The 2-lien of a 2-gerbe

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Motivation

Let G be a sheaf of groups on a space X.

- If G is abelian, what do $H^2(X, G)$, $H^3(X, G)$, and generally $H^n(X, G)$ classify?

- How do we define $H^n(X, G)$ when G is non-abelian?

Giraud answered this for $H^2(X, G)$ and Breen gave a partial solution for $H^3(X, G)$.

Gerbes and their Liens

Definition. • *A (1)-***gerbe** *on a space* X *is a stack in groupoids* $\mathcal{G}$ *on* X *which is locally non-empty and locally connected.*

- *A* **morphism** *of gerbes is just a morphism of the underlying fibered categories.*

- *A gerbe* $\mathcal{G}$ *on* X *is said to be* **neutral** *(or trivial) when the fiber category* $\mathcal{G}_X$ *is non-empty.*

A **lien on a space** X is a collection of sheaves of groups (G_i) corresponding to an open cover (U_i) of X with descent data up to inner conjugation.

In other words a lien on X is an object which is defined locally by a sheaf of groups, but in a category where morphisms between groups differing by inner conjugation are identified.

To every gerbe G on X, we can associate a lien on X, and this association is functorial in G.

Definition. *Let K be a lien on X. A* **gerbe with lien** *K is a gerbe G on X together with an isomorphism of liens $\theta : lien(G) \simeq K$. Two gerbes with lien K are said to be equivalent if there is an equivalence between the underlying fibered categories. We designate by $\mathbf{H^2(X, K)}$ the set of equivalence classes of gerbes with lien K.*

Example *Let G be a bundle of groups on X. The stack $\mathcal{T}ors(G)$ of right G-torsors on X is a gerbe on X. It is (globally) non-empty since its fiber on X always contains the trivial G-torsor on X. It is also locally connected since any G-torsor is locally isomorphic to the trivial G-torsor. Thus $\mathcal{T}ors(G)$ is in fact a neutral gerbe. Its lien is the lien represented by the group G, denoted $lien(G)$.*

Example *Let $1 \to A \to C \to B \to 1$ be an exact sequence of (possible infinite dimensional) Lie groups such that the projection $C \to B$ is a locally trivial A-bundle. Let $p : P \to X$ be a smooth B-bundle over a manifold X. Consider the problem of finding a C-bundle $q : Q \to X$ such that the associated B-bundle $Q/A \to X$ is isomorphic to $P \to X$. The fibered category of local solutions to this is a gerbe on X. The lien associated to this gerbe is isomorphic to the sheaf A_X of smooth A-valued functions.*

2-**Gerbes**

Definition. *A (2)-***gerbe** $\mathcal{P}$ *on a space* X *is a 2-stack in 2-groupoids on* X *which is locally non-empty, locally connected, in which 1-arrows are weakly invertible, and 2-arrows are invertible.*

Example *(***Breen***) Let* L *be a lien on* X*. When is* L *isomorphic to a lien of the form* $lien(G)$ *for some gerbe* G *on* X*? Locally, always! since the lien* L *is locally isomorphic to a lien of the form* $lien(G)$ *for some sheaf of groups* G*, whence it is realized by the neutral gerbe* $Tors(G)$ *corresponding to* G*. Globally this gives a 2-gerbe on* X*.*

The Project

We defined the notion of a 2-lien on a space X.

We have proved some theorems about 2-liens of 2-gerbes which correspond to well known results about liens of gerbes.

[Deligne] Any strict Picard stack $\mathcal{G}$ corresponds to a 2-term complex of abelian sheaves $K^{\cdot} = [\, K^0 \xrightarrow{d} K^1 \,]$. In this case we proved that $\check{H}^3(X, \mathcal{G})$ is isomorphic to the hypercohomology group $\check{H}^3(X, K^{\cdot})$.

The Coequalizer

Definition. *Let $\mathcal{C}$ be a category and $f, g : A \to B$ be two arrows in $\mathcal{C}$. The* **coequalizer** *of f and g is a pair (P, h) consisting of an object P of $\mathcal{C}$ with an arrow $h : B \to P$ such that (1) $hf = hg$; (2) if (D, u) is any other pair such that $u : B \to D$ has $uf = ug$, then $u = u'h$ for a unique arrow $u' : P \to D$.*

If $F : G \times A \to A$ is the action of a group G on a set A then the quotient by the action is just the coequalizer of the arrows $G \times A \underset{F}{\overset{p_2}{\rightrightarrows}} A$. It is this formulation of quotients which lends itself to generalization i.e. we use this approach to define a quotient when a group category acts on a category.

