Dmitry Rumynin - D-modules in positive characteristic

Note Title 4/11/2007

1. Weyl algebras
2. Differential Operators
3. D-modules
4. Localization Theorem

1). F field. Weyl algebra

$$A^n_F = \langle x_1, \dots x_n, \partial_1, \dots \partial_n \mid x_i x_j = x_j x_i,\ \partial_i \partial_j = \partial_j \partial_i,\ \partial_j x_i - x_i \partial_j = \delta_{ij} \rangle$$

A^n_F is a noncommutative Noetherian domain for all F.

$$\varphi : A^n_F \longrightarrow \mathrm{End}_F\, F[x_1, \dots x_n]$$

$$x_i \longmapsto x_i \cdot \_$$

$$\partial_i \longmapsto \frac{\partial}{\partial x_i} \_$$

Char $F=0 \Rightarrow \varphi$ is 1:1.

Im φ : differential operators,

A^n_F is simple, $Z(A^n_F) = F\cdot 1$.

Char $F = p > 0$ $\quad \operatorname{Ker}\varphi = \langle \partial_1^p, \dots \partial_n^p \rangle$

(all pth derivatives vanish)

$\operatorname{Im}\varphi \subsetneq$ differential operators:

only get "small" differential operators

$$Z(A^n_F) = F[x_1^p, \dots, x_n^p, \partial_1^p, \dots \partial_n^p]$$

~ think of as functions on T^*F^n

cotangent bundle to affine space

- A^n_F is Azumaya over its center.

... i.e locally sheaf of matrix algebras:

$$\underset{\text{left mult.}}{F[x_i, \partial_j^p]} \underset{Z(A^n_F)}{\otimes} \underset{\text{right mult}}{A^n_F} = \operatorname{End}_{F[x_i, \partial_i^p]} A^n_F$$

... on this cover A^n becomes End of a vector bundle.

- A^n_F seems to split over Lagrangian subvarieties (conjecture).

e.g on zero section of T^* :

$$\mathrm{Im}\ \varphi = A^n_F / \langle \partial_i^p \rangle \simeq \mathrm{End}_{F[x_i^p]} F[x_i]$$

- $\varphi_{cris} : A^n_F \longrightarrow \mathrm{End}_F [x_i, \frac{x_i^p}{p!}, \ldots, \frac{x_i^{p^m}}{p^m!}, \ldots]$ (divided power polynomials) is 1:1.

So A^n_F are "crystalline differential operators"

2. X smooth algebraic variety

$\mathcal{O}_X$ = functions, $\mathcal{T}_X$ = tangent sheaf

— restricted Lie algebroid:

i.e. 4 operations : $f \cdot \partial$, $[\partial_1, \partial_2]$, $\partial(f)$, & $\partial^{[p]}$ apply ∂ p times.

ex. $\left(\frac{\partial}{\partial x}\right)^{[p]} = 0$: $\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} x^m = m(m-1)\ldots(m-p+1) x^{m-p} = 0$

ex. $\left(x \frac{\partial}{\partial x}\right)^{[p]} = x \frac{\partial}{\partial x}$.

$\mathcal{D}_X := U_{\mathcal{O}_X}(T_X)$ universal enveloping algebroid:

$V \underset{\text{open}}{\subseteq} X \quad \mathcal{D}_X(V) = \langle \mathcal{O}_X(V), T_X(V) : \begin{array}{l} \partial\partial' - \partial'\partial = [\partial,\partial'] \\ \partial \cdot f - f \cdot \partial = \partial(f) \\ f \cdot \partial = f\partial, \ f \cdot f' = ff' \end{array} \rangle$

Note $\partial f^{-1} = f^{-1}\partial + \underbrace{\partial(f^{-1})}_{= -f^{-2}\partial(f)}$

$\Rightarrow$ swap denominators from left to right $\Rightarrow$ good localization.

$\mathcal{D}_X \supset Z_X = Z(\mathcal{D}_X) = \langle f^p, \underset{\in \mathcal{D}_X}{\partial^p} - \underset{\in T_X}{\partial^{[p]}} \rangle$

$\mathcal{D}_X$ is Azumaya over Z_X.

