Graeme Segal - Locality in QFT, I

Note Title 11/14/2006

Philosophy: how do we formulate QFT?

Classical physics: describe world unproblematically
content: point particles,
interaction: fields — functions on spacetime.
We're given spacetime, world = some points & structures on this,
physics gives evolution of these data.

Quantum physics: nothing like this,
"Observables" & "States" —
observe system in state $\rightsquigarrow$ number.

relation to classical physics:
configurations of particles & fields
$\longrightarrow$ ∞ dim manifold, space of states, with flow on it.

QM only starts here, doesn't relate
this ∞ dim manifold of states
to "the world"

manifold of states $\rightsquigarrow$ spectrum of an algebra.

QFT : build in classical ontological structure!
have a spacetime again, as
(pseudo)Riemannian manifold, again trying
to describe things in spacetime...

& moreover everything is local.

QM	QFT
A algebra of observables	X = spacetime manifold
$[\mathcal{O}_x \subset A$ subalgebra$]$	$x \in X \Rightarrow \mathcal{O}_x$ vector space of observables at $x \in X$
state $A \xrightarrow{\theta} \mathbb{C}$	$\mathcal{O}_{x_1} \times \cdots \mathcal{O}_{x_k} \xrightarrow{\text{mult}} A \xrightarrow{\theta} \mathbb{C}$ correlation functions

Locality in classical physics:
Bohm–Aharonov experiment

EM field: has field strength at every pt of spacetime.

Now B.-A. constructed E.M field on an annulus & took particles around a loop -- can arrange field strength everywhere zero, but when electron goes around something happens, depending on the field!

Really need to think of EM field in terms of line bundle L on spacetime w/ unitary connection, $x \mapsto L_x$ + parallel transport along paths

vector bundles are <u>local</u> to us, as are connections

But to a physicist: can't <u>detect</u> L locally no way to know local invariants for a flat connection..... lines are not like numbers, don't arise as readings,

We distinguish between (L, ∇) which is local vs the <u>isomorphism class</u> which is nonlocal: local triviality does not give global triviality

So we're restoring locality by lifting from sets to categories

Topology! cohomology classes likewise non local! Mayer-Vietoris gives obstruction to local vanishing $\Rightarrow$ global vanishing

— but representing cocycles can be local eg diff forms..

$C^\bullet$ cochain complex: introduce category

$\mathrm{Obj} = Z^p$ p-cocycles,

$\mathrm{Mor}(z_0, z_1) = \{ c \in C^{p-1} : z_1 = z_0 + dc \} / d(C^{p-2})$

... so objects are local, & isom classes are $H^p(C^\cdot)$

but also if $C^\cdot \longrightarrow C^{\cdot\prime}$ chain homotopy equivalence $\Rightarrow$ equivalent categories ....
want a category that doesn't depend on choice of complex $C^\cdot$ up to chain homotopy equivalence.

If $p \geq 2$ this <u>doesn't</u> restore locality because of our quotienting:

morphisms not local, since <u>automorphisms</u> are now $H^{p-1}(C^\cdot)$

One solution: go to higher categories

X space. crudest approximation: set $\pi_0 X$

Next invariant: fundamental groupoid $\pi_1 X$, category
Ob = points Mor = paths / homotopy

$\Pi_2(X)$: 2-category ... don't collapse paths up to homotopy.

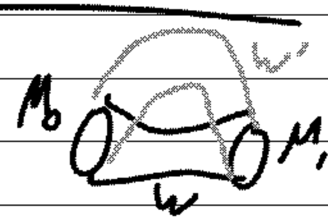
Objects = points

Morphisms $\mathrm{Hom}(x_1, x_2)$ = <u>category</u>
with obj = paths $x_1 \longrightarrow x_2$
mor = homotopy of paths,
up <u>to homotopy</u>

ie $\mathrm{Hom}(x_1, x_2) = \Pi_1\left(\mathrm{Paths}(x_1 \longrightarrow x_2)\right)$
fundamental groupoid.

Composition is not <u>associative</u>, only up to homotopy ... (or natural transformations) .... Lax 2-category.

