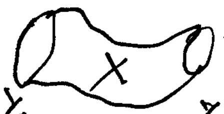
G. Segal : Locality of the space of Quantum States 8/9/04

d-dim QFT :

Data: Y closed cpt $(d-1)$ Riemannian mfld $\longmapsto \mathcal{H}_Y$ vect space



Y_0 X Y_1 Riemannian cobordism $\longmapsto U_X : \mathcal{H}_{Y_0} \longrightarrow \mathcal{H}_{Y_1}$

trace class operator

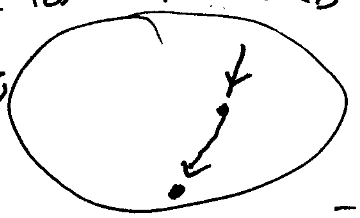
$\mathcal{H}_{Y_0 \sqcup Y_1} \cong \mathcal{H}_{Y_0} \otimes \mathcal{H}_{Y_1}$

Restricting to closed $Y \longrightarrow$ only UV properties, discard IR phenomena

Trace class $\longrightarrow$ compact theory - like compact target σ-model.

$\Rightarrow$ RG flow on such theories: rescale metric by $\mathbb{R}_+$, get 1-parameter trajectories. Hope for a Palais-Smale phenomenon

theory



Hope to come to rest (conformal) at a stationary pt

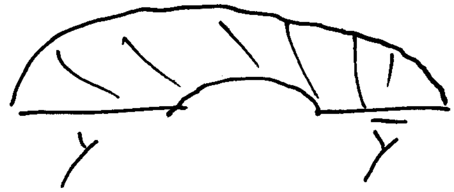
- eg 2D YM, rescale coupling constant. At bottom get trivial theory, at top get topological but noncompact theory - leave space of nice theories.

Dependence on Y: need $\mathcal{H}_Y$ really to depend on inf. nbhd of Y in d-dim manifold... assign $\mathcal{H}$ to germ of d-manifold around Y ... need eg to sew cobordisms!

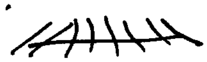
d-dim free field theories (eg massive boson or fermion) ...> $\mathcal{H}_Y$ depends really on $\left[\frac{d-1}{2}\right]$ normal derivatives of metric

- eg in 4 dim need first fundamental form of timeslice.

Role of unitarity:
cylindrical cobordism



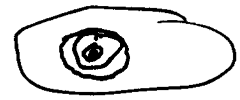
$\Rightarrow \mathcal{H}_Y \otimes \overline{\mathcal{H}_Y} \longrightarrow \mathbb{C}$ (depending on length of cylinder) ... get inner product

This depends on shape of collar: 
shrinking space is complex conjugate to expanding space. So only expect inner product if have collar-reversing isometry!

traditional QFT: $x \in X$ d-dim $\overset{\text{closed}}{\Longrightarrow}$ $\mathcal{O}_x$ fields at x
expectations: $x_1, \dots, x_k$ disjoint $\Longrightarrow$
$$\mathcal{O}_{x_1} \otimes \dots \otimes \mathcal{O}_{x_k} \longrightarrow \mathbb{C} \quad \text{vacuum expectation value.}$$

How does this arise?

cut out discs around points:
$\Sigma = X \setminus \bigcup D_i$ $\quad D_i$ = disc at x_i
is a cobordism $\coprod \partial D_i \longrightarrow \emptyset$
$\Rightarrow$ attempt to define fields at x using discs at x:
$$\mathcal{V}_x = \varprojlim_{x \in D} \mathcal{H}_{\partial D}$$



($D_1 \subset \mathring{D}_2 \Rightarrow$ annulus gives cobordism hence
$\mathcal{H}_{\partial D_1} \longrightarrow \mathcal{H}_{\partial D_2}$, can take inverse limit)

$\longrightarrow \mathcal{V}_x$ is all insertions at x... too big though!

$\varphi \in \mathcal{O}^0$:= "scalar fields in theory":
a rule assigning to a cobordism $X : Y_0 \rightsquigarrow Y_1$
a map $\varphi : \mathring{X} \longrightarrow \mathrm{Hom}_{tr}(\mathcal{H}_{Y_0}, \mathcal{H}_{Y_1})$

(... fields we can insert anywhere in X & change propagation)

s.t. composition of cobordism is composition of φ's & propagation: $X' \circ X \circ X''$ composite $\Rightarrow$

$$\varphi_{X' \circ X \circ X''}\Big|_X = U_{X''} \circ \varphi_X \circ U_{X'}$$

Similarly define higher tensor & jet fields, all give cross sections of bundle $\mathcal{U}_X$ .. naturally these things depending on finite jets of coordinates

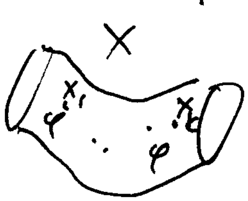
$\Rightarrow$ fields $\mathcal{O}_X \subset \mathcal{U}_X$ all local expressions.