Definition. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let F and G denote functors from $\mathcal{C}$ to $\mathcal{D}$ i.e. we have a diagram $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$. The coequalizer of F and G is a category $Coeq(F,G)$ along with an essentially surjective functor $h : \mathcal{D} \to Coeq(F,G)$ such that $h \circ F \simeq h \circ G$. Further if Z is any other category with a fully faithful functor $j : D \to Z$ with $j \circ F \simeq j \circ G$ then there exists a functor $k : Z \to Coeq(F,G)$ (which is unique up to isomorphism), and a 2-isomorphism α (which is unique once k is fixed), sitting in a commutative diagram:*

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D} \xrightarrow[j]{} Z \xrightarrow[k]{} Coeq(F,G), \qquad h : \mathcal{D} \to Coeq(F,G)$$

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let F and G denote functors from $\mathcal{C}$ to $\mathcal{D}$ i.e. we have a diagram $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$. We wish to prove coequalizers exist.

The objects of $Coeq(F, G)$ will be objects of $\mathcal{D}$.

Let $PreAr(F, G)$ denote the set of formal compositions of

1. $Ar(\mathcal{D})$

2. isomorphisms $F(x) \xrightarrow{\simeq} G(x)$ for all $x \in \mathcal{C}$ and their inverses.

Then there is a smallest equivalence relation $\sim$ on $PreAr(F, G)$ such that

1. the identity axiom holds (i.e. $id \circ \alpha = \alpha$ and $\gamma \circ id = \gamma$ for any arrows α and γ),

2. formal composition of arrows in $Ar(\mathcal{D})$ are related to the actual compositions.

3. for any a, $b \in Ob(\mathcal{C})$ and $t : a \to b$ in $Ar(\mathcal{C})$

$$F(a) \xrightarrow{F(t)} F(b) \xrightarrow{\simeq} G(b) \ \sim \ F(a) \xrightarrow{\simeq} G(a) \xrightarrow{G(t)} G(b)\,.$$

Further, if α_1, α_2 and β_1, β_2 are elements of $PreAr(F, G)$ such that $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$ and suppose $\alpha_2 \circ \alpha_1$ and $\beta_2 \circ \beta_1$ is defined, then $\alpha_2 \circ \alpha_1 \sim \beta_2 \circ \beta_1$.

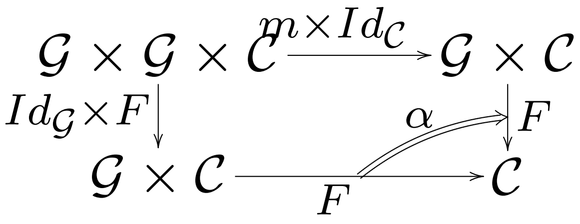
Define $Coeq(F, G)$ as follows: $Ob(Coeq(F, G)) = Ob(\mathcal{D})$ and $Ar(Coeq(F, G)) = PreAr(F, G)/\sim$.

Proposition. *$Coeq(F, G)$ forms a category.*

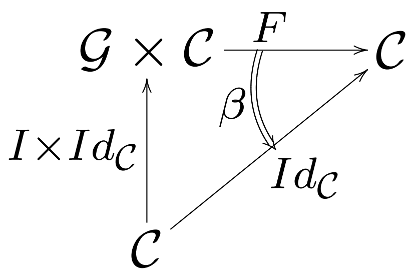
By construction of $Coeq(F, G)$ we have an arrow $h : \mathcal{D} \to Coeq(F, G)$ which is the identity on objects and sends arrows in $\mathcal{D}$ to their corresponding class in $Coeq(F, G)$.

Theorem. *The pair $(Coeq(F, G), h)$ gives the coequalizer of the arrows $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$.*

Definition. *Let $\mathcal{G}$ be a group category and $\mathcal{C}$ be a category. An action of $\mathcal{G}$ on $\mathcal{C}$ is a triple (F, α, β) where $F : \mathcal{G} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor that sits in the following two pasting diagrams:*



(1)



(2)

Further α and β must be compatible with associativity in G.

