{\underset{\varphi}{\leftarrow}} H^*(G/T)$ (affine fibers)
- classical W action. This is isomorphic to $\bar{\mathbb{Q}}_\ell[W]$
- as W-module get regular rep from $H^*(G/T)$
- and φ is equivariant under W action

<u>Borho-MacPherson's Theorem</u> set $\nu = \dim \mathcal{B}$.

i. $\pi_* \bar{\mathbb{Q}}_\ell[2\nu]|_{G_{uni}} \simeq \oplus V_{C,\mathcal{E}} \otimes IC(\bar{C}, \mathcal{E})[\dim C]$
where C is unipotent class, $\mathcal{E}$ is local system on C, G equivariant
... i.e. $\pi^* \bar{\mathbb{Q}}_\ell[2\nu]|_{G_{uni}}$ is semisimple ...

ii. The $V_{C,\mathcal{E}}$ are irreps of W (or 0) and
$\{(C,\mathcal{E}) \mid V_{C,\mathcal{E}} \neq 0\} \xrightarrow{\sim} W^\wedge$.

Remark $(C, \mathcal{E}) \leftrightarrow (u, \rho)$ — Note $\pi : \tilde{G} \to G$ is G-equivariant. G-equiv loc systems on a homogeneous G-space $C \ni u \leftrightarrow$ reps of $A_G(u) = Z_G(u)/Z^0_G(u)$:

$G \times C \to C \qquad \pi_1(G) \to \pi_1(C) \to \pi_0(G_u) \to \pi_0(G) = 1$
— explains this over $\mathbb{C}$. $\qquad \pi_0(G_u) = A_G(u)$

<u>Lemma</u> : $\pi_1 : \tilde{G}_{uni} \to G_{uni}$ is semismall

Consider $Z' = \{(g, B_1, B_2) \mid g \in G_{uni}, g \in B_1, g \in B_2\}$

$p : Z' \to$; $q : Z' \to G_{uni}$

G-orbits $\coprod_W \mathcal{O}_w = \mathcal{B} \times \mathcal{B}$ $\qquad G_{uni}$

vector bundle over $\mathcal{O}_w$

$$Z' = \coprod Z_w, \quad Z_w = p^{-1}(\mathcal{O}_w) \text{ decomp into irred components}$$

$$\dim Z_w = \underbrace{\dim \mathcal{O}_w}_{\dim G - \dim(B \cap wBw^{-1})} + \dim(U \cap wUw^{-1}) = 2\nu \quad (U = N)$$

$$2\nu \geq \dim q^{-1}(C) = \dim C + 2\dim \mathcal{B}_u \quad (\text{some } u \in C)$$

$\Rightarrow \dim \mathcal{B}_u \leq \nu - \frac{1}{2}\dim C$ .. in fact they're equal —

- uses classification of unipotent classes.

Take u regular unipotent, over there map is 1-1.

$\Rightarrow \dim G_{uni} = \dim Z' = 2\nu$.

Semismallness: need $\dim \underbrace{\{x \in G_{uni} \mid \dim \mathcal{B}_x \geq \tfrac{i}{2}\}}_{V} \leq \dim G_{uni} - i$

- suppose otherwise,

$\Rightarrow \dim q^{-1}(V) > 2\tfrac{i}{2} + (2\nu - i) > 2\nu$ contradiction.

Proof of Borho-MacPherson: we know $\pi_* \overline{\mathbb{Q}}_\ell|_{G_{uni}}[2\nu]$ is G-equiv perverse sheaf $= \pi_{1*} \overline{\mathbb{Q}}_\ell[2\nu] = K_1$ (flat base change). It is semisimple since π_1 is proper $\Rightarrow$ decompose it as in i).

ii) W acts on K_1 by autos, must preserve simple parts $\Rightarrow$ acts on multiplicity spaces.

$\mathbb{Q}[W] \xrightarrow{\alpha} \operatorname{End} K_1 = \oplus \operatorname{End} V_{C,\mathcal{E}}$

Claim: α is isomorphism.

$\Rightarrow$ each occurs once and each is irreducible since it's surjective.

Injectivity: localize at identity $e \in G$:

$$\mathbb{Q}[W] \xrightarrow{\alpha} \operatorname{End}(K_1) \to \oplus \operatorname{End} H^i(\mathcal{B})$$

with the composite labelled β. β is injection of W-modules from before $\Rightarrow \alpha$ is injective. (β is regular rep, has no kernel.)

Surjectivity: Show $\dim(\operatorname{End} K_1) \leq |W|$

$\dim \operatorname{End} K_1 = \sum \dim (V_{C,\mathcal{E}})^2$. Write $d_x = \dim \mathcal{B}_x$.

Look at top cohomology, localize decomposition:

$$H^{2d_x}(\mathcal{B}_x) \simeq \oplus V_{C,\mathcal{E}} \otimes \mathcal{H}_x^{\underline{2d_x - 2\nu + \dim C}} IC(\overline{C}, \mathcal{E})$$

numerator here $= 0$

$H^{2d_x}(\mathcal{B}_x)$ has $A_G(x)$ action.

Multiplicity $(\rho : H^{2d_x}(\mathcal{B}_x)) \geq \dim V_{C_x \mathcal{E}_\rho}$:

Via $\rho \leftrightarrow \mathcal{E}$:

$\mathcal{H}_x^0 IC(\overline{C}, \mathcal{E}) = (\mathcal{E}_\rho)_x$ for $x \in C$, which is the representation ρ: it is the stalk of $\mathcal{E}_\rho$.

$$\Rightarrow \sum \dim^2(V_{C,\varepsilon}) \le \sum (m_{x,\rho})^2 = \dim \mathrm{End}_{A_G(x)}\left(H^{2d_x}(\mathcal{B}_x, \bar{\mathbb{Q}}_\ell)\right)$$

$$\| $$

$$\sum_{x \in G_{uni}/\sim} \left| I(\mathcal{B}_x)\times I(\mathcal{B}_x)/A_G(x) \right| = \dim\left(\mathrm{End}_{\mathbb{C}} H^{2d_x}(\mathcal{B}_x, \bar{\mathbb{Q}}_\ell)\right)^{A_G(x)}$$

$\|$ $\qquad$ $I(\mathcal{B}_x)$ = irred components of Springer fiber

$|W|$ $\rightsquigarrow$ follows from study of Z'

$$\begin{tikzcd} & Z' \arrow[dl] \arrow[dr] & \\ \mathcal{B}\times\mathcal{B} & & G_{uni} \supset C \end{tikzcd}$$

Preimages of C, $Z^{-1}(C) = Z_D$, so made up of irred components

so $|W| = \sum_C |I(q^{-1}(C))|$

but $q^{-1}(C) = G \times_{Z_G(u)} \mathcal{B}_u \times \mathcal{B}_u$ $\Rightarrow$ $A_G(u)$ orbits of components of $I(\mathcal{B}_u)$

$I(q^{-1}(C)) = (I(\mathcal{B}_u) \times I(\mathcal{B}_u))/A_G(u)$

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Theorem $K_0 \in D^b_m(X_0, \bar{\mathbb{Q}}_\ell)$ has weights $\le w$ $\Leftrightarrow$ $\forall i$ $\mathcal{H}^i K_0$ has weights $\le w+i$ (same for $\ge w$ ..)