$\boxed{\text{Missing axiom}}$ expect "2D theory ~ Riemannian mfld, 2D CFT ~ Ricci flat mfld, RG flow ~ Ricci flow"

locally moduli of CFT ~ moduli of Ricci-flats....

In traditional QFT: can deform a theory by changing Lagrangian density ---> ie add in a field: $\varphi \in \mathcal{O}$

φ-perturbation $\tilde{U}_X = \sum_{k \geq 0} \int U_{X;\, x_1, \dots x_k} \, dx_1 \cdots dx_k$

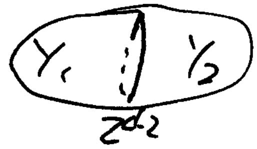


- add over all ways of inserting φ at points $x_1, \dots x_k \in X$ ...

As points collide get unbounded operators such expressions don't tend to exist!

... formally get perturbation this way, but we don't know that any deformation comes this way.... this should be some kind of axiom, need something to make this work.

Need something saying $\mathcal{U}_X$ is constructed locally out of Y...

Locality of $\mathcal{H}_Y$: reconstruct such from decompositions 

out of $\mathcal{H}_{Y_1}, \mathcal{H}_{Y_2}$.

<u>Example</u> 2D TFT $\longleftrightarrow$ commutative Frobenius algebras A, $\theta : A \longrightarrow \mathbb{C}$ s.t. $\langle a, b\rangle := \theta(ab)$ nondegenerate

Semisimple case: $A = \bigoplus_i^k \mathbb{C}e_i$, θ just gives weights for idempotents e_i. So just perturb θ by changing these weights $\theta(e_i)$: $\tilde{\theta}(a) = \theta(e^{\varphi} \cdot a)$

for $\varphi \in A$ ... looks good but it's wrong:
all φ's are <u>scalar</u> fields ... but should perturb by <u>volume</u> forms, 2-form fields.
φ perturbs A to area-dependent theory.

In practice A arises as cohomology of a chain-complex valued field theory (topological string theory)
---> get honest deformations for fields!

$A = H^*(C^\bullet)$ look on disc on fields valued in $C^\bullet$ $C_D = \Omega^\bullet(D, C^\bullet)$
double complex with same cohomology as C:
represent $a \in A$ as $a_0 + a_1 + a_2 + \dots \in \Omega^\bullet(D, C)$
perturb by 2-form component a_2 –
makes a difference over topologically interesting surfaces.

$\Longrightarrow$ Tangent to deformations are Frobenius algebras themselves, Frobenius manifolds.

So have subtlety! how did we know should pass from A to $C^\bullet$?

Topology: can't reconstruct cohomology from a decomposition but can reconstruct from chain complexes of the two parts.

Deformation theory: moduli spaces $\rightsquigarrow$ dgla's
functions on moduli = cochains of dgla
· eg complex structures $\Omega^{0,*}(X, T_X)$ dgla
cochains of Lies give fns on infi ~~def~~ formations
··· positively graded dgla
Rational homotopy theory: commutative dga
$\longleftrightarrow$ dg la, whose cohomology is ~~exact~~
rational homotopy of space. · these are
negatively graded dg Lie algebras

Suggests: Infi deformations of $X \longleftrightarrow H^*(X)$ ... get direct deformations?
In physics have $\infty/2$ complexes instead ..
get interesting deformations from homology classes of X...

<u>Locality of state space:</u>
Expand to 3-tier QFTs

X^d $\begin{tikzcd} Z_0 \arrow[r, bend left, "Y_1"] \arrow[r, bend right, "Y_0"'] \arrow[r, phantom, "X"] & Z_1 \end{tikzcd}$ natural transformation

$Y^{d-1}: Z_0 \rightsquigarrow Z_1$ cobordism $\longmapsto$ functor $F_Y: \mathcal{C}_{Z_0} \to \mathcal{C}_{Z_1}$ additive

closed $Z^{d-2} \longmapsto$ linear category $\mathcal{C}_Z$

(eg if $\partial Y^{d-1} = \phi \Rightarrow F_Y$ is linear functor Vect $\to$ Vect,
$\longleftrightarrow$ vector space $F_Y(\mathbb{C})$).

e.g 3D Chern-Simons (G, k group & level) : (up to rigging) 3D TFT
closed $X^3 \Rightarrow$ CS invariant
closed $Y^2 \Rightarrow$ vector space of conformal blocks

$S^1 = Z^1 \longmapsto$ linear category $\mathcal{C}_{S^1} = \widehat{LG}_k$ - reps (positive energy) semisimple small category.

