Graeme Segal - Locality in QFT 2

Note Title 11/15/2006

Today: Free massive bosonic field theory in d dimensions

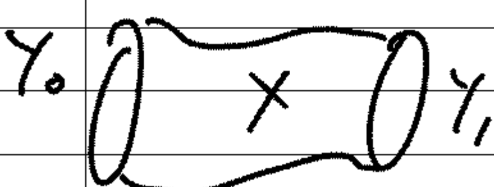
classical description:

X spacetime (Riemannian or pseudo-), $\varphi : X \longrightarrow \mathbb{R}$ scalar field

$$\text{Action} \quad S(\varphi) = \frac{1}{2}\int_X \{ d\varphi . * d\varphi + m^2 \varphi . * \varphi \}$$

$$\updownarrow$$

$$\int_X \|d\varphi\|^2 + m^2 \varphi^2 \, d\mathrm{vol}$$



cobordism: think of $S(\varphi)$ as defining an <u>integral transform</u>

from $\mathcal{H}_{Y_0} = L^2(\Omega^0(Y_0))$ functions on space of fields

$$\Big\downarrow U_X$$

to $\mathcal{H}_{Y_1}$

Φ_0 functional on $\Omega^0(Y_0)$ for $f_1 \in \mathcal{H}_{Y_1}$

Set $(U_X \Phi_0)(f_1)$

$$= \int_{\left\{ \begin{array}{l} \varphi: X \to \mathbb{R} \\ \varphi|_{Y_1} = f_1 \end{array} \right\}} e^{-S(\varphi)} \, \Phi_0(f_0) \, \mathcal{D}\varphi$$

$f_0 = \varphi|_{Y_0}$

How to make sense of this?

These are Gaussian integrals, have good theory.

In physics X would be pseudo-Riemannian, Y would be Riemannian.

For us X is Riemannian - hence replace $e^{iS(\varphi)}$ by $e^{-S(\varphi)}$:

(Riemannian structure comes in through $\times$ operator)

... really working in space of $\mathbb{C}$-valued maps!

try to analytically continue from Riemannian to Lorentzian.

$S(\varphi)$ In local coords : $\swarrow$ d vol

$$g^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \quad \det(g_{ij})^{\frac{1}{2}} dx_1 \ldots dx_d$$

(+ mass term)

Want to allow g^{ij} complex but varying only in open set where $\mathrm{Re}(g^{ij})$ is a <u>positive</u> quadratic form.

Minkowskian: change sign of $g^{0,0}$, makes $(\det g)^{\frac{1}{2}}$ purely imaginary now

--- Minkowskian metrics are on the <u>boundary</u> of the open set of allowable complex metrics,

$\Rightarrow$ physical theory would be a boundary value of our theory.

Gaussian integrals $\int e^{-\frac{1}{2}\langle x, Ax\rangle}\,dx = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}}$

V topological vector space

$q : V \longrightarrow \mathbb{R}$ pos. def. continuous quadratic form

$\rightarrow$ can consider Gaussian integral

$e^{-\frac{1}{2}q(v)}\,dv$ (dv = some Haar measure on V)

Think of $q : V \longrightarrow V^*$ symmetric

$\det q : \Lambda^{top} V \to \Lambda^{top} V^*$

ie $\det q \in (\Lambda^n V)^{\otimes -2}$

[$n = \dim V$]

$(\det q)^{\frac{1}{2}} \in \Lambda^{top} V^{-1}$

$\det q^{-\frac{1}{2}} \in \Lambda^{top} V$

- so need dv valued in $\Lambda^{top} V^*$

so $\int e^{-\frac{1}{2}q(v)}\,dv$ gives a <u>number</u>

In finite dimensions we can define $\int e^{-\frac{1}{2}q(v)}$ polynomial$(v)\, dv$

[e.g. for linear forms $\alpha(v), \beta(v) \in V^*$

$$\int e^{-\frac{1}{2}q(v)} \alpha(v)\beta(v)\, dv = \langle \alpha, q^{-1}\beta \rangle \left(\det \frac{q}{2\pi}\right)^{-\frac{1}{2}}]$$

∞-dim! don't know how to define $\left(\det \frac{q}{2\pi}\right)^{-\frac{1}{2}}$ --- so call it 1, formally can define L^2 (inner product) using above formula

$\longrightarrow$ $L^2_q(V)$ makes sense

To what extent does this depend on q?

When does completion of space of functions $\{e^{-\frac{1}{2}q(v)} \cdot \text{poly}\}$ contain $e^{-\frac{1}{2}\tilde{q}(v)}$?

ie when do $q, \tilde{q}$ define same completed vector space $L^2_q = L^2_{\tilde{q}}$?