Lemma Assume $\mathcal{F}_0$ perverse mixed $/X_0$. Then $\mathcal{F}_0$ has weights $\le w$ $\Leftrightarrow$ $\forall Y_0 \subset X_0$ irred subvariety of dim d $\exists U_0 \subset Y_0$ open dense (Zariski), s.t. weights of $(\mathcal{H}^{-d} \mathcal{F}_0|_{U_0}) \le w-d$.

[Can check weights pointwise, on open & closed pts — this tells us we only have to check in dimension $\le d$ — note $-d$ is perversity of Y_0]

Proof ($\Rightarrow$) is immediate.

($\Leftarrow$) Assume this is not so $\Rightarrow$ $\exists$ simple perverse quotient $\mathcal{G}_0$, pure of weight $w_1 > w$ (by canonical filtration with pure quotients.)

$\mathcal{G}_0 = j_{!*} \mathcal{L}_0[d]$ $\qquad$ $\mathcal{L}_0$ irred loc sys on $U_0 \subset Y_0 \subset X_0$ open smooth affine — as we wish.

$\mathcal{L}_0$ pure of weight $w_1 - d$.

Assume we have a kernel $K_0 \subset \mathcal{F}_0 \twoheadrightarrow \mathcal{G}_0$

$\underline{H}^{-d} \mathcal{F}_0 \to \underline{H}^{-d} \mathcal{G}_0 \to \underline{H}^{-d+1} K_0$

latter vanishes over general point of Y_0 — say on U_0.

$\Rightarrow \underline{H}^{-d} \mathcal{F}_0|_{U_0} \twoheadrightarrow \underline{H}^{-d} \mathcal{G}_0|_{U_0} \to 0$

So $\underline{H}^{-d} \mathcal{G}_0|_{U_0}$ has weights $\leq w$ but we know it's pure of a higher wt $\Rightarrow$ contradiction ∎

<u>Proof of Theorem</u> $\Leftarrow$) immediate: $A \to X \to B$ exact triangle with A, B of wts $\leq w$ $\Rightarrow$ same for X. We can construct K_0 from its cohomologies in this fashion...

$\Rightarrow$) ~~Consider~~ Proof by descending induction in i that ${}^pH^i K_0 \in \mathcal{D}_{\leq w+i}$

Assume true $i > n$. Then ${}^p\tau_{>n} K_0 \in \mathcal{D}_{\leq w}$

Using shifts assume $n = 0$.

$K_0, {}^p\tau_{>0} K_0 \in \mathcal{D}_{\leq w} \overset{?}{\Longrightarrow} {}^pH^0 K_0 \in \mathcal{D}_{\leq w}$

Use exact sequence (Δ) ${}^p\tau_{\leq 0} K_0 \to K_0 \to {}^p\tau_{>0} K_0$

$\Rightarrow {}^p\tau_{\leq 0} K_0 \in \mathcal{D}_{\leq w}$

Consider ${}^p\tau_{>0} K_0 \to {}^p\tau_{\leq 0} K_0 \to {}^pH^0 K_0$

Restrict to $Y_0 \subset X_0$ irred of dim $= d$.

$$\underline{H}^{-d}\, {}^p\tau_{\leq 0} K_0|_{U_0} \to \underline{H}^{-d}\, {}^pH^0 K_0|_{U_0} \to \underline{H}^{-d+1}\, {}^p\tau_{<0} K_0|_{U_0}$$

wt $\leq w-d$ $\qquad\qquad$ $\overset{\shortparallel}{0}$

$\Rightarrow \underline{H}^{-d}\, {}^pH^0 K_0|_{U_0}$ has wts $\leq$ ~~w~~ $w-d$

$\Longrightarrow$ (lemma) ${}^pH^0 K_0 \in \mathcal{D}_{\leq w}$. ∎

Passing from characteristic p to $\mathbb{C}$

Suppose $B = \varinjlim B_i$, all noetherian, $\begin{array}{c} X \\ \downarrow \\ B \end{array}$ of finite presentation

(scheme or morphism of fin type, or finite sheaf, or...)

$\Rightarrow$ This is obtained by base change from some $\begin{array}{c} X_i \\ \downarrow \\ B_i \end{array}$, $i \gg 0$.
X_i essentially unique - two such choices become isomorphic for i large enough.

$X/\mathbb{C}$ scheme of finite type. $\mathbb{C} = \varinjlim A$ over A of finite type $/\mathbb{Z}$ & $\operatorname{Spec} A / \operatorname{Spec} \mathbb{Z}$ smooth
(pass to cofinal family: pass from A to $A[\frac{1}{f}]$, shrinking Spec to make it larger.

X comes from $\begin{array}{c} X_S \\ \downarrow \\ S \end{array}$ $S = \operatorname{Spec} A$, A large enough.
X_S scheme of fin. type. If X was smooth, connected etc can make X_S be such

$f : X \to Y / \mathbb{C}$ also comes from such base change for big enough A.

$\mathcal{T}$ stratification of X also comes from $\mathcal{T}_S$ of X_S, can assume strata smooth & geometrically connected.

$\mathcal{F}/X$ - $\mathbb{Z}/\ell^n$ sheaf comes from $\mathcal{F}_S / X_S$
(constructible sheaves.) NOT true nec. for $\mathbb{Z}_\ell$ sheaves.

Thm (SGA $4\frac{1}{2}$): $f : X \to Y / \mathbb{C}$, $\mathcal{F}/X$ constructible $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaf
$\Rightarrow$ $\begin{array}{c} \mathcal{F}_S \\ | \\ X \end{array} \to Y_S$. Then $\exists\, U \subset S$ s.t. $\forall\, g : U' \to U$

$$\begin{tikzcd} X_{U'} \arrow[r, "g"] \arrow[d, "f"'] & X_U \arrow[d] \\ Y_{U'} \arrow[r, "g"'] & Y_U \end{tikzcd}$$

we have $g^* R^q f_* \mathcal{F}_U \cong R^q f_* g^* \mathcal{F}_U$
$(\mathcal{F}_U = \mathcal{F}_S|_U)$ &
$R^q f_* \mathcal{F}_U$ constructible.

