Tony Pantev - Classical limit, continued

Note Title 1/5/2007

Conjecture: There is an equivalence of categories $D^b(\mathrm{Bun}_{{}^LG}, \mathcal{D}) \xleftarrow{c} D^b(\mathrm{Loc}_G, \mathcal{O})$

s.t. $\forall\, V \in \mathrm{Loc}_G$,

$$\mathcal{H}^\mu(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \boxtimes V^\mu[-d^\mu]$$

(ie points go to Hecke eigensheaves).

Structure sheaves $\mathcal{O}_V$ of points give an <u>orthonormal basis</u> of $D^b(\mathrm{Loc}_G, \mathcal{O})$ in a strong sense.

The conjecture says they go to the common eigenvectors of a family of commuting operators.

* Wrong as stated. Problem: $\mathrm{Bun}_{{}^LG}$ is disconnected

$$\pi_0(\mathrm{Bun}_{{}^LG}) = H^2(C, \pi_1({}^LG)) = \pi_1({}^LG)$$

(topological types of bundles)

$$\mathrm{Bun}_{{}^LG} = \coprod_{\gamma \in \pi_1({}^LG)} \mathrm{Bun}^\gamma_{{}^LG}$$

$$\Rightarrow D^b(\mathrm{Bun}_G, \mathcal{D}) \simeq \prod_{\pi_1({}^L G)} D^b(\mathrm{Bun}_G^\gamma, \mathcal{D})$$

But it's a theorem of Simpson that Loc_G is an irreducible algebraic variety (local systems with reductive monodromy: stability condition) so doesn't break into a product of categories.

Fix: replace Loc_G by the moduli stack of local systems $\mathcal{L}oc_G$.

[unnecessary] simplifying assumption: restrict to substack $\mathcal{L}oc_G^{rs}$ of regularly stable local systems ... stable & have minimal possible automorphism group - & do same on Bun_G

Other solution: look at version of $\mathrm{Bun}_{{}^L G}$ which is generically a space: mod out by generic stabilizers, $\overline{\mathrm{Bun}_{{}^L G}}$ "orbifold version"

$\mathcal{L}oc_G^{rs} \longrightarrow Loc_C^{rs}$ is a gerbe bundle by $Z(G)$

$\Longrightarrow$ category of coherent sheaves decomposes according to characters of generic stabilizer

$$D^b(\mathcal{L}oc_G^{rs}, \mathcal{O}) = \prod_{\gamma \in Z(G)^\wedge} D^b(\mathcal{L}oc_G^{rs}, \gamma)$$

twisted sheaves

But $Z(G)^\wedge \cong \pi_1({}^LG)$ so decomposition works!

Center of ${}^LG \longleftrightarrow$ dg structures on stack of local systems!

if want full stack $Bun_{{}^LG}$ need to include these dg structures on the other side

* Known for $G = \mathbb{C}^*$, $G = SL_2\mathbb{C}$ (Drinfeld)

GL_n: known where to send irreducible local systems.

Deformations of the geometry

$\mathcal{L}oc_G$ is a natural deformation of a simpler space. Specifically we have a specialization in which $\mathcal{L}oc_G$ specializes to $T^*\mathcal{B}un_G^0$

cotangent stack to stack $\mathcal{B}un_G^0 C$. (connected component of the identity)

ie we have a family (flat) of stacks

$$\begin{tikzcd}
T^*\mathcal{B}un_G^0 \arrow[d] & \subset & \mathcal{H} \arrow[d] & \supset & \mathcal{L}oc_G \times \mathbb{C}^* \arrow[d] \\
0 & \in & \mathbb{C} & \supset & \mathbb{C}^*
\end{tikzcd}$$

$\mathcal{H}$ is the moduli stack of z-connections on principal G-bundles, $z \in \mathbb{C}$

Recall: if V is a principal G bundle

$$0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \xrightarrow{\sigma} T_C \to 0$$

- Connections are splittings of σ
- $z \in \mathbb{C}$ a z-connection is a splitting of $z\sigma$ (rescaled Atiyah sequence)

<u>Example</u> $G = GL_n$, V = frame bundle of a vector bundle E

$\nabla : E \longrightarrow E \otimes \Omega^1$ $\quad \nabla(fe) = z\,df\, e + f \nabla e$

$z \neq 0 \Rightarrow$ just a connection

$z = 0$: $\mathcal{O}$-linear map $T_C \longrightarrow \mathrm{ad}\, P$

$\Longleftrightarrow \theta \in \mathrm{ad}\, P \otimes \Omega^1_C$ $\quad$ [integrability : $\theta \wedge \theta = 0$, automatic on curves]

<u>Def</u> (V, θ) with $\theta \wedge \theta = 0$ is a <u>Higgs bundle</u>.

