Mihnea Popa - Fourier-Mukai and Vanishing

Note Title 9/21/2006

(w. G. Pareschi)

Generic Vanishing

X smooth projective $/\mathbb{C}$

Most important basic vanishing:

Kodaira Vanishing L = ample line bundle

$\Rightarrow H^i(\mathcal{O}_X(K_X+L)) = 0 \quad \forall i>0$

(ampleness $\Longleftrightarrow$ differential geometric vanishing)

also have topological proofs, & algebraic proof (mod p) by Deligne-Illusie

K_X+L called adjoint bundle.

Kawamata-Viehweg Vanishing:

If L is nef & big line bundle

$\Rightarrow H^i\, \mathcal{O}_X(K_X+L) = 0 \quad \forall i>0$

L is nef $:=$ $L \cdot C \geq 0$ for
C any curve on X

$\overset{\text{Kleiman}}{\Longleftrightarrow}$ $L^d \cdot V \geq 0$ for $V \subset X$ any
subvariety of dim d

$\longrightarrow$ closure of ample cone...

nef + big: assume also $L^n > 0$
($n = \dim X$)

.... e.g. pullbacks of ample line
bdles by birational, or generically
finite, maps

Real improvement by K-V: $\mathbb{Q}$-divisor
version...

Grauert-Riemenschneider vanishing:

$f: X \to Y$ surjective, X, Y projective
& X smooth, generically finite
$$\Rightarrow \quad R^i f_* \omega_X = 0 \quad \forall i > 0$$
[local vanishing theorem]

In fact $R^i f_* \mathcal{O}_X(K_X + L) = 0$
$\forall i > 0 \quad \forall L$ nef

Proof of G-R from K-V:
$\mathcal{F}$ sheaf on X

<u>Lemma</u> $R^i f_* \mathcal{F} = 0 \iff H^i(\mathcal{F} \otimes f^* A) = 0$
$\forall A >> 0$ on Y

.... use Leray spectral sequence
$$E_2^{ij} = H^j(R^i f_* \mathcal{F} \otimes A) \Rightarrow H^{i+j}(X, \mathcal{F} \otimes f^* A)$$

Serre Thm. A: $E_2^{ij} = 0 \quad j > 0$
$$\Rightarrow H^0(R^i f_* \mathcal{F} \otimes A) \simeq H^i(\mathcal{F} \otimes f^* A)$$

Serre Thm B: <u>globally generated</u> if no higher cohomology $\Rightarrow$ sheaf vanishes if H^0 term is also zero

To prove G-R, now need

$H^i(\omega_X \otimes \underset{\text{nef}}{L} \otimes \underset{\text{big+nef}}{f^*A}) = 0$ which is

Kawamata-Viehweg.

More generally, if dim of general fiber of f is k $\Rightarrow$

$R^i f_* \omega_X = 0 \quad \forall i > k$.

Same argument works by induction on k: pull back ample line bundle under such a map $\Rightarrow$ get vanishing above the dimension.

Q: Can we have vanishing with less positivity?

<u>Green-Lazarsfeld</u>: have vanishing if involve Albanese map.

e.g. what about K_X for any X smooth projective?

$$X \xrightarrow{a} A = \mathrm{Alb}(X) = H^0(\Omega_X^1)^\vee / H_1(X,\mathbb{Z})$$

integration over 1-cycles!
universal map to an abelian variety.

Dual of $\mathrm{Alb}(X) = \mathrm{Pic}^0(X)$

<u>Theorem</u> (G-L "generic vanishing theorem")
If $\dim X = d$ & $\dim a(X) = d-k$
then $\operatorname{codim} V^i(\omega_X) \geq i-k \ \forall i$

$V^i(\omega_X) = \{ \alpha \in \mathrm{Pic}^0(X) : h^i(\omega_X \otimes \alpha) \neq 0 \}$
ith cohomological support locus ~~of~~ ω_X.
$h^{d-i}(\alpha^{-1})$

Special case: $k=0$ (a is gen. finite.
e.g subvariety of abelian variety)
$\Longrightarrow \operatorname{codim} V^i(\omega_X) \geq i \ \forall i$

in particular for general $\alpha \in \mathrm{Pic}^0$
$H^i(\omega_X \otimes \alpha) = 0 \quad \forall i > 0$

(for bigger k get generic vanishing starting from k^{th} cohomology)

--- Kodaira-type vanishing for

$K_X = K_X + \mathcal{O}_X$ rather than $K_X + (L$ positive$)$

Remarks 1. Exist results about structure of $V^i(\omega_X)$ (G-L, Simpson, Beauville, Ein-Lazarsfeld)

.... unions of torsion translates of subabelian varieties

– so e.g. if Pic^0 simple $\Rightarrow$ either points or all of Pic^0

Equality obtained in very special cases ..