Moral: our functors f_*, $f_!$, f^*, $f^!$, $\underline{\operatorname{Tor}_p}$, $\underline{\operatorname{Ext}^p}$ for $\mathbb{Z}/\ell^n\mathbb{Z}$ constructible sheaves descend to S for A large enough, commuting with base change (on U).

Recall that the derived category of constr $\overline{\mathbb{Q}}_\ell$ sheaves was a limit of $\mathcal{D}^b_{\mathcal{T},\mathcal{L}}(X, \overline{\mathbb{Q}}_\ell) \subset \mathcal{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$
$\mathcal{T}$ stratification of X, $\mathcal{L}$ family of irred loc sys on strata of $\mathcal{T}$.

- also replacing $\overline{\mathbb{Q}}_\ell$ by $\mathbb{Z}/\ell^n\mathbb{Z}$ in above.

$X, T, \mathcal{L} / \mathbb{C} \rightsquigarrow X_S, T_S, \mathcal{L}_S$

Need $\forall F, G$ of the form $j_! L$, $j : T_S \hookrightarrow X_S$, $L \in \mathcal{L}_S(T)$
$\underline{Ext}^q(F, G)$ is compatible with all base changes $S' \to S$
& $R^p a_* \underline{Ext}^q(F, G)$ loc. constant, compatible with base change. — true if we shrink S.

$S = \operatorname{Spec} A$, choose dvr, strictly Henselian V s.t. $A \subset V \subset \mathbb{C}$.

$$\begin{array}{ccc} \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} V \\ \Psi & & \Psi \\ x & \longleftarrow & s \end{array}$$

with residue field $k(s)$ finite.

Henselian: category of étale sheaves on $\operatorname{Spec} V \leftrightarrow$ étale sheafs on closed point. Strictness: $k(s)$ alg closed, $k(S) = \overline{k(x)}$.

Now pull back to $V \Rightarrow X_V, T_V, \mathcal{L}_V$. Take special fiber $X_s, T_s, \mathcal{L}_s$ (small s).

$$X \xrightarrow{u} X_V \xleftarrow{i} X_s \qquad u \text{ base change } V \subset \mathbb{C}.$$

Lemma
$$D^b_{T,\mathcal{L}}(X, \mathbb{Z}_\ell) \xleftarrow{u^*} D^b_{T,\mathcal{L}}(X_V, \mathbb{Z}_\ell) \xrightarrow{i^*} D^b_{T,\mathcal{L}}(X_s, \mathbb{Z}_\ell)$$
are equivalences of categories

Proof For $\mathbb{Z}/l^n\mathbb{Z}$ sheaves we have
$H^p(\_, \underline{Ext}^q(F, G))$ are the same for X, X_V, X_s. ■

Now consider perverse sheaves: $X, T, \mathcal{L}$, $L \in \mathcal{L}(T)$
$j : T \hookrightarrow X$, $R^q j_* L|_{T'} \in \langle \mathcal{L}(T') \rangle$
needed for def of perverse t-structure (true after refinement). — can then achieve this over S after shrink, so that $R^q j_*$ commutes with base change $\Rightarrow$ so our equivalences of categories (lemma) are compatible with perverse t-structures.
(Perv. truncation uses $R^q j_*$ et al.)

<u>Definition</u> $X/\mathbb{C}$, F simple perverse sheaf of $\mathbb{C}$-vect spaces$/X(\mathbb{C})$
$\Rightarrow$ F is of geometric origin if $F \in$ smallest set containing $\mathbb{C}$ on pt & ~~which~~ is stable under taking constituents of ${}^pH^i T$, $T = Rf_*, Rf_!, Rf^*, Rf^!$ $(f: X \to Y/\mathbb{C})$, ${}^pH^i(A \otimes^L B)$, ${}^pH^i \underline{RHom}(A,B)$.

<u>Definition</u> $K \in D_c^b(X(\mathbb{C}), \mathbb{C})$ is semisimple of geometric origin if $K \simeq \oplus F_i[n_i]$, F_i simple of geom. origin

<u>Theorem (Decomposition Thm.)</u> : If $f: X \to Y/\mathbb{C}$ proper, K semisimple of geometric origin $\Rightarrow$ $Rf_* K$ is such.

<u>Lemma</u> F simple perv. $\overline{\mathbb{Q}_\ell}$-sh. of geom origin
$S = \mathrm{Spec}(A \subset \mathbb{C})$ sufficiently large, then under $D_{c,\ell}(X, \overline{\mathbb{Q}_\ell}) \leftrightarrow D_{c,\ell}(X_s, \overline{\mathbb{Q}_\ell})$
$F \longleftrightarrow F_S/X_S$ obtained from F_0/X_0 with F_0 pure $\quad \mathbb{F}_q$

<u>Proof</u> True for constant sheaf, must check after applying various functors, take components of resulting mixed perverse sheaves $\Rightarrow$ pure sheaf again. $\square$

$K - K_s \qquad \Rightarrow f_* K_s = \oplus \ldots$
$\quad K_0$ pure $\qquad f_* K_0$ pure

S. Kirillov - <u>Kazhdan-Lusztig Theory</u> 12/9

$k = \overline{\mathbb{F}_p}$ $\quad F$ - Frobenius $\quad G$ - reductive connected alg group $/k$
B - Borel, T - torus, W - Weyl $\quad G, B_0, T$ split - defined $/\mathbb{F}_p$
Maximal split situation : F acts trivially on W.

$\mathcal{B}$ - variety of all Borels. $\quad \mathcal{B} \times \mathcal{B}/G \simeq W$
$(B_1, B_2) \rightsquigarrow (B_0^w, B_0)$: write $B_1 \xrightarrow{w} B_2$

$\mathcal{B} \times \mathcal{B} = \bigcup O_w$

$B_1 \xrightarrow{w_1} B_2 \xrightarrow{w_2} B_3 \not\Rightarrow B_1 \xrightarrow{w_1 w_2} B_3$: true

if $\mathrm{length}(w_1 w_2) = \mathrm{length}(w_1) + \mathrm{length}(w_2)$

$\mathcal{B} = \bigcup_w \mathcal{B}_w$

$\mathcal{B}_w = \{B \mid B \xrightarrow{w} B\} \simeq k^{\ell(w)}$

$\overline{\mathcal{B}_w} = \bigcup_{y \le w} \mathcal{B}_y$.