[Stack of Higgs bundles on a compact Kähler manifold is the cotangent to the stack of bundles

(hard in general, easy on curves)

Curve case: $T_V \mathrm{Bun}_G = H^1(C, \mathrm{ad}\, V) = H^0(C, \mathrm{ad}\, V \otimes \Omega^1)^*$

So $T_V^* \mathrm{Bun}_G^\circ = H^0(C, \mathrm{ad}\, V \otimes \Omega^1)$

$\mathcal{H}_0 = \mathcal{H}iggs_G^\circ = T^* \mathrm{Bun}_G^\circ$.

Next: specialize category of $\mathcal{D}$-modules:

Sheaf $\mathcal{D}$ is a sheaf of filtered algebras (by order of diff'ops) $\Longrightarrow$ Rees specialization to the associated graded, $gr\,\mathcal{D} = \mathrm{Sym}\,T$

Specifically we have $\mathcal{R} \longrightarrow Bun_{{}^LG} \times \mathbb{C}$

s.t. $\mathcal{R}|_{Bun_{{}^LG} \times z} = \begin{cases} \mathcal{D} & z \neq 0 \\ gr\,\mathcal{D} & z = 0 \end{cases}$

$\mathcal{R} = \sum z^i P_i \quad : P_i \in \mathcal{D}^i.$

So get specializations

$$D^b(Loc_G, \mathcal{O}) \qquad\qquad D^b(Bun_{{}^LG}, \mathcal{D})$$

$$D^b(\mathcal{H}iggs^\circ_G, \mathcal{O}) \qquad\qquad D^b(Higgs_{{}^LG}, \mathcal{O})$$

stack of topologically trivial Higgs bundles — Higgs moduli space all topological types

<u>Expect</u> $C^{cl} : D^b(\mathcal{H}iggs^\circ_G, \mathcal{O}) \xrightarrow{\sim} D^b(Higgs_{{}^LG}, \mathcal{O})$

taking pts to "automorphic" $\mathcal{O}$-modules

General classical conjecture

$\exists \quad \underline{\mathcal{C}} : D^b(\mathcal{H}iggs_{G}, \mathcal{O}) \xrightarrow{\sim} D^b(\mathcal{H}iggs_{{}^LG}, \mathcal{O})$

on full stack of Higgs bundles

Specializing to $\mathcal{C}^{cl}$.

Part 3

Note $\mathcal{H}iggs_G$ has components labelled by $\pi_1(G)$ & stabilizers labelled by $Z(G)$

$$\Rightarrow D^b(\mathcal{H}iggs_{G}, \mathcal{O}) = \prod_{\substack{\alpha \in \pi_1(G) \\ \beta \in Z(G)^\wedge}} D^b(\mathrm{Higgs}_G^{\alpha}, \mathcal{O}, \beta)$$

Similarly

$$D^b(\mathcal{H}iggs_{{}^LG}, \mathcal{O}) = \prod_{\substack{\beta \in \pi_1({}^LG) \\ \alpha \in Z({}^LG)^\wedge}} D^b(\mathrm{Higgs}^{\beta}_{{}^LG}, \mathcal{O}, \alpha)$$

Classical limit of Hecke operators

Note that $D^b(\mathrm{Higgs}_{{}^LG}, \mathcal{O}) = D^b(\mathrm{Bun}_{{}^LG}, S^\bullet T)$

(not quite true on full stack Bun_G : need to kill generic stabilizers)

But $QCoh(\mathrm{Bun}_{{}^LG}, S^\cdot T)$

$$= \left\{ (E, \varphi) \middle/ \begin{array}{l} E \in QCoh(\mathrm{Bun}_{{}^LG}), \\ \varphi : E \longrightarrow E \otimes \Omega^1_{\mathrm{Bun}} \ \ \mathcal{O}\text{-lin} \\ \& \ \varphi \wedge \varphi = 0 \end{array} \right\}$$

... Higgs sheaves on $\mathrm{Bun}_{{}^LG}$

Idea: Deform $(\mathrm{Hecke}_x^\mu, IC)$ to correspondences for Rees modules on $\mathrm{Bun}_{{}^LG} \times \mathbb{C}$ to get $(\mathrm{Hecke}_x^\mu, \underline{I}^{\mu, cl})$

where $\underline{I}^{\mu, cl}$ is a Higgs sheaf (or complex) on Hecke_x^μ.