2. Proof is Hodge theory with coeffs in a flat bundle ... transcendental

Q: Is there a vanishing theorem for $K_X + L$ L nef

... can't use Hodge theory here ..

What about multiples mK_X
(Siu's theorems on def. invariance of plurigenera)

Will provide such general results for all adjoint bundles; mK_X versions don't hold.

Theorem (Ein-Lazarsfeld)
(A, Θ) irreducible p.p.a.v.
$\Rightarrow$ Θ is normal & has rational singularities.
(*)

[Z has rational singularities, in char 0,
$\iff$ $\forall$ resolution of singularities
$Y \xrightarrow{f} Z$ $\quad Rf_* \mathcal{O}_Y = \mathcal{O}_Z$ ...
so behaves as if it's smooth .....]

Kollár : (A, Θ) log-canonical
E-L : Θ has canonical singularities
$\overset{\text{Elkik}}{\iff}$ rational

Sketch of proof: $\exists$ ideal sheaf on A which measures failure of $(*)$; take embedded resolution of singularities

$$\begin{array}{ccc} X & \subseteq & Y \\ \downarrow & & \downarrow \pi \\ \Theta & \subseteq & A \end{array}$$

$\operatorname{adj}(\Theta) = \pi_* \mathcal{O}_Y(K_Y - \pi^*\Theta + \tilde{\Theta})$

$\tilde{\Theta}$ = proper transform of Θ

.. comes from negative exceptional things.. $\operatorname{adj}(\Theta) \subset \mathcal{O}_A$

& its vanishing $\Longleftrightarrow$ canonical singularities

Facts

1. exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(\Theta) \otimes \operatorname{adj}(\Theta) \to f_*\omega_X \to 0$$

2. $\operatorname{adj}(\Theta) = \mathcal{O}_A \Longleftrightarrow (*)$

$\Rightarrow$ For any $\alpha \in \operatorname{Pic}^0(X)$ get exact sequences

$0 \longrightarrow \alpha \longrightarrow \mathcal{O}_A(\Theta) \otimes \alpha \otimes \mathrm{adj}(\Theta) \longrightarrow f_* \omega_X \otimes \alpha \longrightarrow 0$

$H^i \alpha = 0 \quad \forall i \quad \forall \alpha \neq \mathcal{O}_A$

$\Rightarrow$ can ignore cohomology!

$H^0(\mathcal{O}_A(\Theta) \otimes \alpha \otimes \mathrm{adj}(\Theta)) \cong H^0(f_* \omega_X \otimes \alpha)$

Claim: $\cancel{H}$ 0

... by G-R $\quad H^i(f_* \omega_X \otimes \alpha) = H^i(\omega_X \otimes f^* \alpha)$

By G-L for general α, $H^i = 0$

— generic vanishing says on H^0 is same as an Euler characteristic:

$h^0(f_* \omega_X \otimes \alpha) = \chi(\omega_X) > 0$

so our funny sheaf has sections.

Now $H^0(\mathcal{O}_A(\Theta) \otimes \alpha \otimes \mathrm{adj}(\Theta)) \hookrightarrow H^0(\mathcal{O}(\Theta) \otimes \alpha)$

Principal polarization $\longleftarrow$ $\overset{\text{IS}}{\mathbb{C}}$

$\Rightarrow$ zero locus of $a_i^*(\Theta) \in |\Theta + a|$
$\forall a \in A$
so is empty. ∎

Another POV $\mathcal{P}$ Poincaré bundle for X

$$\begin{tikzcd}
 & X \times \hat{A} \arrow[dl] \arrow[d] \arrow[drr] & & \\
X \arrow[d, "a"'] & A \times \hat{A} \arrow[dl] \arrow[dr] & \mathcal{P} & \hat{A} \arrow[d, equal] \\
A & & \hat{A} = \mathrm{Pic}^0 A = \mathrm{Pic}^0 X &
\end{tikzcd}$$