Cobordism theory:
category of n-dim manifolds
with morphisms = cobordisms



Make into 2-category

$$\begin{tikzcd} M_0 \arrow[r, bend left=50, "W"] \arrow[r, bend right=50, "W'"'] \arrow[r, phantom, "Y \Downarrow"] & M_1 \end{tikzcd}$$

Back to QFT

Want to encode correlation functions

Motivation from path integral:

X = spacetime, $\varphi \in \Phi_X$ = "fields" on X

"probability measure" $e^{-S(\varphi)}\,\mathcal{D}\varphi$

Observables = local functionals on fields, can measure infinitesimally

e.g.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & M \arrow[dr, "f"] & \\ & & \mathbb{R} \end{tikzcd}$$

can eg take $\varphi \longmapsto f(\varphi(x))$ & derivatives

Correlations of $F_1 \in \mathcal{O}_{x_1}, \dots, F_k \in \mathcal{O}_{x_k}$

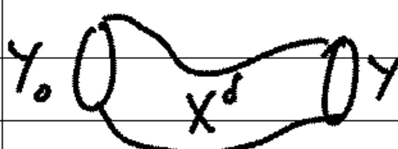
are $$\int_{\Phi_X} F_1(\varphi)\cdots F_k(\varphi)\, e^{-S(\varphi)}\,\mathcal{D}\varphi$$

What axioms do these satisfy?

Segal
(following Witten) inclined to say
a d-dim QFT is a rule

closed oriented d-manifold $X \longmapsto U_X \in \mathbb{C}$

Closed oriented $(d-1)$ manifold Y
$\longmapsto$ topological vector space $\mathcal{H}_Y$

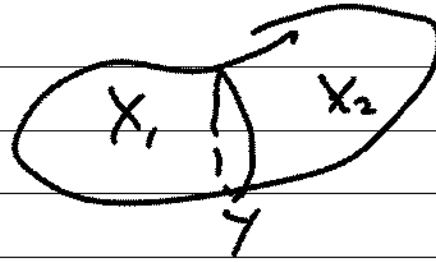


cobordism $\longmapsto$
$U_X : \mathcal{H}_{Y_0} \longrightarrow \mathcal{H}_{Y_1}$
(trace class)

- $\underline{\sqcup} \longmapsto \otimes$ (monoidal functor)
- concatenation of cobordisms $\longmapsto$ composition of morphisms

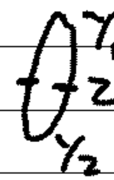
$\rightsquigarrow$ can extract system of correlation functions from this

Is this a reasonable formulation?
haven't encoded <u>enough</u> locality

Can cut 

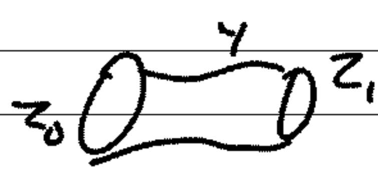
but can't still reduce to uniform little bits: each slice will still have topology ... to get to a truly local picture need higher codimensions ...

need to reconstruct $\mathcal{H}_Y \simeq \mathcal{H}_{Y_1} \underset{Z}{\ast} \mathcal{H}_{Y_2}$

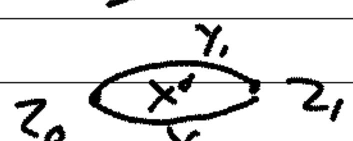
for 

$= Y$

Z^{d-2} closed $d-2$ manifolds $\rightsquigarrow$ (topological) linear category $\mathcal{C}_Z$



$\rightsquigarrow$ functor $\mathcal{H}_Y : \mathcal{C}_{Z_0} \longrightarrow \mathcal{C}_{Z_1}$



$\rightsquigarrow$

$$\begin{tikzcd} \mathcal{C}_{Z_0} \arrow[r, bend left, "\mathcal{H}_{Y_2}"] \arrow[r, bend right, "\mathcal{H}_{Y_1}"'] \arrow[r, phantom, "\Downarrow \mathcal{H}_X"] & \mathcal{C}_{Z_1} \end{tikzcd}$$

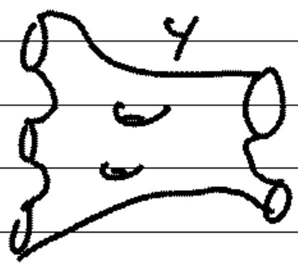
<u>Example</u> Chern-Simons, 3d TFT

G compact, $k \in \mathbb{Z}_+$ level

$\Leftrightarrow$ central extension

$$\mathbb{T} \to \widetilde{LG} \to LG$$

$S^1 \longmapsto$ category of pos. energy finite type reps of LG at level k



$\longmapsto$ functor $\Phi_Y : \mathrm{Rep}_k LG^{\times 3} \longrightarrow \mathrm{Rep}_k LG^{\times 2}$

go down to a point!