Then use $\underline{I}^{\mu, cl}$ as a kernel of an integral transform

$$\mathrm{Bun}_G \leftarrow \mathrm{Hecke}_x^{\mu} \rightarrow \mathrm{Bun}_G$$

$$H^{\mu,cl}_x(E,\varphi) = q^{\mu}_{x!}\left(p^{\mu,*}_x(E,\varphi) \otimes I^{\mu,cl}\right)$$

... Hecke operators on Higgs stacks.

Classical limit GLC, 2:

$$H^{\mu,cl} c^{cl}(\mathcal{O}_{(V,\theta)}) = c^{cl}(\mathcal{O}_{(V,\theta)}) \boxtimes \left(\rho^{\mu}(V), \rho^{\mu}(\theta)\right)[-d^{\mu}]$$

Real issue here, main missing piece in Arinkin's program to prove GLC by quantization.

... get wrong answer by taking gr of a good filtration on IC ..... get operators that don't generate Rep G ... works for minuscule μ, so can check when these generate ...

<u>However</u> $IC_{\mathrm{Hodge}_x^{\mathrm{an}}}$ is a mixed Hodge module, & taking the Rees module for the Hodge filtration gives a specialization I^{Hicc}.
(can't calculate this though ...)

There is another answer we can write locally using Cartan–Carrière local duality get abelianized answer

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$\boxed{\text{Abelianization}}$ The duality equivalence $\mathbb{C}$ is a manifestation of Cartier duality for commutative group stacks.
This requires us to interpret Higgs_G as a commutative group stack

The abelianization of Higgs_G is the Hitchin map

If we fix generators $p_1, \ldots p_r$ (homogeneous) of $\mathbb{C}[\mathfrak{g}]^G$
degs $d_1 \ldots d_r$
$\longrightarrow$

$h : \mathcal{H}iggs_G \longrightarrow H^0(S^{d_1}\Omega^1_C) \oplus \cdots$
$\cdots \oplus H^0(S^{d_r}\Omega^1_C)$

$(V,\theta) \longmapsto (p_1(\theta), \ldots\ p_r(\theta))$

& $\dim \mathcal{H}iggs_G = 2\dim(\text{base})$

Example: $G = Gl_n\mathbb{C}$. $\det(\theta - \lambda\,id) = \sum (-1)^n \lambda^n p_i(\theta)$

$h : Bun_{Gl_n} \longrightarrow H^0(\Omega^1) \oplus H^0(\Omega^{1\otimes 2}) \oplus \cdots H^0(\Omega^{1\otimes n})$

The fibers of h on $\mathcal{H}iggs^{n(g-1)}_{Gl_n}$

over $\alpha_1, \ldots \alpha_n$ is the compactified Jacobian of the curve $\overline{C}_\alpha : \{\sum(-1)^i \lambda^i \alpha_i = 0\} \subset T^*C$

λ = tautological section in $p^*\Omega^1_C$

... spectral curve, n-sheeted cover of C.

(reduced part of any fiber, even the 0 fiber, is a group! but have lots of other components)

<u>General G</u> Given (V,θ) Higgs sheaf

$\Rightarrow$ construct a cover $\tilde{C} \to C$ (cameral cover) which is a W-Galois cover

& the Hitchin base for G = moduli space of such covers, & fibers = Prym's of cameral covers

To define $\tilde{C} \to C$ look at $\mathfrak{g} \longrightarrow \mathfrak{g}/\!\!/G$

