David Nadler - Constructible Sheaves & Fukaya Category

Note Title 1/10/2007

(with E. Zaslow)

X real analytic compact oriented manifold

$$D_c(X) \xrightarrow[\text{equiv of triangulated categories}]{\sim} DF(T^*X)$$

constructible derived category — derived Fukaya category of the symplectic manifold T^*X

Remarks 1. These are actually shadows of underlying A_∞ categories & the theorem is an A_∞ quasi-equivalence

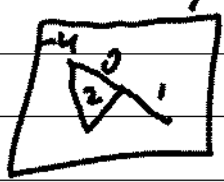
2. Relation to geometric Langlands
RHS: category of [exact Lagrangian] A-branes
exact: canonical 1-form $\theta = \Sigma \xi_i \, dx_i$ on T^*X
with $d\theta = \omega$: $\omega|_{\text{Lagrangian}} = 0$,
exact Lagrangian means $[\theta|_L] = 0 \in H^1(L)$

— doesn't include Hitchin fibers!
(work in progress for nonexact Lagrangians)

LHS = regular holonomic $\mathcal{D}$-modules
by the Riemann-Hilbert correspondence
in the complex analytic case.

Classical story:

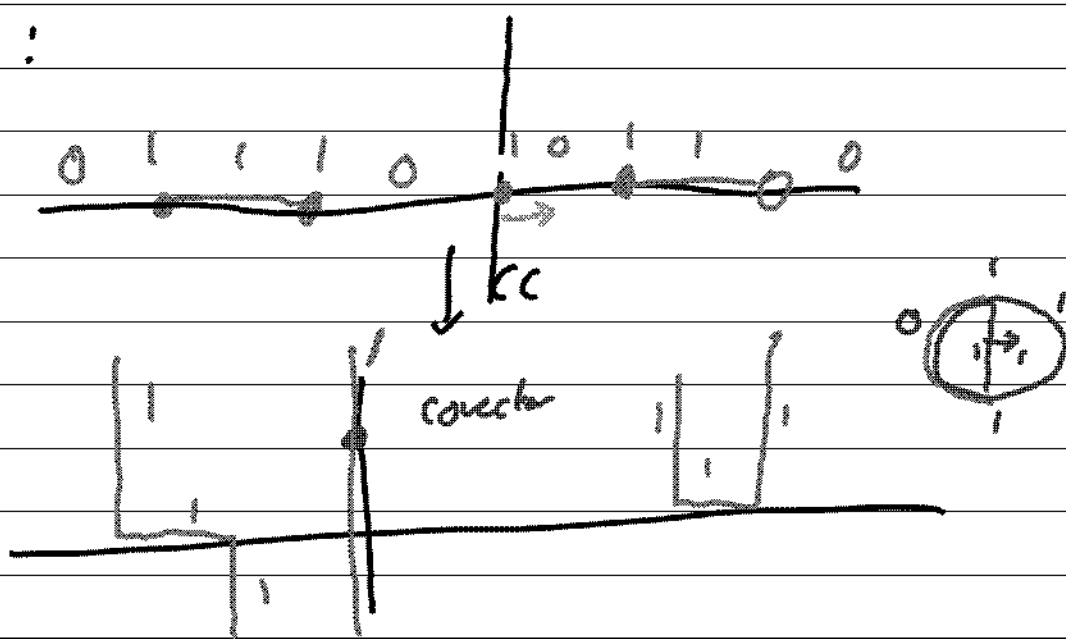
Grothendieck group of $D_c(X)$ = constructible functions on X : decompose X into subanalytic sets & assign #s to each



$$\begin{array}{ccc} D_c(X) & \xrightarrow[\text{char. of stalks}]{\text{take Euler}} & \text{Const. Fns}(X) \\ & \swarrow \text{Char. cycle} & \end{array}$$

RHS: Conical Lagrangian cycles in T^*X

$X = \mathbb{R}$:



Theorem (Schmid - Vilonen):

