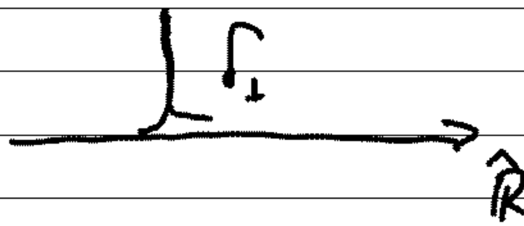
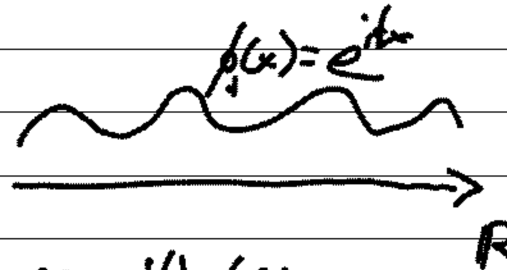
Edward Frenkel - Ramifications of Geometric Langlands

Note Title 1/10/2007

Langlands Correspondence = Fourier Transform (categorified, derived)





$(\partial_x - it)\,\phi_t(x) = 0$

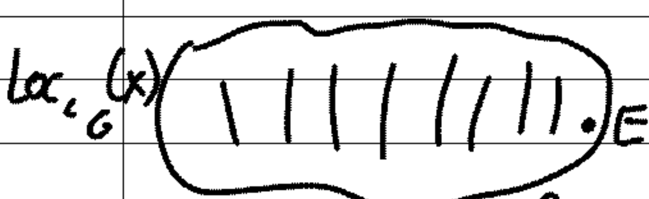
$\Rightarrow$ $\mathcal{D}$-module

Geometric Langlands :

X smooth projective curve / $\mathbb{C}$

G reductive Lie group / $\mathbb{C}$ (mostly: simple, conn., simply connected)

Bun_G = moduli stack of G-bundles on X



$\longmapsto$ Aut_E $\mathcal{D}$-modules on Bun_G

parameter space for "nice" D-modules ... analogs of e^{itx} .... on Bun_G with which we can do harmonic analysis

${}^LG$: Langlands dual group. $Loc_{{}^LG}$ = stack of ${}^LG$ local systems on X

${}^LG$ local system = ${}^LG$ bundle with flat connection
(eventually allow ∇ meromorphic)

$E \longmapsto \mathrm{Aut}_E$ Hecke eigensheaf with eigenvalue E.

Aside: in classical Langlands program
$E \rightsquigarrow$ representation $\mathrm{Gal}(\bar{F}/F) \longrightarrow {}^LG$
(which is unramified ... factors through π_1)

$\mathrm{Aut}_E \rightsquigarrow$ automorphic function, which is a Hecke eigenfunction

Aut_E has been constructed for $G = GL_n$, E irreducible
$n=2$ Drinfeld $n>2$ Gaitsgory, Frenkel, Vilonen
G simple, $E =$ an oper for ${}^LG$
.... Beilinson - Drinfeld

[GL_1: Aut_E is a flat line bundle on $\mathrm{Bun}_{GL_1} = \mathrm{Pic}$]

Today: Generalization of Beilinson-Drinfeld construction to the ramified setting.

<u>Physics</u> Kapustin-Witten: relate Langlands correspondence to 4D SUSY Yang-Mills theory .... assigned to a compact Lie group G_c & a coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

S-duality: $YM(G_c, \tau) \simeq YM({}^L G_c, -\frac{1}{\tau})$

Consider YM on 4-manifolds $M = X \times \Sigma$, in limit of small $X \Rightarrow$ get σ-model on Σ valued in $\mathcal{M}_{X,G}$ moduli of Higgs bundles

$\mathcal{M}_{X,G} \approx T^* Bun_G$.

Higgs bundle: $(\mathcal{F}, \eta)$ $\mathcal{F}$ holomorphic G-bundle
$\eta \in H^0(X, \mathrm{ad}_{\mathcal{F}} \otimes \Omega)$

$\mathcal{M}_{X,G}$ = moduli of (semi)stable Higgs bundles
$\supset \mathcal{M}^{\circ}_{X,G}$ locus where $\mathcal{F}$ is stable (η arbitrary)
$\shortparallel$
T^* of moduli of stable bundles.

Nonabelian Hodge Theory $\mathcal{M}_{X,G} \simeq \mathrm{Loc}_G X$

different complex structures in a hyperkähler structure.

$$
\begin{tikzcd}
\mathrm{Loc}_{{}^LG}(X) \arrow[rr] \arrow[d, phantom, "\wr"] & & \mathrm{Bun}_G \\
\mathcal{M}_{X,{}^LG} \arrow[dr, "{}^Lp"'] & & \mathcal{M}_{X,G} \simeq T^*\mathrm{Bun}_G \arrow[dl, "p"] \\
& \mathrm{Hitch}_G \cong \mathrm{Hitch}_{{}^LG} \arrow[u, phantom, "\text{Hitchin's integrable system}"] &
\end{tikzcd}
$$

generic fibers are tori

dual torus fibrations

T-duality

Step I Suppose E belongs to a "nice" locus in $\mathrm{Loc}_{{}^LG} X \implies$ apply nonabelian Hodge theory to it & get $\widetilde{E} \in \mathcal{M}_{X,{}^LG}$

Step II Apply T-duality to $\widetilde{E}$: get a line bundle $(\mathcal{L},\nabla)$ w/ flat unitary connection on $p^{-1}h \subset \mathcal{M}_{X,G}$: object of Fukaya category ("A-brane")

Step III Attach to $(\mathcal{L}, \nabla)$ a $\mathcal{D}$-module on Bun_G, & check it's a Hecke eigensheaf wrt E $\rightsquigarrow$ our Aut_E.