$\Rightarrow$ map of bundles

$$\underset{\mathrm{ad}\, V}{\overset{\|}{V \times_{\mathrm{ad}} \mathfrak{g}}} \longrightarrow V \times_{\mathrm{ad}} \mathfrak{g}/\!\!/G = \underline{\mathfrak{g}/\!\!/G} \quad \text{trivial bundle}$$

Similarly $\mathrm{ad}\, P \otimes \Omega^1_C \xrightarrow{\nu} (\mathfrak{g} \otimes \Omega^1_C)/\!\!/G$

Now $(\mathfrak{g} \otimes \Omega^1)/\!\!/G \simeq \Omega^1_C \otimes (\mathfrak{t}/W)$

$\Rightarrow$ W-Galois cover $\Omega^1 \otimes \mathfrak{t} \longrightarrow \Omega^1 \otimes \mathfrak{t}/W$

The (V,θ) cameral cover is the pullback

$$\begin{tikzcd}
\tilde{C} \arrow[r] \arrow[d] & \mathrm{tot}\,(\Omega^1 \otimes \mathfrak{t}) \arrow[d] \\
C \arrow[r, "\nu(\theta)"] & \mathrm{tot}\,(\Omega^1 \otimes \mathfrak{t}/W)
\end{tikzcd}$$

Hitchin base $B_G = \Gamma(C, (\Omega^1_C \otimes \mathfrak{t})/W)$

Hitchin map $h: \mathcal{H}iggs_G \longrightarrow B_G$

$(V,\theta) \longmapsto (\tilde{C}_{(V,\theta)} \longrightarrow C)$

Over $B_G \times C$ we have a universal cameral cover $\tilde{\mathcal{C}} \longrightarrow B_G \times C$

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[r] \arrow[dr] & B_G \times C \arrow[d] \\ & B_G \end{tikzcd}$$

Theorem (Abelianization) [Donagi-Gaitsgory]

The universal cameral cover $\tilde{\mathcal{C}} \to B_G \times C$ together with G determine an affine abelian group scheme $\mathcal{T} \longrightarrow B_G \times C$ so that

a. The Hitchin map $h: \mathcal{H}iggs_G \longrightarrow B_G$ is a principal homogeneous stack over the group stack $Tors_{\mathcal{T}}$

b. Every choice of a theta-characteristic on C determines a section of h

(Part a true in any dimension, part b true only for curves)

---- literally this is true not for Higgs bundle but for <u>regularized</u> Higgs fields $\mathcal{H}iggs_G^{reg} = \left\{ (V, \theta, \Gamma) : \begin{array}{l} (V,\theta) \text{ Higgs} \\ \& \ \Gamma \subset \text{ad } V \text{ sheaf} \\ \text{of regular centralizers} \\ \text{with } \theta \in \Gamma \otimes \Omega^1 \end{array} \right\}$

If we restrict to

$\mathcal{H}iggs_G^0 \longrightarrow \mathcal{B}_G^0$

($\mathcal{B}_G^0$ = cameral covers with a <u>simple</u> Galois ramification)

then everything is <u>regular</u> ($\Gamma = Z_{\mathfrak{g}}(\theta)$)

& theorem holds for $\mathcal{H}iggs_G^0 \longrightarrow \mathcal{B}_G^0$.

<u>Rmk</u> The Hitchin section associated to a theta characteristic ξ is a geometric analogue of Kostant's section $\mathfrak{g} \underset{s}{\rightleftarrows} \mathfrak{g}/\!/G$

s lands in $\mathfrak{g}^{rs} \subset \mathfrak{g}$.
It's built out of an sl_2 triple e, f, h with e, f regular nilpotents
$s(\mathfrak{g}/\!/G) = f + Z_{\mathfrak{g}}(e)$

<u>Rmk</u> In most cases the sheaf $\mathcal{T}$ of abelian groups is given by
$$\mathcal{T}\Big|_{\{\tilde{C}\}\times C} = \pi_*\left(\mathrm{char}(G) \otimes \mathcal{O}^*_{\tilde{C}}\right)^\vee$$
(for all types except B)

Note: $H^1(C, \mathcal{T}_{\{\tilde{C}\}\times C})$ is a Prym variety for $\tilde{C}$. If $\tilde{C} \to C$ étale this would be
$$\left(\mathrm{char}\, G \otimes \mathrm{Pic}\, \tilde{C}\right)^\vee$$