<u>Hecke algebra</u>

1. $P = \{$functions $f : \mathcal{B}^{F^r} \to \overline{\mathbb{Q}}_\ell$

$q = p^r$ $\mathcal{B}^{F^r}$ Borels defined $/ \mathbb{F}_q$.

$\tau_w : P \to P$

$$(\tau_w f)(B) = \sum_{B' \xrightarrow{w} B} f(B')$$

<u>Theorem</u> $\mathcal{H} = \bigoplus \mathbb{Q} \langle \tau_w \rangle$ satisfies:

1. $\mathcal{H} \simeq \mathrm{End}_{G^{F^r}} P$
2. s-simple reflections $\Rightarrow$ $\mathcal{H}$ is generated by τ_s

$\tau_w \cdot \tau_s = \tau_{ws}$ $\quad \ell(ws) = \ell(w) + 1$

$\tau_s^2 = q\,\cancel{\tau_s} + (q-1)\tau_s$

2. $\mathcal{H} = G^{F^r}$-invariant functions on $\mathcal{B}^{F^r} \times \mathcal{B}^{F^r}$

$$(f_1 \cdot f_2)(x) = \sum_{\substack{y \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \\ p_{13}(y) = x}} f(p_{12}(y)) f(p_{23}(y))$$

$\mathcal{B}^{F^r} \times \mathcal{B}^{F^r} \times \mathcal{B}^{F^r}$

$\downarrow p_{ij}$

$\mathcal{B}^{F^r} \times \mathcal{B}^{F^r}$

$T_w =$ char. functions of O_w satisfy same relations as τ_s.

$\overline{} : \mathcal{H} \to \mathcal{H}$. Structure constants are polynomials in q, so we can consider $\mathcal{H}$ as an algebra over $\mathbb{Q}[q^{\pm 1}]$.

$\overline{} : T_w \longmapsto T_{w^{-1}}^{-1}$, $\quad q \longmapsto q^{-1}$.

$\Rightarrow \quad T_i^{-1} = q^{-1} T_i + (q^{-1} - 1)$

setting $\tilde{T}_w = + q^{-\frac{\ell(w)}{2}} T_w \Rightarrow \overline{\tilde{T}_w} = \tilde{T}_w + \sum_{y < w} (\ldots) \tilde{T}_y$

Define $R_{y,w} \in \mathbb{Z}[q]$ by $\overline{T_w} = \sum_{y \le w} q^{-\ell(y)} R_{y,w}(q^{-1}) T_y$

<u>Theorem</u> (Kazhdan-Lusztig) $\exists!$ basis C'_w in $\mathcal{H}$ s.t.

1. $\overline{C'_w} = C'_w$ $\quad$ 2. $C'_w = \tilde{T}_w + \sum_{y < w} (\ldots) \tilde{T}_y$

with $(\ldots)$ polys in $q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}]$.

We rewrite the last formula as $C'_w = \sum_{y \le w} q^{\frac{\ell(y)-\ell(w)}{2}} P_{y,w} T_y$

where $P_{y,w} \in \mathbb{Z}[q]$, $\deg P_{y,w} \le \frac{\ell(w)-\ell(y)-1}{2}$

(follows from estimates on powers of q in C'_w ...)

— Kazhdan-Lusztig polynomials.

$P_{y,w}(1)$ describe multiplicity $[M_{w\lambda} : L_{y\cdot\lambda}]$

$\lambda \in P_+$ regular.

$\mathcal{B} = \bigcup \mathcal{B}_w$. Denote $(\bar{\mathbb{Q}}_\ell)_w$ constant sheaf on $\mathcal{B}_w$.

$\mathcal{B}_w \xrightarrow{j} \bar{\mathcal{B}}_w$. Set $\mathcal{F}_w = j_{!*}(\bar{\mathbb{Q}}_\ell)_w$

$\mathcal{H}^i_x(\mathcal{F}_w)$ $x \in \mathcal{B}_y$, $y \le w$: $\mathcal{F}_w$ is coh. constructible w.r.t. $\mathcal{B}_w$

<u>Theorem</u> ①. $\mathcal{H}^i_x \mathcal{F}_w = 0$; i odd.

②. Eigenvalues of F^{*r} on $\mathcal{H}^{2i}_x(\mathcal{F}_w)$ ($x \in \mathcal{B}_y^{F^r}$) are p^{ir}.

③ $P_{y,w}(q) = \sum_i \dim \mathcal{H}_x^{2i}(\mathcal{F}_w) \cdot q^i$

$= \mathrm{Tr}(F^{*r}, \mathcal{H}^*_x(\mathcal{F}_w))$

$\Rightarrow$ all coefficients of K-L polynomials are <u>non-negative</u> integers.

<u>Proof</u> Idea: find U - nbhd of $\mathcal{B}_y$ in $\bar{\mathcal{B}}_w$ and calculate $\mathrm{Tr}(F_r^{*r}, H^*_c(U, \mathcal{F}_w)) = A$.

First, calculation & def of U:

$\alpha^y = \{ B \mid B \text{ is opposite to } B_0^{yw_0} \} \simeq k^{\ell(w_0)}$

w_0 = long word $\qquad B \xrightarrow{w_0} B_0^{yw_0}$

- big cell

Set $U = \bar{\mathcal{B}}_w \cap \alpha^y$ - nbhd of $\mathcal{B}_y$, affine set

- can choose so that $\alpha^y \ni 0 \leftrightarrow B_0^y$

$U = \coprod_{y \le z \le w} \mathcal{B}_z \cap \alpha^y$ (?)

(Lefschetz principle)

$\Rightarrow A = \sum_{p \in U^{F^r}} \mathrm{Tr}(F^{*r}, \mathcal{H}^*_p(\mathcal{F}_w)) =$

$= \sum_{y \le z \le w} \sum_{p \in (\mathcal{B}_z \cap \alpha^y)^{F^r}} \mathrm{Tr}(F^{*r}, \mathcal{H}^*_p(\mathcal{F}_w))$

Prove by induction on y; descending from w: so now assume proven for all $y < z \leq w$.

$$= \sum_{p \in (\mathcal{B}_y \cap a^y)^{F^r}} Tr(F^{*r}, \mathcal{H}^*_p(\mathcal{F}_w))$$

$$+ \sum_{y < z \leq w} \sum q^i \cdot |\mathcal{B}_z \cap a^y|^{F^r} \cdot \dim \mathcal{H}^{2i}_z(\mathcal{F}_w)$$

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$q = p^r$ **Recap** $\mathcal{F}_w = j_{!*}(\bar{\mathbb{Q}}_l)_w$ $\mathcal{B} = \bigcup \mathcal{B}_w$

We're proving: for $y \leq w$ $x \in \mathcal{B}_y^{F^r} \subset \mathcal{B}$.

