C. Teleman - The Structure of Semisimple 2D TFT   4/20/05
(hope it's true!)

The Mumford Conjecture $\Longrightarrow$ complete structure formula for semisimple 2D TFT
$\Uparrow$
('02, Madsen-Weiss)
form of Madsen-Tillmann
conjectured by

I. 2D Topological Field Theory (Atiyah)

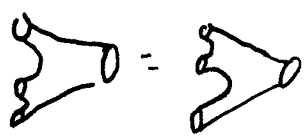
Oriented circle $S^1 \rightsquigarrow A$ (finite dim) vector space $/k$

multiplicative $S^1 \sqcup S^1 \sqcup S^1 \rightsquigarrow A^{\otimes 3}$

 topological surface $\Sigma \Rightarrow$ linear map $A^{\otimes 3} \xrightarrow{Z(\Sigma)} A^{\otimes 2}$
oriented

sewing of surfaces $\rightsquigarrow$ composition of maps.

 $\Rightarrow A \otimes A \longrightarrow A$ commutative, associative multiplication



 $: k \to A$ unit
$: A \xrightarrow[\theta]{} k$ trace (k-linear)

$\beta : A \otimes A \to A \to k$ nondegenerate pairing.

— Commutative Frobenius algebra $/k$.

<u>Theorem</u> 2D TFT $\Longleftrightarrow$ a commutative Frobenius algebra

A is called semisimple if $A \otimes_k \bar{k}$ semisimple ring
ie $=$ $\oplus$ copies of $\bar{k}$ as a ring.

i.e $A_{\mathbb{C}} \cong \oplus\, \mathbb{C} \cdot e_i \qquad e_i e_j = \delta_{ij} e_i$

$\theta : A \to k \qquad \theta_{\mathbb{C}} : \mathbb{C} e_i \to \mathbb{C}$

$\theta_i = \theta(e_i) \in \mathbb{C}$, nonzero by nondegeneracy of θ.

— just need to specify ~~some~~ numbers θ_i ..

 : $A \to A$ is multiplication by $\sum \theta_i^{-1} e_i$,
invertible: this invertibility is equivalent to semisimplicity.

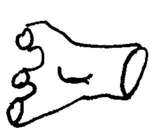
In general $Z($  $) = \alpha \in A$,
 = multiply by α. $\alpha = \sum \theta_i^{-1} e_i$ in semisimple case.
[if $A/\text{nil rad} = \mathbb{C} \Rightarrow$ some power of $\alpha = 0$
– other extreme!]

Stabilizing: add lots of handles 
allows to increase genus without losing information.

Deforming semisimple A: A has a natural basis of projectors – unique vectors up to permutation – & $\theta_i \neq 0$ for each basis element.
To deform with parameters: make the θ_i parameter dependent $\theta_i = \theta_i(t)$. Could also move basis itself: if something else fixes the vector space get element of $\mathrm{End}\, A$ depending on parameter & equal to 1 at basepoint.

II. Field theories in families: surfaces with boundary now varying in families: apply sewing axiom.

VI. Smooth surfaces, parametrized boundaries:

 $\hookrightarrow F$ surface bundle, fixed boundary

$$\begin{array}{c} F \\ \downarrow \\ B \end{array} \quad \text{for each } b \in B \text{ want } \quad A^{\otimes p} \xrightarrow[Z(F_b)]{} A^{\otimes q}$$

$$Z(F) \in H^*(B, \mathrm{Hom}_k(A^{\otimes p}, A^{\otimes q}))$$

A is constant so θ's constant ← topological info on variation. ~~Assume~~ Require functorial in B

Genus g, p incoming, q outgoing $\Rightarrow$ universal family over an orbifold :

... to remember circles cap with disc, remember center & unit tangent vector.

$\mathcal{M}_g^{p+q}$ compact R.S. of genus g, $p+q$ marked points.

$$\begin{array}{ccc} (S^1)^{p+q} & \longrightarrow & \tilde{\mathcal{M}}_g^{p+q} \\ & & \downarrow \\ & & \mathcal{M}_g \end{array}$$

remember unit tangent vectors, with $\Sigma_p \times \Sigma_q$ equivariance permuting points.

$\rightarrow$ need cohomology of $\tilde{\mathcal{M}}_g^{p+q}$.

Sewing : $B_1 \times B_2 \longrightarrow \mathcal{M}_{g_1+g_2}^{p_1+p_2+q_1+q_2-2}$

$Z \otimes Z \longleftarrow Z$. Factorization

looks like nbhd of boundary of compactified moduli space

V.2 Unparametrized boundaries : don't require parametrization of boundary circles.