We're asking to pass from $\left(\det \frac{q}{2\pi}\right)^{-\frac{1}{2}}$ to $\left(\det \frac{\tilde{q}}{2\pi}\right)^{-\frac{1}{2}}$:

to pass between must multiply by $\det\left(\tilde{q} q^{-1}\right)^{-\frac{1}{2}}$

So need $\tilde{q} q^{-1}$ to be sufficiently close to identity: $\tilde{q} q^{-1} = 1 + T$

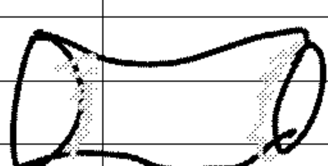
trace class

$\Rightarrow$ then $\det(\tilde{q} q^{-1})$ makes sense: $\prod (1+\lambda_i)$ converges for λ_i eigenvalues of T trace class.

So spaces $L^2_q = L^2_{\tilde{q}}$ when $\tilde{q} q^{-1} \in 1 + \text{trace class}$!

Isomorphism $L^2_q(V) \simeq L^2_{\tilde{q}}(V)$
not compatible with normalization $\|e^{-\frac{1}{2}q}\| = 1$
— multiplies norm by $\det(\tilde{q}q^{-1})$...
so really we've only defined the Hilbert space projectively $\longleftrightarrow$ inner product takes values in a <u>determinant line</u> not $\mathbb{C}$

Classical scalar fields: $S(\varphi)$ critical
solutions of field equations
$$\Delta\varphi + m^2\varphi = 0 .$$



Look at such classical φ near boundary: determined by their Cauchy data — value + normal derivative

$$\varphi , \quad *d\varphi|_Y \in \Omega^{d-1}(Y_0)$$

i.e classical Cauchy data live

in $\Omega^0(Y_0) \oplus \Omega^{d-1}(Y_0) =: \Sigma_{Y_0}$

naturally dual :

i.e Σ_{Y_0} is a symplectic vector

space associated to Y_0.

$W_X := \big($Space of classical solns on $X\big)$

$\hookrightarrow \Sigma_{\overline{Y_0}} \oplus \Sigma_{Y_1} =: \Sigma$ symplectic direct sum where we've reversed orientation of Y_0

& the image is

a closed Lagrangian subspace

in the Minkowskian case

(where we use $i * d\varphi|_{Y_0}$ as Cauchy data

making it a real form)

For complex metrics $W_X \subset \Sigma_{\mathbb{C}}$,

stays maximal isotropic

but becomes positive isotropic!

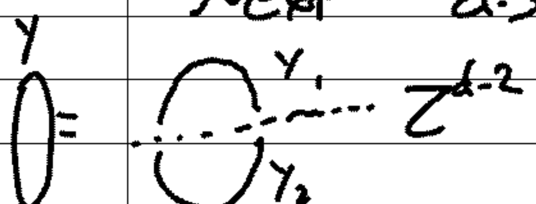
use W_x $\qquad$ S = symplectic form on Σ_x

$i\, S(\bar{w}, w) > 0$

$(S(\bar{w}, w) = -S(w, \bar{w}) \Rightarrow$ pure imaginary)

On the boundary get real isotropic

Next discuss locality of $\mathcal{H}_Y$ in Y

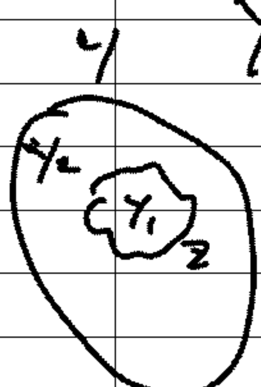


$\mathcal{H}_Y \overset{?}{=} \mathcal{H}_{Y_1}$ "$\times$" $\mathcal{H}_{Y_2}$

Physical significance:

Y = space, Y_1 = region in space, "accessible to us"

Y_2 = its complement, "far away"



Quantum mechanics!

A_Y algebra of operators on $\mathcal{H}_Y$

$a \in A_Y$ $\quad$ $\xi \in \mathcal{H}_Y$ $\Rightarrow$ expected value $\langle \xi, a\xi \rangle$

Locality in regions of Y:
have operators A_{Y_i} supported in Y_i

& they should <u>commute</u> in A_Y
since Y_0, Y_1 disjoint

$$A_{Y_1} \hookrightarrow \mathcal{H}_Y \hookleftarrow A_{Y_2} \text{ "unknown"}$$

we want to study $\mathcal{H}_Y$ just in terms of our local operators ----
we want to see to what extent this picture is associated only to Y_1
---- $\mathcal{H}_Y \circlearrowleft_{A_{Y_2}}$ is determined by Y_1 up to isomorphism:
A_{Y_2} is a "white noise" determined purely by Y_1 ----

Back to free theory: $\mathcal{H}_Y = \mathbb{C}^2(\Omega^\circ(Y))$

quantization of symplectic space Σ_Y

Let $\mathrm{Heis}(\Sigma_Y)$ = Heisenberg group of Σ_Y

$$\mathbb{T} \to \mathrm{Heis}(\Sigma_Y) \to \Sigma_Y$$

$$g_\xi \longmapsto \xi$$

$$\boxed{g_{\xi_1} \cdot g_{\xi_2} = e^{iS(\xi_1, \xi_2)} \cdot g_{\xi_1 + \xi_2}}$$

$\mathcal{H}_Y$ is described as an irreducible representation of $\mathrm{Heis}(\Sigma_Y)$

.... $L^2(V)$ is a unitary rep of V & V^*

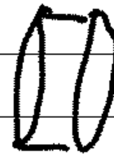
with Heisenberg commutation rules $\underset{T}{\text{translating}}$ $\underset{M}{\text{mult.}}$

$$T_v M_{e^{i\alpha}} = e^{i\alpha(v)} M_{e^{i\alpha}} T_v$$

For every quadratic form $q : V \to V^*$ on V get such a representation (graph (q) is an isotropic subspace).