$j: U \subset X$ open

F const. fn or sheaf

$m: X \to \mathbb{R}$ defining fn for $\partial U \subset X$

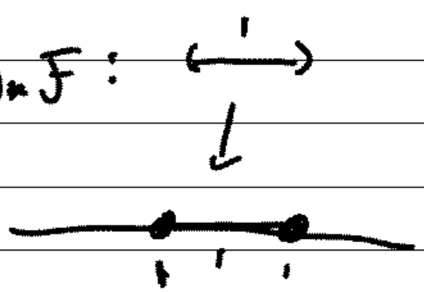
$$CC(Rj_* F) = \lim_{\varepsilon \to 0^+} \left(CC(F) + \varepsilon \Gamma_{\frac{dm}{m}} \right)$$

Γ = graph of 1-form $\frac{dm}{m}$

Our picture:

$\Gamma_{\frac{dm}{m}}$

$Rj_* F$:



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2 more properties of CC:

1. CC is an isomorphism from constructible fns to conical Lagrangian cycles ... as abelian groups
2. Dubson - Kashiwara formula:

$$\chi(F) = CC(F) \cap \Gamma_{df} \quad \text{for any fn } f$$

..... underlying functoriality of $D_c(X) \longrightarrow$ Fukaya.

Fukaya Categories

T^*X is an exact symplectic manifold $(\omega = d\theta)$

Objects: exact Lagrangian submanifolds $L \subset T^*X$
with brane structure:

0. equip L with a local system $\mathcal{E} \to L$
1. Grading: obstruction to grading is Maslov class in $H^1(L)$
(gradings $\in \mathbb{Z}$ - torsor)
⟿ allows us to lift $\mathbb{Z}/2$ gradings to $\mathbb{Z}$ gradings
2. Relative pin structure .... needed to orient moduli spaces ($\mathbb{Z}/2$-worth of choices)

It will turn out that hamiltonian isotopies give isomorphic objects ...

Morphisms: given branes $L_0, L_1 \rightsquigarrow$
graded vector space $\hom(L_0, L_1)$
$= \bigoplus_{p \in L_0 \cap L_1} \mathbb{C}\langle p \rangle$ (graded by Maslov class)

.... eventually will be able to perturb Lagrangians: contractible space of choices that will enable us to get transverse intersections.

Compositions of morphisms:

1. $\mu_1 \circlearrowright \hom(L_0, L_1)$ with $\mu_1^2 = 0$

2. $L_0, L_1, L_2 \Rightarrow \mu_2 : \hom(L_0, L_1) \otimes \hom(L_1, L_2) \longrightarrow \hom(L_0, L_2)$

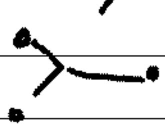
& μ_2 is a chain map

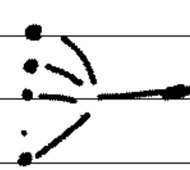
$\vdots$

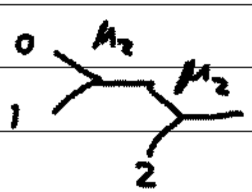
k. $L_0, L_1, \dots, L_k : \mu_k : \bigotimes_{i=0}^{k-1} \hom(L_i, L_{i+1}) \longrightarrow \hom(L_0, L_k)$

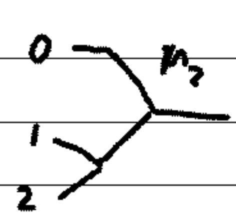
Schematic way to think: 

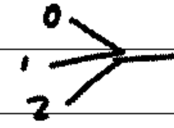
connects intersection pts

μ_2 

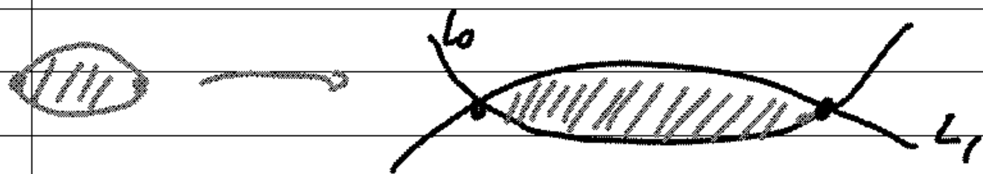
μ_k : 

e.g. 