$\Rightarrow$ $\mathcal{H}^i_x(\mathcal{F}_w) = 0$ i odd & eigenvals of F^r in $\mathcal{H}_x^{2i}(\mathcal{F}_w)$ are q^i.

We've fixed w and do induction on y.

$U = \overline{\mathcal{B}_w} \cap a^y$ - nbhd of $\mathcal{B}_y$ in $\overline{\mathcal{B}_w}$.

U is stable under the torus action & the only fixed point is the origin in the affine space a^y under some identification:

$(x_1, \ldots x_n) \to (a_i^{\lambda_i} x_i \ldots)$ $\lambda_i > 0$

We're calculating $Tr(F^r, H^*_c(U, \mathcal{F}_w))$
$= Tr(F^r, \mathcal{H}^*_x(\mathcal{F}_w)) + (\underset{y<z\leq w}{*})$

$U = \bigcup_{y \leq z \leq w} (\mathcal{B}_z \cap a^y)$

More precisely first term is $Tr(F^r, \mathcal{H}^*_x(\mathcal{F}_w)) \cdot \#(\mathcal{B}_y \cap a^y)^{F^r}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = q^{l(y)}$

\# of fixed points - can be calculated recursively to be some polynomials in q - rest of terms $*$ also polynomial in q

Now use Verdier duality:

$F^r \longrightarrow F^{-r}$ $\quad H^*_c \longrightarrow H^* \quad \Longrightarrow$

Tate twist in Verdier duality

$$\mathrm{Tr}(F^r, H^*_c(U, \mathcal{F}_w)) = q^{\ell(w)}\, \mathrm{Tr}(F^{-r}, H^*(U, \mathcal{F}_w))$$

But $\mathcal{F}_w$ is conic sheaf for torus action $\Rightarrow$ replace global homology by local homology at origin

$\Rightarrow \quad q^{\ell(w)}\, \mathrm{Tr}(F^{-r}, \mathcal{H}^*_{B^y}(\mathcal{F}_w))$

$B^y = x$ standard Borel twisted by y – fixed point of torus.

<u>Lemma</u> E. vals of Frobenius in $\mathcal{H}^i_{B^y}(\mathcal{F}_w)$ have absolute value $q^{i/2}$.

<u>Proof</u> Uses Deligne's result on purity & conical structure. ∎

This identity holds for all r

<u>Corollary</u> $\lambda = q^{i/2}$, and $\frac{i}{2} \in \mathbb{Z}$ (so no odd homologies). ∎

<u>Theorem</u> $\sum_i \dim \mathcal{H}^{2i}_x(\mathcal{F}_w)\, q^i = P_{y,w}(q)$

<u>Proof</u> Let $\mathcal{C}$ be the derived category of mixed sheaves on $\mathcal{B} \times \mathcal{B}$, constructible wrt orbit stratification. (constant on strata)

$\mathcal{C} \xrightarrow{\varphi} \mathcal{H}$ = functions on $\mathcal{B} \times \mathcal{B}$, G-invariant.

$\mathcal{F} \longmapsto f_{\mathcal{F}}(x) = \mathrm{Tr}(F^r, \mathcal{H}^*_x(\mathcal{F}))$

$(\mathbb{Q}_\ell)_w \longmapsto T_w$ char. function of an orbit.

L - Tate sheaf: constant sheaf with $F^r|_L = q$

<u>Theorem</u> Under φ, Verdier duality is identified with the complex involution $\bar{\ }$ on $\mathcal{H}$

<u>Proof</u> For sl_2: $\bigcirc$ $\quad D(\mathbb{Q}_\ell) = L^{-1}\mathbb{Q}_\ell$ ← const sheaf on $\mathcal{B}$

pass to functions $(T_\sigma + 1) \mapsto q^{-1}(T_\sigma + 1)$

$\Longleftrightarrow T_\sigma \mapsto T_\sigma^{-1}$ which gives ~~our~~ involution.

In general prove for reflections, extend by multiplication. (reflection case reduces to sl_2) ∎

Define convolution: $\begin{array}{c} \mathcal{B}\times\mathcal{B}\times\mathcal{B} \\ \downarrow\downarrow\downarrow \\ \mathcal{B}\times\mathcal{B} \end{array}$ If $\mathcal{F}, \mathcal{G}$ are sheaves in $\mathcal{C}$, $\mathcal{F} * \mathcal{G} := p_{13\,!}\,(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{G})$

($! = *$ since all is proper)

<u>Claim</u> $\varphi(\mathcal{F} * \mathcal{G}) = \varphi(\mathcal{F}) \cdot \varphi(\mathcal{G})$ & $D(\mathcal{F} * \mathcal{G}) = D(\mathcal{F}) * D(\mathcal{G})$.

First is equivalent to Lefschetz fixed point formula, trace theorem ...

Thus our sheaves $\mathcal{F}_w \overset{\varphi}{\longmapsto} \tilde{C}_w$ with

1. $\overline{\tilde{C}_w} = \tilde{C}_w$ 2. $\tilde{C}_w = T_w + \sum_{y < w} \ldots\ T_y$

(from support of $\mathcal{F}_w$)

3. Estimates on power of q in coefficients – come from <u>perversity</u> of sheaf – homologies vanish for some indices. Also use eigenvalues of Frob on perverse sheaves.

$\Longrightarrow$ $\tilde{C}_w$ is the K-L basis $\blacksquare$

<u>Cor</u> Coefficients of K-L polynomials are non-negative.

$P_{y,w}$ give nice basis of regular basis of Hecke algebra. For other reps, described by purely combinatorial data ("W-graphs") get similar combinatorial bases

SL_n case: these are $\mathrm{Ind}_{S_k \times S_{n-k}}^{S_n} \underline{1}$

$\longleftrightarrow$ Grassmannian ..

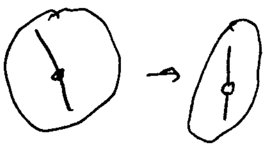
Then we get formulas for Hecke reps which depend well on q, which we can then specialize to ~~good~~ interesting values of q.

Elementary construction of perverse sheaves 12/13

MacPherson – Vilonen, Inv. Math 84/1986

X = top. stratified space. p = middle perv, $\forall$ strata even dim.