<u>Theorem</u> (Donagi-Pantev) • Given G & ${}^L G$
B_G & $B_{{}^L G}$ depend only on $\mathfrak{g}$ & ${}^L\mathfrak{g}$
& there is a natural isomorphism $B_G \xrightarrow{\sim} B_{{}^L G}$
taking B_G^0 to $B_{{}^L G}^0$

• Under this isomorphism

$$\mathcal{H}iggs^{\circ}_{G} \longrightarrow B^{\circ}_{G} \simeq B^{\circ}_{{}^LG} \longleftarrow \mathcal{H}iggs^{\circ}_{{}^LG}$$

are Cartier dual commutative group stacks.

<u>Def</u> Given a commutative group stack $\mathcal{H} \to B$ the Cartier dual is defined as

$$\mathcal{H}^D := \mathrm{Hom}_{gr}(\mathcal{H}, B\mathcal{O}^*_B)$$

$$= \mathrm{Ext}^1_{\substack{\text{cx of} \\ \text{ab. shvs}}}(\mathcal{H}, \mathcal{O}^*_B)$$

<u>NOT</u> an anti-involution in general...
also not all extensions as shvs of groups are <u>representable</u> as gp stacks (eg for extensions of $\mathbb{G}_a$ by $\mathbb{G}_m$) ... in our setting (Ext to $\mathbb{G}_m$) they <u>are</u> all representables

<u>If</u> $\mathcal{H}$ is filtered with graded pieces abelian varieties, f.g abelian grs, affine tori or the

Beilinson
ie 1-motives

classifying stacks of the last two $\Rightarrow$
subcategory is preserved & duality is
an involution.... same for formal completions.

<u>Examples</u> • $A \to B$ family of abelian
varieties $/B$ $\Rightarrow$ A^D = dual family
of abelian varieties.

• If $H \to B$ family of affine tori
$\Rightarrow$ $H^D = B[\operatorname{char} H] \to B$ classifying
stack of character lattices

• If $\Lambda \to B$ family of free abelian groups
$\Rightarrow$ $\Lambda^D = B(\operatorname{Hom}(\Lambda, \mathcal{O}^*))$ classifying
space of torus.

If $\mathcal{H}$, $\mathcal{H}^D$ are dual & $\mathcal{H}^{DD} = \mathcal{H}$
$\Rightarrow$ $\exists$ universal extension $\quad P \in \operatorname{Ext}^1(\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{H}^D, \mathcal{O}^*)$
which can be viewed as a sheaf
$$\mathcal{P} \in \operatorname{Coh}(\mathcal{H} \times_B \mathcal{H}^D)$$

$\Rightarrow$ Fourier-Mukai equivalence $D^b(\mathcal{H}) \xrightarrow[\sim]{\mathbb{F}_p} D^b(\mathcal{H}^D)$

Below $(\ )^0$ means identity component.

There are short exact sequences (or dist. triangles) of commutative group stacks

(1) $0 \to \mathcal{H}iggs_G^0 \longrightarrow \mathcal{H}iggs_G \longrightarrow \pi_1(G) \to 0$

connected component

(2) $0 \to BZ(G) \longrightarrow \mathcal{H}iggs_G^0 \longrightarrow Higgs_G^0 \to 0$

${}^L(1)$ $0 \longrightarrow BZ({}^LG) \longrightarrow \mathcal{H}iggs_{{}^LG} \longrightarrow Higgs_{{}^LG} \longrightarrow 0$

${}^L(2)$ $0 \to Higgs^0_{{}^LG} \to Higgs_{{}^LG} \longrightarrow \pi_1({}^LG) \to 0$

$(1)^D = {}^L(1)$ & $(2)^D = {}^L(2)$.

So $(\mathcal{H}iggs_G^0)^D = Higgs_{{}^LG}$

Argument is Poincaré duality on base with coefficients

Main part:

$$\mathrm{Higgs}^{\circ}_{G} \to B \leftarrow \mathrm{Higgs}^{\circ}_{^LG}$$

Do Poincaré duality with coefficients in a certain perverse sheaves given by Hodge theory applied to τ .....

Other ingredient: self duality of Pic & Abel-Jacobi (Hecke symmetries) ....

.... nonalgebraic argument: use identification of Jacobian as vector space / lattice.