From now on F algebraically closed

Frobenius twist of X: $X^{(1)} := (X \to pt \to pt)$
$\alpha^{1/p} \leftarrow\!\shortmid \alpha$

don't change X, but change map $X \to pt = \operatorname{Spec} F$ by Frobenius

Geometric Frobenius $F_X : X \to X^{(1)}$
on functions $f \mapsto f^p$.

[Another way of thinking : $\mathcal{O}_X \supset \langle f^p \rangle = \mathcal{O}_{X^{(1)}}$]

$$\operatorname{Spec} Z_X = T^*X^{(1)} \xrightarrow{\pi} X^{(1)}$$

$$Z_X \supset \mathcal{O}_X^{(1)}$$

D_X gives rise to $\mathcal{D}_X :=$ sheaf on $T^*X^{(1)}$ associated to D_X.

$\mathcal{D}_X$ is locally free of rank $p^{2\dim X}$

Azumaya on $T^*X^{(1)}$.

Local splitting $\quad T^{*,(1)}X := X \times_{X^{(1)}} T^*X^{(1)} \xrightarrow{p_2} T^*X^{(1)}$

$$\langle x_i, \partial_i^p \rangle \qquad\qquad \langle x_i^p, \partial_i^p \rangle$$

Can replace D_X-modules by modules over the Azumaya algebra $\mathcal{D}_X$ over $T^*X^{(1)}$: localize over the cotangent bundle!

3. D_X-Coh : D_X-modules, q-coherent as $\mathcal{O}_X$-mod + locally finitely generated as D_X-modules.

$= \left\{ (M, \nabla) \quad \begin{array}{l} M \text{ quasicoherent sheaf,} \\ \nabla = \text{flat connection} \end{array} \right\}$

<u>Remark</u> In char. p, connections have p-curvatures $R_p(\nabla) = \nabla_\partial^{\,p} - \nabla_{\partial^{[p]}}$

$\longleftrightarrow$ action of $\partial^p - \partial^{[p]} \in D_X$

$R_p(\nabla) : F_X^*(T_X) \otimes M \longrightarrow M.$

Slogan: computing p-curvature $\Longleftrightarrow$ measuring support of M on $T^*X^{(1)}$.

<u>Def</u> X is $\mathcal{D}$-affine if

$$D^b(D_X\text{-coh}) \underset{D_X \overset{L}{\underset{R\Gamma(X,D_X)}{\otimes}} -}{\overset{R\Gamma(-)}{\rightleftarrows}} D^b(R\Gamma(X, D_X)\text{-mod}_{fg})$$

are inverse equivalences of categories.

e.g. if X is affine.

Beilinson-Bernstein: in char. 0 flag varieties are $\mathcal{D}$-affine

Theorem G reductive group (connected)
$X = G/B$ flag variety, $p > \max\left(\begin{smallmatrix}\text{coxeter \#s of} \\ \text{simple components} \\ \text{of } G\end{smallmatrix}\right)$
$\Rightarrow$ X is $\mathcal{D}$ affine

Proof $A = R\Gamma(X, \mathcal{D}_X)$

$$D^-(A) \underset{R\Gamma}{\overset{LL}{\rightleftarrows}} D^-(\mathcal{D}_X\text{-coh})$$

adjoint functors.

Witchcraft $\longrightarrow$ $R\Gamma(LL(A)) = A$
$\longrightarrow$ $LL\, D^b(A) \in D^b(\mathcal{D}_X)$

If these hold: rest of proof is straightforward:
it follows $R\Gamma \circ LL \simeq Id_{D^b(A)}$
$\Rightarrow$ semi-orthogonal decomposition

$$D^b(\mathcal{D}_X) = \underset{\substack{\in \\ a}}{Im\, LL} \oplus \underset{\substack{\in \\ b}}{Ker\, R\Gamma}$$

[$R\mathrm{Hom}(a, b) = 0$.
& $\forall c$ has decomposition triangle] $\quad a \to c \to b \to a[1]$

• $\mathcal{D}_X$ is "Calabi-Yau"
i.e. Serre functor is shift by $\dim T^*X$
$R\mathrm{Hom}(a,b) = R\mathrm{Hom}(b, S[\dim T^*X])$
$\Rightarrow$ decomposition is actually orthogonal
but variety is connected & quasiprojective
$\Rightarrow$ no such exist, so $\underline{\mathrm{Ker}\, R\Gamma = 0}$.