$P(X)$ = perverse sheaves, constructible

Inductive construction of $P(X)$ from category of local systems.
Suffices to construct $P(X)$ from $P(X \setminus S)$, loc system on S

where S is closed stratum (remove strata to make it closed)

$\mathcal{A}, \mathcal{B}$ categories, $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ functors
$\begin{tikzcd} \cdot \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "T"] & \cdot \end{tikzcd}$

T nat. transformation $\Longrightarrow$

$\mathcal{C}(F, G, T) = \{$ objects: (A, B, m, n) $A \in \mathcal{A}$, $B \in \mathcal{B}$

with
$$\begin{tikzcd} FA \arrow[rr, "TA"] \arrow[dr, "m"'] & & GA \\ & B \arrow[ur, "n"'] & \end{tikzcd}$$

Morphisms: $(A, B) \longrightarrow (A' B')$
pairs (a, b) with
$A \xrightarrow{a} A'$, $B \xrightarrow{b} B'$

$$\begin{tikzcd} FA \arrow[rr] \arrow[d] \arrow[dr] & & GA \arrow[d] \\ FA' \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[ur] & GA' \\ & B' \arrow[ur] & \end{tikzcd}$$

1. If $\mathcal{A}, \mathcal{B}$ abelian categories, F right exact, G left exact
$\Longrightarrow \mathcal{C}(F, G, T)$ is abelian, and forgetful functor
$\mathcal{C}(F, G, T) \longrightarrow \mathcal{A} \times \mathcal{B}$ is exact.

<u>Proof</u> $\mathcal{C}$ is additive from additivity of $\mathcal{A}, \mathcal{B}$.
Construct kernels: take
$$\begin{tikzcd} \ker F\cancel{A} \arrow[rr] \arrow[dr] & & \ker \cancel{GA} \\ & \ker b \arrow[ur] & \end{tikzcd}$$
- <u>not</u> in our category:
ker FA needs to come from category $\mathcal{B}$
- try
$$\begin{tikzcd} F(\ker a) \arrow[rr] \arrow[dr] & & G(\ker a) \\ & (\ker b) \arrow[ur] & \end{tikzcd}$$
in our category

and maps to above diagram
G left exact $\Longrightarrow G(\ker a) \simeq \ker G a$

. . .

Ex. Given two open sets, take our functors to be
$j_!$ & j_* restricted to other set $\Longrightarrow$ this
construction gives sheaves on the union.

Assume S = closed, contractible. Choose x = basepoint, $\dim S = 2d$.

2. $\mathcal{D}^b_c(S) \longrightarrow \mathcal{D}^b(\text{Vect space})$: fiber at x,
is an equivalence (S contractible). Inverse:
take constant extension. Just need to prove that all
extensions in $\mathcal{D}^b_c(S)$ are trivial

— Ext of complexes only in deg 0 are cohomologies, but
S contractible ... ~~extend...~~ maps determined by cohomologies

3. Identify $\mathcal{D}^b_c(S)$ with $\mathcal{D}^b(\text{Vect})$.

$Q^\bullet \in P(X)$, $i : S \hookrightarrow X$, $j : X \setminus S \hookrightarrow X$

S still contractible, $i_* i^! Q^\bullet \to Q^\bullet \to j_* j^* Q^\bullet \xrightarrow{[1]}$

$\Rightarrow Q^\bullet$ is determined (up to noncanonical iso) by
an element in $\mathrm{Ext}^1_{P(X)}(j_* j^* Q, i_* i^! Q^\bullet)$

$$\simeq \mathrm{Ext}^1_{\mathcal{D}(S)}(i^* j_* j^* Q, i^! Q) \simeq \bigoplus_{\mathbb{Z}} \mathrm{Hom}\left(\underline{H}^i(i^* j_* j^* Q), \underline{H}^{i+j}(i^! Q)\right)$$

By perversity, $\underline{H}^{<-d}(i^! Q^\bullet) = 0$, $\underline{H}^{>-d}(i^* Q^\bullet) = 0$

$$i^! Q \longrightarrow i^* Q \longrightarrow i^* j_* j^* Q \longrightarrow \quad \text{in } \mathcal{D}(S)$$

(*) $\Rightarrow 0 \to \underline{H}^{-d-1}(i^* Q) \to \underline{H}^{-d-1}(i^* j_* j^* Q) \xrightarrow{\partial} \underline{H}^{-d}(i^! Q)$
$\longrightarrow \underline{H}^{-d}(i^* Q) \to \underline{H}^{-d}(\text{—}) \xrightarrow{\partial} \underline{H}^{-d+1}(i^! Q) \to 0$
$\longrightarrow \cdot \xrightarrow{\sim} \cdot \longrightarrow 0 \longrightarrow \cdot \xrightarrow{\sim} \cdot \longrightarrow 0$

$\Rightarrow Q^\bullet$ is determined up to iso by $Q/_{X \setminus S}$ and by above
exact sequence (*).

4. S contractible, T = (small enough) closed tub. neighborhood.
$\pi_S : T \longrightarrow S \ni x$.

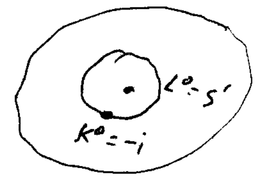
<u>Def</u> $\underline{D^0 = \pi_S^{-1}(x)}$: stratified space, closed in X. — normal slice

$\underline{L^0 =} \partial D^0$ = link at x.

<u>Def</u> A <u>~~perverse~~ link</u> K° = any closed subset of L_0 s.t.
$\forall\ p \in P(X \setminus S)$, $\mathbb{H}^{>-d}(K^\circ, P) = 0 = \mathbb{H}^{<-d}(L^0, K^\circ; P)$
— relative : restrict to complement of K°, take compact support.

5. Theorem Perverse links always exist.

e.g. $X = \mathbb{C}^* \sqcup \{0\}$



K^0 any point on circle

LHS is cohomology of local system in wrong degree,

same for RHS - 0^{th} compact cohomology..

6. Set $\mathcal{A} = P(X - S)$ $\mathcal{B} = Vect$. $F, G : \mathcal{A} \to \mathcal{B}$, $T : F \to G$:

$$F(P^\cdot) = \mathbb{H}^{-d-1}(K^0, P), \quad G(P^\cdot) = \mathbb{H}^{-d}(L^0, K^0; P)$$

$$\underbrace{\longrightarrow}_{T = \partial \text{ boundary map}}$$

F is right exact, G left exact by 4) (def. of perv. link)

$\Rightarrow$ $\mathcal{C}(F, G, T)$ is abelian.