$\bullet \longmapsto$ 2-category with one object & automorphisms $= \mathrm{Rep}_k LG$.

Composition given by fusion $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

Should we start from points at bottom or from points.

QFT basic structure! bundle of $\mathcal{O}_x$ & the operator product expansion

$x \to x'$ move together

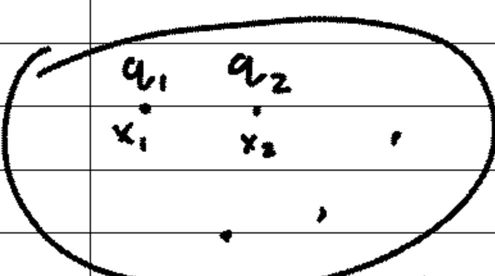
$\psi_x \in \mathcal{O}_x \quad \psi_{x'} \in \mathcal{O}_{x'}$

$\rightsquigarrow$ get an expansion

$$\psi_x * \psi_{x'} = \sum_i \psi_x^{(i)} (x-x')^i$$

Analogy: QFT $\longleftrightarrow$ iterated loop spaces/config. spaces

Look at configs of points of X labelled by an auxilliary space Φ



& let points collide as labels amalgamate

... eg label by integers which add under collision $\Longrightarrow$

$\coprod \mathrm{Sym}^d X$

Can make $(Q)^n$ bundle over $(X^d)^n$

but want to glue together : use structure of d-fold loop space

1-fold loop space $\Longleftrightarrow$ A_∞ space $Q \times Q \to Q$
homotopy associative product

$\Longleftrightarrow$ gives topology on configuration space :

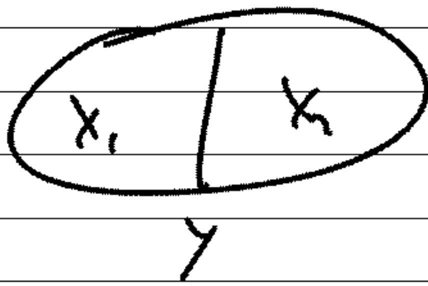
$x_1, \dots x_n$ $\qquad Q^n \hookrightarrow \mathcal{C}_n$

d-fold loop space structure on Q
$\Longleftrightarrow$ factorization structure over X^d.

$X^d_{\text{space}} \rightsquigarrow$ Q-labelled config space $C_Q(X)$, space

closed $d-1$ manifold $Y^{d-1} \rightsquigarrow$ 1-fold loop space
closed $d-2$ " $Z^{d-2} \rightsquigarrow$ 2-fold loop space
etc

with composition laws just as in QFT:



$X_1, X_2 \longleftrightarrow$ spaces with action of 1-fold loop space $C(Y)$
$= C(\partial X_1)$

$$C(X) = C(X_1) \times_{C(Y)} C(X_2)$$

and so on down.

Q d-fold loop space $\rightsquigarrow B^d Q$
d-fold classifying space
= Q labeled configs on the d-disc, rel the boundary.

Q! Theorem $C(X) \simeq \mathrm{Maps}(X, B^d Q)$

... so reconstruct everything from what we assigned to a point, i.e. Q.

Try to do converse: given auxiliary space M, study $\mathrm{Map}(X, M)$

$Y \rightsquigarrow \mathrm{Map}(Y, \Omega M)$

= maps of nbhd of Y to M mapping boundary to basepoint...

to pt $\rightsquigarrow$ $Q = \Omega^d M$.

Y

Is it true that $\mathrm{Map}(X; M) = \mathrm{Map}(X, B^d \Omega^d M)$?, only true if M is d-connected...

Examples: K-theory of X $= \mathrm{Map}(X, \mathbb{Z} \times BU)$

Algebraic Local model: X smooth variety can model using <u>coherent sheaves</u>

$C(X)$ = cont fns on X

$\mathrm{Map}(X, \mathbb{Z} \times BU) \longleftrightarrow$ f.g. projective $C(X)$-modules

fin gen $C(X)$-modules

$\longleftrightarrow$ fin many points &

fin dim vector space at each

Theorem: space of these is the space of maps $\mathrm{Map}(X, (\mathbb{Z} \times BU)^{(d)})$

d-fold connected cover!

connective K-theory of X.

to pass between them need observables not localized at points $\longleftrightarrow$ failure of connectivity of target