∎

PART 2

• Intro • G • $G^\vee$

Intro $\quad D_X\text{-Coh} \longleftrightarrow \mathcal{D}_X\text{-Coh}$
$\qquad X \qquad\qquad T^*X^{(1)}$

1. How to use it?
2. How to lift information to characteristic zero?

2 : witchcraft

1 : Science. eg. Use gerbe geometry

Find $Y \subset T^*X^{(1)}$ s.t. $\mathcal{D}_X|_Y$ splits.

ie $\exists V_Y$ vector bundle s.t. $\mathcal{D}_X|_Y \simeq \operatorname{End} V_Y$

eg • $Y \subset T^*X^{(1)}$ open no good

..... $\operatorname{End} V_Y$ has lots of zero divisors

but $\mathcal{D}_X$ has none! so can't split.

• $Y = X^{(1)} \subset T^*X^{(1)}$: splits

• Conjecture : $Y \subset T^*X^{(1)}$ Lagrangian

$\Rightarrow \mathcal{D}_X|_Y$ splits. $(V = F_* \mathcal{O}_X)$

– true for graphs of sections or conormals to subvarieties

• Braverman-Bezrukavnikov:

Y = generic fiber of $h : T^*Bun_n \to Hitch_n$

$\Rightarrow \mathcal{D}_X|_Y$ splits.

- G : connected reductive group / $\mathbb{F}$
  char $\mathbb{F} = p >$ max (Coxeter #s)

$B \subset G$ Borel, $X = G/B$ flag variety
$\mathfrak{g} = \mathrm{Lie}\, G \circlearrowleft [p]$ restricted Lie algebra.

ex. $\mathfrak{g} \subset \mathfrak{gl}_n$ $x^{[p]} = x^p \in \mathrm{Mat}_{n\times n}(\mathbb{F})$
[note $\mathrm{Tr}(x^p) = \mathrm{Tr}(x)^p$]

$$\mathcal{U} = \mathcal{U}(\mathfrak{g}) \supset Z(\mathcal{U}) \supset \underbrace{Z_p}_{p\text{-center}} = \langle x^p - x^{[p]} \rangle = \mathcal{O}(\mathfrak{g}^{*(1)})$$

Harish-Chandra center $Z_{HC} = \mathcal{U}^G \simeq \mathcal{O}(\mathfrak{g}^*)^G$.

Back to Beilinson-Bernstein localization:

$$R\Gamma(X, D_X) = \Gamma(X, D_X) = \mathcal{U}^0 = \mathcal{U}/(Z_{HC}^+)$$

quotient by augmentation ideal of HC center.

$X = \{\mathfrak{b} \subseteq \mathfrak{g}\}$ Borel subalgebra

$T_\mathfrak{b} X = \mathfrak{g}/\mathfrak{b}$, $T^*_\mathfrak{b} X = \mathfrak{b}^\perp = \{\alpha \in \mathfrak{g}^* : \alpha|_\mathfrak{b} = 0\}$

$T^*X \xrightarrow{\mu} \mathcal{N} \subset \mathfrak{g}^*$ closed

$(\mathfrak{b}, \alpha) \longmapsto \alpha$

moment map for G action.

$D_X \leftarrow U^0 \twoheadleftarrow U$

$\mathcal{O}_{T^*X^{(1)}} \leftarrow Z_p/Z_{p+}^G \twoheadleftarrow Z_p$ on p-centers!

as sheaf on $X^{(1)}$

get exactly Frobenius-twisted Springer map!

LL, $R\Gamma$ are $\mathcal{O}(\mathcal{N}^{(1)})$-equivariant

<u>Theorem</u> $\forall \chi \in \mathcal{N}^{(1)}$, $D^b(U^0\text{-mod}_\chi)$ is equivalent to $D^b(\mathcal{D}_X\text{-coh}_{\mu^{-1}(\chi)})$

$\simeq D^b(T^*X^{(1)}\text{-coh}_{\mu^{-1}(\chi)})$

... $\mathcal{D}_X$ splits on formal neighborhood of Springer fibers, so get representations of fixed infl character $\longleftrightarrow$ coherent sheaves on Springer fibers

(splitting comes from Steinberg module)

Different splittings give different t-structures.