Theorem (S contractible) $P(X) \xrightarrow{\phi} \mathcal{C}(F,G,T)$ is equivalence:

$$\phi : Q^\cdot \longmapsto \left(\mathbb{H}^{-d-1}(K^0, Q|_{X-S}) \xrightarrow{\partial} \mathbb{H}^{-d}(L^0; K^0; Q|_{X-S}) \right)$$

$$\xrightarrow{\partial} \mathbb{H}^{-d}(D^0; K^0, Q^\cdot) \nearrow$$

Proof

1. exactness: $Q \to Q|_{X-S}$ exact. & also $\mathbb{H}^{*}(D^0, K^0, Q^\cdot)$ vanishes in all degrees but $-d$ $\Rightarrow$ exact

2. equivalence: on objects: $Q \longleftrightarrow Q|_{X-S}$ and exact sequence (*) as before.

$$A \xrightarrow{t} B \qquad 0 \to \ker m \to \ker t \xrightarrow{m} \ker n \to \operatorname{cok} m$$

$$m \searrow C \nearrow n \quad \Longrightarrow \quad \to \operatorname{cok} t \to \operatorname{cok} n \to 0$$

exact

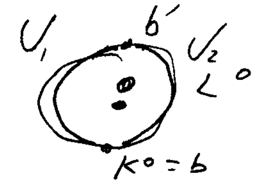
- this gives our 6-term exact sequence

$\mathbb{H}^*(D, L) = H^{-d}(i^! Q)$

Given A, B vector spaces; to give C is same as to give such an exact sequence.

$\Rightarrow$ faithfulness of functor: exact + equiv on objects

equiv on morphisms - harder...



Example $X = \mathbb{C}^* \cup \{0\}$

Perverse sheaves $Q^\cdot$ on X

$$\longleftrightarrow \left(Q^\cdot|_{\mathbb{C}^*},\ H^{-1}(b, Q|_{\mathbb{C}^*}) \xrightarrow{\partial} H^0_{compact}(S^1 \setminus b, Q^\cdot|_{\mathbb{C}^*}\right)$$

$$\overset{m}{\rightsquigarrow} H^0(D, b; Q) \nearrow_n$$

Denote $Q^\cdot|_{\mathbb{C}^*} = \mathcal{L}[1]$ $\quad a =$ monodromy $\mathcal{L}_b \to \mathcal{L}_b$

(invertible)

$$\Rightarrow \quad \mathcal{L}_b \xrightarrow{T} H^1_{comp}(S^1 \setminus b, \mathcal{L}) \simeq \mathcal{L}_{b'} \quad \text{(Poincaré)}$$

$$\mathcal{L}_b \xrightarrow{(id, -id)} H^1_c(U_1, \mathcal{L}) \oplus H^1_c(U_2, \mathcal{L}) \nearrow \mathcal{L}_{b'}$$

$H^1_c(U_1, \mathcal{L}) \simeq \mathcal{L}_b$, $\quad H^1_c(U_2, \mathcal{L}) \simeq \mathcal{L}_b \xrightarrow{(id, a)} \mathcal{L}_b \xrightarrow{\text{monodromy}} \mathcal{L}_{b'}$

$$\wr\!\wr \quad H^0_c(S^1 \setminus b \setminus b', \mathcal{L}[1])$$

$T = -id + a$

i.e. $\mathcal{L}_b \xrightarrow{T = -id + a} \mathcal{L}_b$, $\quad \mathcal{L}_b \xrightarrow{m} B \xrightarrow{n} \mathcal{L}_b$ $\quad \Leftrightarrow$ diagram $A \xrightarrow{-1+a} A$, $\quad A \xrightarrow{m} B \xrightarrow{n} A$

$$\Leftrightarrow \quad A \underset{n}{\overset{m}{\rightleftarrows}} B, \quad nm = -1 + a$$

$$\Leftrightarrow \quad (A, B, m, n) \text{ s.t. } A \underset{m}{\overset{n}{\leftrightarrows}} B, \quad nm + 1 = a \text{ invertible}$$

— Representations of a quiver.

12/16

Fourier-Deligne Transform & Kazhdan-Lusztig

K-L: Gluing of perverse sheaves & discrete series reps.
J.G&P. Vol 5 n.1 1988

Deligne-Fourier transform: (restrict to symplectic vector space)

char $k = p$ $\quad \psi: \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$

$\Rightarrow$ Artin-Schreier $\mathcal{L}_\psi$ loc. constant $\overline{\mathbb{Q}}_\ell$-sheaf
$\quad\quad |$
$\quad \mathbb{G}_{a,k}$

from $x \mapsto x^p - x$,

$\mathbb{F}_p$-covering of $\mathbb{G}_a$, push forward by ψ.

$V, \langle,\rangle$ symplectic vector space, $\dim 2d$ $\quad \langle,\rangle : V \times V \longrightarrow \mathbb{G}_a, k$

$\Rightarrow$ pullback $\mathcal{L}_\psi(\langle,\rangle)$.

Fourier transform $\quad \mathcal{F} = \mathcal{F}_\psi : D^b_c(V, \bar{\mathbb{Q}}_\ell) \circlearrowleft$

$\mathcal{F}(K) = p_{1!}(\mathcal{L}_\psi(\langle,\rangle) \otimes p_2^* K[2d](d))$

Theorem 1. $\mathcal{F}^2 \cong id$ (Sign comes from having symplectic form)

2. $D\mathcal{F}_\psi = \mathcal{F}_{\psi^{-1}} D \qquad$ 3. $\mathcal{F}$ is t-exact wrt perverse t-structure

($\Rightarrow$ $\mathcal{F}$ induces involution of $Perv(V)$).

Main technical lemma: get same transform if we replace $p_{1!}$ by p_{1*} .. $\Rightarrow$ D statement.

– this is a type of properness statement for A-S correspondence.

– similar to involutions coming from an automorphism of V...

Not hard to see $\mathcal{F}$ is right t-exact – using estimates on coh. amplitude of $!, *$ functors. Then exactness follows from 1. or from 2. .. ∎

Suppose $U \overset{j}{\hookrightarrow} V$ open $\Rightarrow$ $\mathcal{F}_! : D(U) \circlearrowleft$ right t-exact.

$\mathcal{F}_! = j^*\mathcal{F} j_!$. Have natural transformation

$\mathcal{F}_!^2 \longrightarrow id \quad : \quad \underbrace{j^*\mathcal{F}\underbrace{j_! j^*}_{\to id}\mathcal{F} j_!}_{\to id}$

$\Rightarrow {}^p\mathcal{F}_! : Perv(U) \circlearrowleft$ right t-exact

Consider category of pairs

(A, B), $\quad A, B \in Perv(U)$. $\quad \alpha : {}^p\mathcal{F}_! A \longrightarrow B$,

$\beta : {}^p\mathcal{F}_! B \longrightarrow A$ with

$$\begin{tikzcd} {}^p\mathcal{F}_!^2 A \arrow[r, "{}^p\mathcal{F}_!\alpha"] \arrow[d, "can"'] & {}^p\mathcal{F}_! B \arrow[dl, "\beta"] \\ A & \end{tikzcd} \quad \circlearrowleft$$

& other one $A \leftrightarrow B$, $\alpha \leftrightarrow \beta$

$\Rightarrow$ abelian category by gluing (as last time) $Glue(U, \mathcal{F})$

Example $\dim(V - U) < \frac{\dim V}{2}$

$\Rightarrow Glue(U, \mathcal{F}) \xrightarrow{\sim} Perv(V)$

Natural functor $\quad \mathrm{glue}(U,F) \longleftarrow \mathrm{Perv}(U)$

$$(j^*A, j^*FA) \longmapsfrom A$$

which is equivalence when A is big enough

(complement shouldn't contain anything Lagrangian $\Rightarrow$ Can't have A supported on $V \setminus U$ with FA also such.)