Alb

G-L conjecture (a priori stronger than G-L thm)

$$R^i p_{\hat{A}*} \mathcal{P} = 0 \quad \forall i \neq d = \dim X$$

(k = 0 or sm finite

$$\| $$

$$R^i p_{\hat{A}*}(p_X^* \mathcal{O}_X \otimes \mathcal{P})$$

Fourier-Mukai of structure sheaf of X

Mukai: $\mathbb{R}\hat{S} : \mathbb{D}(A) \xrightarrow{\sim} D(\hat{A})$

$\bullet \longmapsto R\, p_{\hat{A}*}\, (p_A^*(\cdot) \otimes P)$

equivalence of derived categories

Hacon: use Fourier-Mukai to prove Green-Lazarsfeld conjecture

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Why does GL conjecture $\Rightarrow$ GL theorem

1. Base change argument:

$\operatorname{codim} V^i(\omega_X) \geq i \iff$

$\operatorname{codim} \operatorname{Supp} R^i \hat{S}\, \omega_X \geq i$

2. Grothendieck duality:

$\mathbb{R}\Delta\,(\,\mathbb{R}\hat{S}\ \mathcal{O}_X[d]) \simeq \mathbb{R}\hat{S}\ \omega_X$

$R\Delta(\cdot) = \mathbb{R}\mathcal{H}om(\cdot, \mathcal{O}_{\hat{A}})\ [g]$

G-L conjecture just says
$\mathcal{F} = \mathbb{R}\hat{S}\mathcal{O}_X[d]$ is just a sheaf,
not complex

3. $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_{\hat{A}}) \simeq R^i\hat{S}\omega_X$

any sheaf $\mathcal{F}$ codim Supp $\mathcal{E}xt^i \geq i$
on Cohen-Macaulay spaces...
Auslander - Buchsbaum
$\longrightarrow$ get G-L theorem

X, Y [Cohen-Macaulay varieties], projective
will say smooth for simplicity

P locally free [or any complex]
$\downarrow$
$X \times Y$

$$\mathbb{R}\Phi : D(X) \longrightarrow D(Y)$$

$$\bullet \longmapsto \mathbb{R}p_{Y*}(p_X^* \otimes P)$$

- has a left adjoint

$$\mathbb{R}\Psi = \mathbb{R}\Psi_{Y \xrightarrow{P^\vee \otimes p^* \omega_X[g]} X} : D(Y) \to D(X)$$

i.e. $\mathrm{Hom}_{D(Y)}(A, \mathbb{R}\Phi B) \simeq \mathrm{Hom}_{D(X)}(\mathbb{R}\Psi A, B)$

<u>Def</u> $\mathcal{F}$ a sheaf on X (or any object of $D(X)$)

is GV (generic vanishing) or

GV_k for some $k \geq 0$ wrt P

if $\operatorname{codim} \operatorname{Supp} R^i\Phi\mathcal{F} \geq i$ (or $\geq i-k$) $\forall i$

$\iff$ $\operatorname{codim} V^i_P(\mathcal{F}) = \{y \in Y : h^i(\mathcal{F} \otimes P_y) \neq 0\}$

(weaker than regularity: $\underline{>i}$ strict version)

Theorem TFAE:

1. $\mathcal{F}$ is GV (or GV_k)

2. $H^i(\mathcal{F} \otimes (R\psi L[g])^\vee) = 0$
$\forall i > 0 \quad (\forall i > k) \qquad \forall L \gg 0$

[.... $R\psi L[g]^\vee$ is just a vector bundle by base change

$E = \hat{L}^\vee$ is a positive vector bundle
& "$E \to 0$ as $L \to \infty$"]

3. $R^i\Phi(R\Delta\mathcal{F}) = 0 \quad \forall i \neq d$
$(\forall i \notin [d-k, d])$

Interpretation:

$1 \iff$ G-L Thm for $\mathcal{F} = \omega_X$

$3 \iff$ G-L Conjecture for $\mathcal{F} = \omega_X$

$2 \iff$ Condition studied by Hacon

e.g. apply to $f: X \longrightarrow Y$ as Fourier transform P = graph

$\mathbb{R}f_* = \mathbb{R}\Phi$

$\mathbb{L}f^* = \mathbb{R}\Psi$

(2) says $R^i f_* \mathcal{F} = 0$

$\iff H^i(\mathcal{F} \otimes f^* L) = 0 \quad \forall L >> 0.$

Applications: Can obtain generic vanishing from
- classical vanishing
- properties of moduli spaces