χ gives max. ideal $\mathfrak{m}_\chi \subset \mathcal{O}(\mathcal{N}^{(1)})$

$M \in U^0\text{-mod}_\chi \iff \exists n \quad \mathfrak{m}_\chi^n M = 0.$

Let Y_χ = set of simple modules in $U^0\text{-mod}_\chi$.

$M \in Y_\chi \Rightarrow \mathfrak{m}_\chi M = 0. \Longleftrightarrow$
irred module for f.dim algebra $U^0_\chi = U^0/\mathfrak{m}_\chi$.

$Y_\chi = \operatorname{Irr} U_\chi^0$

<u>Corollary</u> $|Y_\chi|$ = Euler characteristic of $\mu^{-1}(\chi)$
- conjecture of Lusztig

G acts on U^0. Let G_χ = maximal reductive subgroup of $\operatorname{Stab}_G(\chi)$,

acts on U^0_χ. $\Longrightarrow G_\chi \circlearrowright Y_\chi$.

In fact Y_χ is an "extended" G_χ-set:
pick module M.

$\mathrm{Stab}_{G_\chi} M$ acts on $\mathrm{End}(M)$, which is a direct summand of $\overline{U}_\chi^0$
$= U_\chi^0 / \mathrm{Rad}\, U_\chi^0$

$\Rightarrow$ get central extension by F^* of the stabilizer of each $M \in Y_\chi$.

$\mathrm{Rep}\, G_\chi \circlearrowright \overline{U}_\chi^0\text{-mod}_{G_\chi}$
by tensor product.

- Y_χ can be recovered from the categorical picture.

- $G^\vee$ picture $LG^\vee \supset L_+G^\vee \supset I$ Iwahori:

$I = (LG^\vee_+ \longrightarrow G^\vee)^{-1}(B)$

$\mathcal{F}\ell = LG^\vee / I$. Convolution no longer commutative!

$I \backslash Fl \longleftrightarrow W_{aff} = W \ltimes \Lambda.$

$$Z : D^b_{LG_0}(Gr, D) \longrightarrow D^b_I(Fl, D)$$

central functor

Let $P_I \subseteq D^b_I(Fl, D)$

category of equivariant perverse sheaves

Simple objects $IC(Fl_w)$ $w \in W_{aff}$

Have 3 Kazhdan-Lusztig pre orders
(not necessarily symmetric)

$\geq_L$ $\geq_R$, $\geq_{LR}$ $\Rightarrow$ 3 equivalence
relations $\sim_L$, $\sim_R$, $\sim_{LR}$

$x \in \mathcal{N} \Rightarrow$ canonical distinguished involution
$d_x \in W_{aff}$

$\Rightarrow$ order reversing bijection

$$W_{aff} / \sim_{LR} \longleftrightarrow \mathcal{N}/G$$
$$[d_x] \longleftrightarrow G \cdot x$$

$\forall w \in W_{aff}$

$$\mathcal{P}_I^{\leq w} = \langle IC(Fl_s) : [s] \leq [w] \rangle$$

$$\mathcal{P}_I^{x} = \mathcal{P}_I^{\leq_{LR} d_x} / \mathcal{P}_I^{<_{LR} d_x}$$

$\cup$

$\mathcal{M}_x$ = semisimple objects in $\mathcal{P}_I^x$

– monoidal, structure comes from taking perverse cohomology of convolution

$\mathcal{M}_x \supseteq \mathcal{L}_x = \langle IC(Fl_w) : w \underset{L}{\sim} d_x \rangle$

canonical left cell

$\cup$

$\mathcal{A}_x = \langle$ subquotients of $IC(Fl_{d_x}) * Z(V),\ V \in Rep\, G \rangle$

$\uparrow$

$Rep\, G$

In fact $\mathcal{A}_x \simeq Rep\, G_x$.

A_x acts on L_x by conductiy

$\implies$ recover an extended G_x-set Y_x' from this picture.

<u>Conjecture</u> $Y_x \simeq Y_x'$ as extended G_x-sets.