So can only sew families over space where we specify isomorphisms between boundary circles

---> asking for cohomology of $\mathcal{M}_g^{p+q}$.

$$Z_g^{p,q} \in H^*(\mathcal{M}_g^{p+q}, \mathrm{Hom}(A^{\otimes p}, A^{\otimes q}))$$

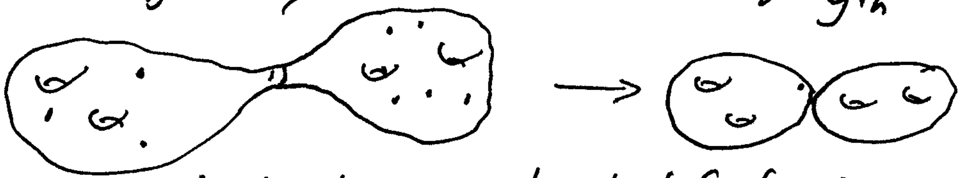
given point in $\mathcal{M}_g^{p+q}$ $\mathcal{M}_h^{r+s}$

can't sew : need tubular nbhd of stratum in $\mathcal{M}_{g+h}$ given by such ~~bound~~ sewings : take circle bundles

$$\begin{array}{ccc} S^1 & \hookrightarrow & \tilde{\mathcal{M}} \\ & & \downarrow \\ & & \mathcal{M}_g^{p+q} \times \mathcal{M}_h^{r+s} \end{array}$$

S^1 of identification of marked circles

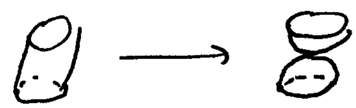
$\tilde{\mathcal{M}}$ = product of two circle bundles modulo diagonal circle action = spherical neighborhood of a boundary divisor in $\mathcal{M}_{g+h}^{p+q+r+s-2}$



Get classes almost defined over Deligne-Mumford space: strings live over a circle bundle.

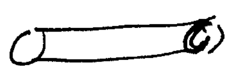
V3. Allow "nodal" curves: require classes $Z_g^{p+q} \in H^*(\overline{M}_g^{p+q}; \mathrm{Hom}(A^{\otimes p}, A^{\otimes q}))$

Factorization law at boundary:

So need to understand family  $t = xy$

cylinder has $\mathbb{C}^*$ automorphism: really family over $\mathbb{CP}^\infty$ $\Longrightarrow$ class in

identity map $1 \in H^*(\mathbb{CP}^\infty, \mathrm{Hom}(A,A))$ $\qquad$ $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty, \mathrm{Hom}(A,A))$

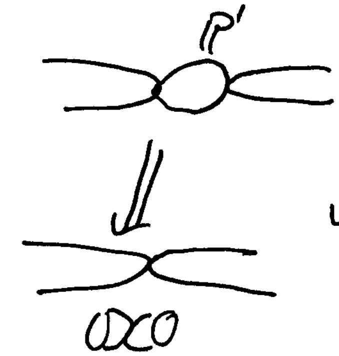
 

need class $F(\psi_1,\psi_2)$ which specializes to 1: $\psi_1, \psi_2 \in H^2(\mathbb{CP}^\infty)$

$F(\psi_1,\psi_2) \in \mathrm{End}\, A[[\psi_1,\psi_2]]$

$F(\psi,\psi) = \mathrm{Id}$: ie extend identity to function of two variables.

Such F in principle gives rule for extending to boundary

 gluing : $\Rightarrow$ 

not stable curve

want to be able to erase this $\mathbb{P}^1$ (destabilizing)

$F(\psi_1,\psi_2)\,F(\psi_2,\psi_3) = F(\psi_1,\psi_3)$

$\Rightarrow F(\psi_1,\psi_2) = f(\psi_1)\,f(\psi_2)^{-1}$ for some function f.

Also justified by locality of gravity reduces. product of factors for each copy of curve.

$\Rightarrow$ [Extension to boundary requires $f(\psi) \in \mathrm{End}\,A\,[[\psi]]$.
Could make change of variables everywhere in theory making $f(\psi)$ into 1.

in practice $A = H^*_{Floer}(LX) = H^*_{Floer}(x)$

$A[[\psi]] = H^*_{S^1}(x) = H^*(x)[[\psi]]$

III. Tautological classes and Mumford conjecture.

$\Sigma_g \hookrightarrow C_g = M_g^1$ universal surface. On Σ_g have canonical bundle K

$\downarrow \pi$ Euler class $e(K) = \psi$

M_g

Morita $\kappa_i = \pi_* \psi^{i+1}$. $\kappa_0 = 2g-2$

κ_i has degree $2i$.