On Σ_Y have a class of isotropic subspaces: boundary values of classical solutions on X for different X with $\partial X = Y$

— & corresponding class of forms only depends on normal data to X along Y

— use Lagrangian  coming from $Y \times [0,1]$

— get same Hilbert space up to scaling provided we fix the germ of the metric along Y.

Need to normalize these scalars:

put $\| e^{-\frac{1}{2} q} \| = \det_\zeta \left(\frac{q}{2\pi} \right)^{-\frac{1}{2}}$:

... use ζ-function determinant:
has property $\det(q^{-1}\tilde{q})\det_\zeta q = \det_\zeta \tilde{q}$
whenever $q^{-1}\tilde{q} \in 1+$ trace class.

$G_Y = \mathrm{Heis}(\Sigma_Y) \curvearrowright \mathcal{H}_Y$ irreducible unitary rep

For $Y_1 \subset Y$ can define subgroup
$G_{Y_1} \subset G_Y$ consisting of group elements
supported in interior of Y_1

$\mathcal{H}_Y$ has a collection of vectors $e^{-\frac{1}{2}q}$
corresponding to quadratic forms.
& any such $e^{-\frac{1}{2}q}$ is cyclic for G_{Y_1}
(case of Reeh-Schlieder theorem in QFT)

So can't have $\mathcal{H}_Y = \mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$:

time evolution of system $\in$ completed group algebra of $\mathcal{G}_Y$,

can write as sum of pieces in Y_1 & Y_2

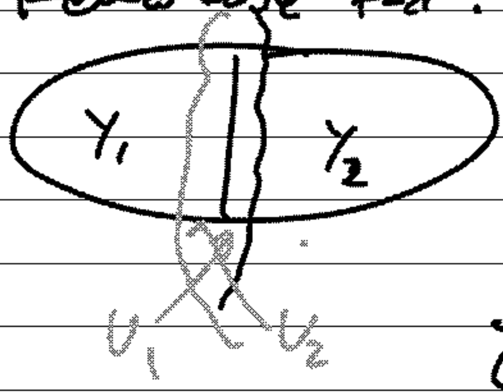
$\Rightarrow$ time evolution would preserve decomposition $\Rightarrow$ two parts Y_1, Y_2 have no interaction!

OTOH $\mathcal{H}_Y \big|_{\mathcal{G}_{Y_1} \times \mathcal{G}_{Y_2}}$ remains irreducible

$\to$ would have expected in finite dim that $\mathcal{H}_Y$ decomposes as $\otimes$ product.

But $\mathcal{H}_Y$ is decomposable under $\mathcal{G}_{Y_1}$ & any invariant subspace is isomorphic to the whole space — Type III representation

Remarkable fact: enlarge Y_1, Y_2 a little



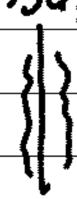
$$\mathcal{G}_{U_2} \circlearrowright \mathcal{H}_{Y_1 \cup U_2} \circlearrowleft \mathcal{G}_{Y_1}$$

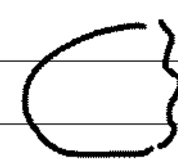
... Given representations

$\mathcal{H}_L$ of $\mathcal{G}_{U_2}$ & $\mathcal{H}_R$ of $\mathcal{G}_{U_1}$

can define Connes tensor product

$$\mathcal{H}_L \underset{(\mathcal{G}_{U_1}, \mathcal{G}_{U_2})}{\times} \mathcal{H}_R$$

$\mathcal{H}_0 \longleftrightarrow$ 

$\mathcal{H}_L \longleftrightarrow$ 

$\mathcal{H}_R \longleftrightarrow$ 

want to define $\mathcal{H}_L \underset{\mathcal{H}_0}{\otimes} \mathcal{H}_R$

$$\mathrm{Hom}_{\mathcal{G}U_R}(\mathcal{H}_0, \mathcal{H}_L) \underset{\mathcal{G}U_L}{\otimes} \mathcal{H}_0 \underset{\mathcal{G}U_R}{\otimes} \mathrm{Hom}_{\mathcal{G}U_L}(\mathcal{H}_0, \mathcal{H}_R)$$

(complete wrt a norm)

$$\|$$

$$\mathcal{H}_L \underset{\mathcal{H}_0}{\otimes} \mathcal{H}_R$$

$$\wr|$$

$$\mathcal{H}_Y$$

We're in situation where a group is acting cyclically,

$$\mathcal{G}U_R \subset \mathrm{End}_{\mathcal{G}Y_1}(\mathcal{H}_L)$$

& $\mathcal{G}U_R$ generates this End ... it's its own double commutant

.... Type III rep theory