- like reps of a finite group.

[good treatment in thesis of H. B. Posthuma]

Category $\leftrightarrow$ Space $\rightsquigarrow$ chain complex...

$\mathcal{C}_{S^1}$ is actually interesting! consider cylinder

$\mathcal{C}_{S^1} \longrightarrow \mathcal{C}_{S^1}$

functor for cylinder from $\mathcal{C}_{S^1} \longrightarrow \mathcal{C}_{S^1}$, which is equivalent to the identity

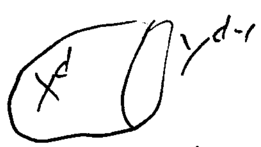
To get ~~exact fact~~ equivalence to the identity need conformal structure on cylinder, get action of Diff S^1 on identity functor (conformal structures $\sim$ Diff S^1)

$\rightarrow$ Diff S^1 action on $\widehat{LG}_k$ - reps

Free boson theory: $\varphi : X^d \longrightarrow \mathbb{R}$

$\mathcal{H}_{Y^{d-1}} = L^2(C^\infty(Y, \mathbb{R}))$ functions on space of C^∞ fns

... expressions like $P(\varphi)\, e^{-\frac{1}{2}\langle \varphi, A\varphi \rangle}$



$\Phi_X(\varphi) = (*)\, e^{-S(\varphi_{cl})}$

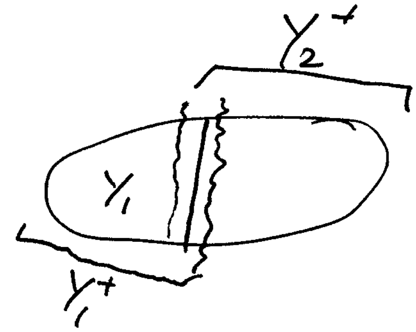
φ_{cl} : solution of classical field eqn over X, with action $S(\varphi_{cl}) = \frac{1}{2}\int_Y \varphi \frac{\partial \varphi_{cl}}{\partial n}\, dy$

Operator $A : \varphi \longmapsto$ normal derivative of classical solution associated to φ... first order ΨDO with symbol depending on normal derivatives of metric.

- collar $d\ell\; e^{\langle A \rangle}$ with A's ΨDOs with first few terms in symbol determined by metric of collar.

Problem: ΨDOs not local, can't cut & push

$\Rightarrow$ assign Hilbert spaces to collars $\mathcal{H}_{Y_1^+}, \mathcal{H}_{Y_2^+}$ with action of operators on extra ribbons



Need orientation reversing isometry of collar $\Rightarrow$

$$\mathcal{H}_Y \cong \mathcal{H}_{Y_1^+} * \mathcal{H}_{Y_2^+}$$

$*$ "fusion" the fields on overlap

(Wassermann, Jones) $\mathcal{L}G$ $\quad S^1 \to G$, $\quad \mathcal{H}$ positive energy projective rep of $\mathcal{L}G$.

Cut loop into parts

:

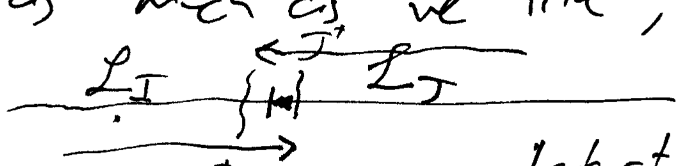
$\mathcal{L}G \hookrightarrow \mathcal{L}_I G \times \mathcal{L}_J G$: loops identity outside I or J, two commuting subgroups

Get equivalence of reps of $\mathcal{L}G$ & $\mathcal{L}_I G \times \mathcal{L}_J G$..

So expect $\mathcal{H} = \oint_\alpha \mathcal{H}_\alpha' \otimes \mathcal{H}_\alpha''$ integral of reps (sum or integral). Actually this only happens for Type I reps, not for ours...

for any $\xi \in \mathcal{H}$, $\mathcal{L}_I$ or $\mathcal{L}_J$ orbit will be dense — can't extricate the reps

--- reps are of Type III_1 : decompose as much as we like, all pieces isomorphic.



Look at reps of $\mathcal{L}_I$ with commuting action of extra tiny bit $I^+ \setminus I$, same for J,

now can tensor over overlap.