__Theorem__ (Harer; Madsen-Weiss - conjectured by Mumford, Tillmann)
In degree $< g$ $H^*(M_g;\mathbb{Q}) = \mathbb{Q}[\kappa_i]$

in "infinite genus" $H^* = \mathbb{Q}[\kappa_i]$.

Theorem (Tillmann, Bodigheimer) Each marked point contributes a factor of $\mathbb{CP}^\infty$ to cohomology, generated by ψ class ψ_i = Euler class of K along that factor.

On M_g^n, classes available for given degree & large g are K_i, $\psi_1, \dots \psi_n$.
Incidentally: if add parametrized boundary component don't change cohomology (up to deg g)
(don't see this in genus zero!)

Structure Theorem V1. A semisimple, $\theta_1, \dots \theta_n$ structure consts (smooth surfaces w/ parametrized boundaries)
all $Z(\tilde{M}_g^{p,q}; \mathrm{Hom}(A^{\otimes p}, A^{\otimes q}))$ defined by single class

$$Z_\infty = \exp\left(\sum_{i>0} a_i K_i\right)$$

: group like cohomology class of stable mapping class group.

$$Z = \theta \left(\underbrace{\cdots}_{p \text{ input}} \cdot Z_\infty \cdot \Theta_g \cdot \underbrace{\cdots}_{q \text{ output}}\right).$$

where $\Theta_g = \theta_1^{\bullet - g} e_1 + \dots + \theta_N^{\bullet - g} e_n$:

$$Z(\Sigma_g) = \sum \theta_i^{1-g} = \theta\left(\sum \theta_i^{-g} e_i\right).$$

(can include K_0 in Z_∞ get rid of θ_i's :
answer for constant surface is just the K_0.

2. Unparametrized boundaries: need to specify second datum
$E(\psi) \in \mathrm{End}\, A[[\psi]]$ $\quad 1 + \psi E_1 + O(\psi^2)$

On ~~$M_g^{p,q}$~~

$\Rightarrow$ class in $H^*(\mathcal{M}_g^{p+q}, \mathrm{Hom}(A^p, A^q))$

insert $E(\psi_k)$ for each incoming point

& $E(\psi_k^{-1})$ for each outgoing point.

– just change of variables once & for all.

V3. Now have E, F. can change variables to make $F=1 \longrightarrow$ inductive procedure to extend class in higher cohomology, depends on E. (involves ψ classes of nodes)

Coates-Givental Twisted GW invariants:

X symplectic $\quad \mathcal{M}_g^n(X) \xrightarrow{E = ev} X^n$

$x_1, \dots x_n \in H^*(X)$

$$GW = \int_{\overline{\mathcal{M}}_g^n(X)} E^*(x_1 \boxtimes \dots \boxtimes x_n) \cap \text{virtual fundamental class}$$

or can integrate just over fibers $\overline{\mathcal{M}}_{g,n}(X) \longrightarrow \overline{\mathcal{M}}_{g,n}$

resulting classes are supposed to have this TFT structure.

Twisted GW: $V \to X$ vector bundle

$$\begin{tikzcd} \overline{\mathcal{M}}_g^{n+1} \arrow[r, "EV_{n+1}"] & X^n \times X \\ & x_1 \dots x_n \quad\quad V \end{tikzcd}$$

· calculate

$$\begin{tikzcd} E^*V \arrow[d] & \\ \overline{\mathcal{M}}_g^{n+1}(X) \arrow[d] & \text{curve} \arrow[l, hook'] \\ \mathcal{M}_g^n(X) & \end{tikzcd}$$

$K^0(\overline{\mathcal{M}}_g^n(X)) \ni \underline{\mathrm{Index}}\, E^*V$

Now take multiplicative char class $K^0 \to H^*$,
get $c(V) \in H^* \overline{\mathcal{M}}_g^n(X)$

$$\int_{\overline{\mathcal{M}}_g^n(X)} c(V) \wedge E^*(\gamma_1 \otimes \ldots \otimes \gamma_n) \wedge \text{virt}$$

V-twisted GW invariants

~~For~~ ~~Below~~ Calculate in terms of a class
$Z_\infty = \exp(\Sigma\, a_i k_i)$ $a_i \longleftrightarrow$ Chern classes of V

So $Z_\infty \longleftrightarrow$ Givental's quantum Lagrangian.

So twisting by a vector bundle has the effect of adding a Z_∞ like term.

[What are GW invariants of BG ?
twist by line bundle (Verlinde algebra) $\Rightarrow A$ semisimple.]

Prescription: rather than taking GW invariants (pushforward to pt) push forward to $\overline{\mathcal{M}}_g^n$, get collection of classes: don't integrate them (need complicated Kontsevich-Witten formulas) but just see simple structure.

Mumford: in semisimple case all classes we need are necessarily