Definition. *Let $\mathcal{G}$ be a group category acting via the maps (F, α, β) on a category $\mathcal{C}$. We define the* **quotient of $\mathcal{C}$ by $\mathcal{G}$** *to be the category that represents the coequalizer of the diagram $\mathcal{G} \times \mathcal{C} \underset{F}{\overset{p_2}{\rightrightarrows}} \mathcal{C}$ in the 2-category CAT.*

Now generalize these constructions to $STACKS$ using the universal properties of the coequalizer.

Example *Let* $Coeq(F, G)$ *denote the following fibered category. To each open* $U \subset X$, $Coeq(F, G)(U) := Coeq(F_U, G_U)$. *Let* $V \hookrightarrow U$ *be an inclusion of open sets in the space* X. *To define the restriction functors* f^* *for* $Coeq(F, G)$ *consider the following diagram:*

$$\begin{array}{ccccc} \mathcal{C}(U) & \overset{F_U}{\underset{G_U}{\rightrightarrows}} & \mathcal{D}(U) & \xrightarrow{\varphi_U} & Coeq(F, G)(U) \\ c^* \downarrow & & \downarrow d^* & & \downarrow f^* \\ \mathcal{C}(V) & \overset{F_V}{\underset{G_V}{\rightrightarrows}} & \mathcal{D}(V) & \xrightarrow{\varphi_V} & Coeq(F, G)(V) \end{array} \qquad (3)$$

Also, as part of the data, we have 2-arrows $\alpha : F_V \circ c^* \Rightarrow d^* \circ F_U$ *and* $\beta : G_V \circ c^* \Rightarrow d^* \circ G_U$.

The 2-lien of a 2-gerbe

It is an object that is given locally by a group stack, with 2-descent given up to inner equivalence.

Proposition. *For every* 2*-gerbe* $\mathcal{G}$*, there exists a* 2*-lien* L *of* $\mathcal{G}$ *characterized up to a canonical* 2*-equivalence.*

Thus we are justified in saying "the" 2-lien of a 2-gerbe $\mathcal{G}$.

Definition. *A group category* G *is said to be a strict Picard category when its group law is endowed with a commutivity isomorphism* $S_{X,Y} : XY \to YX$ *which is functorial in* X *and* Y *and sits in a commutative square*

$$\begin{array}{ccc} X1 & \xrightarrow{S} & 1X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle S} \\ X & = & X \end{array} \tag{4}$$

and two hexagonal "associativity" diagrams. In addition $S_{Y,X} \circ S_{X,Y} = 1_{XY}$ *for all* X, Y *in* G *and* $S_{X,X} = 1_X$ *for all* X. *A* gr*-stack* $\mathcal{G}$ *is said to be a strict Picard stack if it is endowed with a commutativity natural transformation* S *for the group law* $S_{X,Y}$ *that induces for each open* $U \subset X$ *the structure of a strict Picard category on* $\mathcal{G}(U)$.

Proposition. *Let* $\mathcal{G}$ *be a gr-stack on* X*. A 2-gerbe* P *on* X *is a* $\mathcal{G}$*-2-gerbe if and only if its 2-lien is locally equivalent to* $lien_2(\mathcal{G})$*.*

Proposition. *Let* $\mathcal{G}$ *be a Picard stack on* X*. A gerbe* P *on* X *is an abelian* $\mathcal{G}$*-2-gerbe if and only if* $lien_2(P)$ *is equivalent to* $lien_2(\mathcal{G})$*.*

$\mathcal{P}$: connected $\mathcal{G}$-2-gerbe (for each pair of objects x, y in some fiber 2-category $\mathcal{P}_U$, the set of arrows in $\mathcal{P}_U$ from x to y is non-empty), where $\mathcal{G}$ is Picard.

Definition. *We designate by $H^3(X, L)$ the set of equivalence classes of 2-gerbes with 2-lien L, and by $\breve{H}^3(X, L) \subset H^3(X, L)$ the set of equivalence classes of* connected *2-gerbes with 2-lien L.*

Lemma. *(Deligne) For every Picard stack P there exists a complex $K^\cdot \in C(X)$ such that $P \simeq ch(K)$.*

Theorem.

$$\breve{H}^3(X, \mathcal{G}) \xrightarrow{\sim} H^3(X, K^\cdot) \tag{5}$$