Finite group setting: $A = \mathbb{C}[G]$, G finite.

M_1 rt A-module, M_2 left A-module

$$M_1 \underset{A}{\otimes} M_2 = \bigoplus_{\substack{P \text{ irrep of } G \\ (\text{left})}} \cancel{\mathrm{Hom}_{\text{left } A\text{-mod}}(P, M_2)}$$

$$\mathrm{Hom}_{rt}(P^*, M_1) \otimes \mathrm{Hom}_{\text{left}}(P, M_2)$$

On this get "correct" inner product:

$M = \bigoplus_P P \otimes \underbrace{\mathrm{Hom}_G(P, M)}_{M_P} \ni f_1, f_2$ P-multiplicity space

$\langle f_1, f_2 \rangle 1_P = (f_1^* f_2)$ — trace renormalized by dimension d_P

— this gives correct inner product, not the trace, important distinction in ∞ dim.

— gives "correct" inner product on $M_1 \underset{A}{\otimes} M_2$ from direct sum decomp (in Type I setting), not $\otimes$ inner product.

E_1, E_2 f.d. vector bundles over space X with measure

$E_1 \rightarrow X \leftarrow E_2$ $\quad \mathcal{H}_1 = \Gamma_{L^2}(E_1)$, $\mathcal{H}_2 = \Gamma_{L^2}(E_2)$ modules$/C^\infty(X)$.

Want to reconstruct ~~L²~~ $\Gamma_{L^2}(E_1 \otimes E_2)$.

Would like "=" $\mathcal{H}_1 \underset{C^\infty(X)}{\otimes} \mathcal{H}_2$ ....

but no continuous linear maps $\mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \Gamma_{L^2}(E_1 \otimes E_2)$

(product of L^2 isn't L^2!)

Let $\mathcal{H}_0 = L^2(X)$.

$$\mathrm{Hom}_{C^\infty(X)}(\mathcal{H}_0, \mathcal{H}_1) \otimes \mathrm{Hom}_{C^\infty(X)}(\mathcal{H}_0, \mathcal{H}_2)$$

$$\| \qquad\qquad\qquad\qquad\qquad$$

$$L^\infty(E_1) \qquad \cancel{\otimes} \qquad L^\infty(E_2)$$

$$\swarrow$$

$$\Gamma_{L^\infty}(E_1 \otimes E_2) \subset \Gamma_{L^2}(E_1 \otimes E_2)$$

[maps of L^2 commuting with C^∞ are L^∞ ..]

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loop group acs $\quad \mathcal{H}_1 \quad J_0 \quad I_0 \quad \mathcal{H}_2$

comes fusion:

"identity object" $\mathcal{H}_0$ bimodule with automorphism flipping the two sides

$$\mathcal{L}_{I_0} \circlearrowright \mathcal{H}_0 \circlearrowleft \mathcal{L}_{J_0}$$

$\Rightarrow \mathcal{H}_1 \times \mathcal{H}_2 \qquad \mathrm{Hom}_{\mathcal{L}_{I_0}}(\mathcal{H}_0, \mathcal{H}_1) \otimes \mathrm{Hom}_{\mathcal{L}_{J_0}}(\mathcal{H}_0, \mathcal{H}_2)$

equalized over identification of left & right.

& then complete in correct inner product

basic rep of $\mathcal{L}_G$ = $\mathcal{H}_0$ characterized as irrep of two copies of $\mathcal{L}_{I_0}$ with inner product & involution interchanging the two ---> basic rep of loop group on

$\mathcal{H}_0$ basic rep of S^1, consider wrt I_0 action

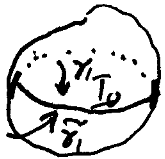
$$\mathrm{End}(\mathcal{H}_0) \xrightarrow{\Theta} \mathbb{C} \qquad A \mapsto \langle \Omega, A\Omega \rangle$$

θ is not a trace: $\theta(\gamma_1 \gamma_2) \neq \theta(\gamma_2 \gamma_1)$

But θ has KMS property!

$\theta(\gamma_1 \gamma_2) = \theta(\gamma_2 \tilde{\gamma}_1)$ wrt automorphism $\gamma \mapsto \tilde{\gamma}$ of dense subgroup

automorphism ~~by~~ for γ's which are bdry values on cut Riemann sphere, $\tilde{\gamma}$ comes from analytically continuing



$\mathcal{L}_{I_0} \supset$ subgroup of boundary values of holomorphic maps on cut Riemann sphere.
On this subgroup have automorphism passing around Riemann sphere — generator of von Neumann modular group....