G connected, simply conn, semi-simple split$/k$. $T \subset B$, $U = [B,B]$ etc.

$X = G/U$. $\qquad U = \prod_{\alpha \in R^+} X_\alpha \qquad R^+$ positive roots

$R(w) = \{ \alpha \in R^+ \mid w\alpha \in -R^+ \}$

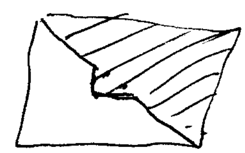
$U_w = \prod_{\alpha \in R(w)} X_\alpha \qquad$ e.g. $U_s = X_{\alpha_s} \qquad s$ simple

$$U = U_s \cdot U_{w_0 s}$$

$M_s = \langle X_{-\alpha_s}, X_{\alpha_s} \rangle$ subgroup $\simeq SL_2 \subset G$.

If we fix isom $X_{\alpha_s} \simeq \mathbb{G}_a \Rightarrow$ unique iso $M_s \simeq SL_2$.

Consider projection $\quad G/U \longrightarrow G/M_s U_{w_0 s}$



(Note $M_s U_{w_0 s} = U_{w_0 s} M_s$)

$$\begin{tikzcd} & V_s = G/U_{w_0 s} \times_{M_s} k^2 \arrow[d] \\ G/U \arrow[ur, "j_s"] \arrow[r] & G/M_s U_{w_0 s} = Y_s \end{tikzcd}$$

k^2 - SL_2 rep

- rank 2 bundle over $G/M_s U_{w_0 s}$.

Inclusion above comes from $\quad G \longrightarrow G/U_{w_0 s} \times k^2$

$$g \longmapsto \left(gU_{w_0 s}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

(under $U_s \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$)

e.g. $\quad G = SL_2, \quad G/U = k^2 \setminus 0, \quad G/M_s U_{w_0 s} = pt \ldots$

In general j_s is complement to zero section

V_s has canonical G-invariant symplectic form

$$\langle , \rangle : V_s \times_{Y_s} V_s \to \mathbb{G}_a$$

(from sl_2-invariant form)

$\mathcal{F}_{s,!} : D^b_c(X) \circlearrowleft$ $\qquad (X = G/U)$

" $j_s^* F_{V_s} j_{s!}$

<u>Theorem</u> $w = s_1 s_2 \cdots s_\ell$ shortest decomp (ℓ = length w)

$\Rightarrow F_{w!} = \mathcal{F}_{s_1 !} \circ \cdots \circ \mathcal{F}_{s_\ell !}$ independent of choice of decomposition $\Longrightarrow$ action of braid ~~group~~ monoid on $D^b_c(X)$.

<u>Digression</u> : group action on category : $g \in G \longmapsto$ functor $F_g : \mathcal{C} \to \mathcal{C}$ $\quad g_1, g_2 \in G \Rightarrow \alpha_{g_1, g_2} : F_{g_1} F_{g_2} \xrightarrow{\sim} F_{g_1 g_2}$ satisfying cocycle condition.

Braid monoid $B^+_{W,S}$

To give action of $B^+_{W,S}$ on $\mathcal{C}$ $\Longleftrightarrow$ (Deligne) data:

for $w \in W \rightsquigarrow F_w$ functor

$w_1, w_2 \in W$ s.t. $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \Rightarrow F_{w_1} \circ F_{w_2} \xrightarrow{\sim} F_{w_1 w_2}$

+ associativity condition for triples with $\ell(w_1 w_2 w_3) = \sum \ell(w_i)$

We want to have: $\forall w_1, w_2 \in W$ $\quad F_{w_1 !} \circ F_{w_2 !} \longrightarrow F_{w_1 w_2 !}$

morphism + associativity

Need to check for $sw = ws'$ $\quad \ell(sw) = \ell(w) + 1$

$$\begin{array}{ccc} s \cdot w \cdot s' & \Longrightarrow & s \cdot s \cdot w \\ \downarrow & & \downarrow \\ w \cdot s' \cdot s' & \xrightarrow{?} & w \end{array} \qquad \text{unproved.}$$

Proof of theorem – write geometric kernels $K(w)$ on $X \times X$ for $F_{w!}$.

$\mathcal{A}$ – ab. category, glued from $|W|$ copies of Perv(G/U) & functors ${}^p\mathcal{F}_{w!}$: Perv$(X) \circlearrowleft$ right-exact : $(\{A_w\}, w \in W)$

$w, w' \Rightarrow$ morphism ${}^p\mathcal{F}_{w'!} A_w \to A_{w'w}$ + commutativity (compatibility with above diagrams.)

$\rightsquigarrow$ like reps of quiver with $|W|$ vertices.

W acts on $\mathcal{A}$ commuting with G action, & T action behaving correctly under W.

$w \in W \Rightarrow \mathcal{A}_w = \{A \in \mathcal{A}$ with iso $wFr^* A \xrightarrow{\sim} A\}$
"Deligne-Lusztig variety" type construction.

<u>Conj</u> $\mathcal{A}$ has finite cohomological dimension, $Ext^i(A, A')$ fin dim$/k$.

$\Rightarrow$ action on Ext groups.

$\Rightarrow$ $K^0(\mathcal{A}_w)$, $\langle , \rangle$ pairing, action of twisted Frobenius

$A, A' \in \mathcal{A}_w \Rightarrow wFr^* A \xrightarrow{\sim} A$ & for A'

$\Rightarrow$ $\varphi_i : Ext^i_{\mathcal{A}}(A, A') \circlearrowleft$,

$\langle , \rangle = \Sigma(-1)^i \operatorname{Tr} \varphi_i$

<u>Conjecture</u> $K^0(\mathcal{A}_w) / \ker \langle , \rangle$ is finite dimensional vector space with $G \times T_w$ action. Given characters of $T_w \Rightarrow$ reps of G : conjecturally whole discrete series.