Theorem If $\dim X = d$ & $\dim a(X) = d-k$

L = any nef line bundle

$\Rightarrow \operatorname{codim} V^i(\mathcal{O}_X(K_X+L)) \geq i-k \quad \forall i$

when $k=0 \Rightarrow$ in particular for α general

$H^i(\mathcal{O}_X(K_X+L) \otimes \alpha) = 0 \quad \forall i>0$

$$
\begin{tikzcd}
 & & \mathcal{P} & & \\
 & & X \times \hat{A} \arrow[dl] \arrow[dd, "r"] \arrow[dr] & & \\
\hat{X} \arrow[r] \arrow[dd, "b"', "\substack{\text{base}\\ \text{change}}"] & X \arrow[dd, "a"] & & \hat{A} \arrow[dd, equal] & M \gg 0 \\
 & & \hat{A} \times \hat{A} \arrow[dl] \arrow[dr] & & \\
\hat{A} \arrow[r, "\varphi_M"'] & A & & \hat{A} &
\end{tikzcd}
$$

Mukai: look at isogeny $\varphi_M : \hat{A} \to A$
given by line bundle M

then $\varphi_M^* \widehat{M^\vee}^\vee \cong \oplus M$ copies of M

Want to show $H^i(\omega_X \otimes L \otimes \widehat{M^\vee}^\vee) = 0$

Turns out for some reason our F-M transform
can be calculated by base change

$\Rightarrow$ suffices to check

$$H^i(\hat{X}, \omega_{\hat{X}} \otimes \underbrace{\varphi^* L}_{\text{nef}} \otimes \underbrace{b^* M}_{\substack{\text{big line} \\ \text{(or vanishing-ish)}}})$$

$\kappa(L)$ = Iitaka dimension

codim $V^i(\mathcal{O}_X(K_X+L)) \geq i - k + \kappa_L$

where $\kappa_L = \begin{cases} 0 & \text{if } \kappa(L) = -\infty \\ \kappa(L|_F) & \text{if } \kappa(L) \geq 0 \end{cases}$

$\to$ general fiber of a

big line $\iff$ maximal Iitaka dimension

Every standard vanishing thm $\Longrightarrow$
generic vanishing thms:
get GV for

- $R^j f_* \omega_Y \otimes L_{nef}$ Kollár vanishing

[Kollár: $Rf_*\omega_Y \underset{q\text{-is}}{\cong} \oplus R^i f_* \omega_Y[-i]$]
"formality"

- $\Omega^j_X \otimes L$ Akizuki-Nakano

- $\omega_X \otimes E$ $\quad$ E nef v.bundle
- $\omega_X^{\otimes m}$ $\quad$ if X minimal
- Siu/Nadel version: $\omega_X \otimes L \otimes \mathcal{I}(D)$

D is $\mathbb{Q}$-divisor

- Replace Pic^0 by moduli space:
  M fine compact moduli space

F nef

$$D(A) \xrightarrow{R\hat{S}} D(\hat{A})$$

$\Uparrow$

F GV $\iff$ $R\hat{S}(R\Delta F) =$

$= R^g \hat{S}(R\Delta F)$ is a sheaf $\mathcal{G}$

$\Uparrow$

F M-reg $\implies$ ample

codim Supp $\geq i$ $\forall i > 0$ ("GV_{-1}")

$\iff$ $\mathcal{G}$ torsion-free

$\mathcal{G}$ torsion free $\iff$

$$0 \to \mathcal{G} \to E_1 \to E_2 \to 0$$

E_1, E_2 locally free:

ie 1-syzygy sheaf

Next: $\mathcal{G}$ reflexive $\iff$ 2-syzygy sheaf
gives strong GV_{-2}

k^{th} syzygy $\rightsquigarrow GV_{-k} \ldots$?

<u>Conjecture</u> For (A, Θ) abelian variety
$\exists\, Y \subset A$ s.t. $[Y] = \frac{\Theta^m}{m!}$ (some $m \geq 2$)

$\iff Y = W_d \subset J(C)$ Jacobian
or $Y = F \subset J^3(X)$, $[F] = \Theta^3/3!$
Fano surface of lines in cubic 3-fold.

... analog of classification of
surfaces of minimal degree in $\mathbb{P}^n$:
either rational normal scrolls, or
$\mathbb{P}^5 \supset$ Veronese surface

Cohomological criterion: Y of min deg
$\iff I_Y$ is Castelnuovo-Mumford 2-regular:
$h^i(I_Y(2-i)) = 0 \quad \forall i > 0$

Now want an abelian variety version of this

Conjecture: $I_Y(\Theta)$ GV $\iff [Y] = \frac{\Theta^m}{m!}$

Sue m.