Both boundaries of 

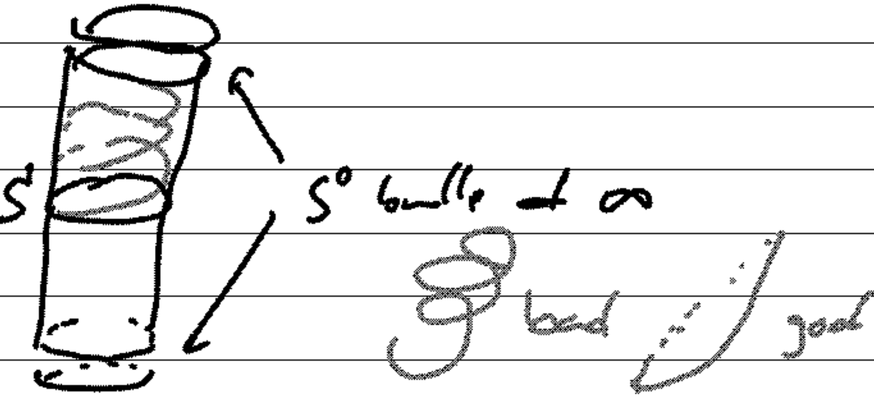
Pictures come from considering holomorphic discs:
pick compatible a.c. structure J on T^*X
coming from a Riemannian metric on X
then count discs with boundaries on our Lagrangians



A_∞ composition maps:
1. come from counting discs
2. relations come from considering boundaries of one-dim moduli spaces
.... eg $\mu_1^2 = 0$: boundary of 1-dim moduli spaces of discs gives μ_1^2 ..

New wrinkle: we'll consider noncompact branes in T^*X.
To get a good theory: $L \subset T^*X$ allowable if
$\overline{L} \subset \overline{T^*X} = T^*X \cup$ sphere bundle at ∞
is a subanalytic subset.

eg $X = S^1$

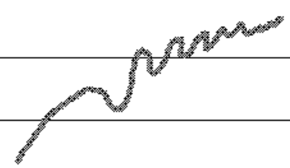


$\Rightarrow$ 1. $\bar{L} \cap S_\infty X$ is Legendrian

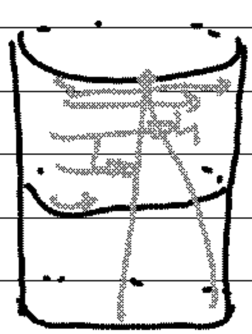
2. metric on $T^*X \Rightarrow$

$|\xi| : L \longrightarrow \mathbb{R}$ will have no critical points for $|\xi| >> 0$

... rules out



Intersections at infinity:



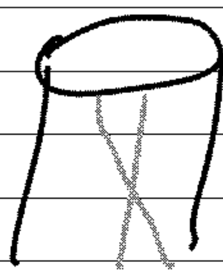
Given $L_0, L_1 \ldots L_k$ need to perturb them, insist on the following strategy: fix g on X Riem. metric ($\Rightarrow$ acc. J)

$\Rightarrow$ geodesic flow on T^*X

Choose small $0<\delta_0<\ldots<\delta_k$ & perturb by normalized geodesic flow of ∞
(use hamiltonian $|\xi|$ not $|\xi|^2$)

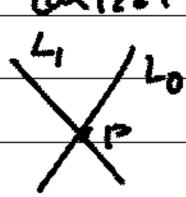
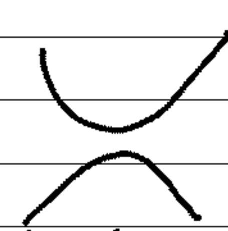
$\rightsquigarrow$ with this procedure can calculate $\hom(L_i, L_j)$ for $i<j$
$\rightarrow$ get nonsymmetric homs:

one direction get no intersection



other direction get one

Technical aspects: 1. to ensure compact moduli of discs need area bounds, but not uniform:
.... to calculate higher & higher products need to perturb Lagrangians to be closer & closer to being conical ..... - asymptotically conical geometry.

2.  $\mathrm{cone}(p)$ =  connect sum

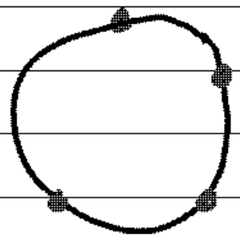
Can't always do geometrically, so do algebraically: pass to triangulated envelope

Informal Discussion

$$D_c(X) \underset{\mu}{\overset{\pi}{\rightleftarrows}} DF(T^*X)$$

$X = S^1$

$D_c(S^1)$:



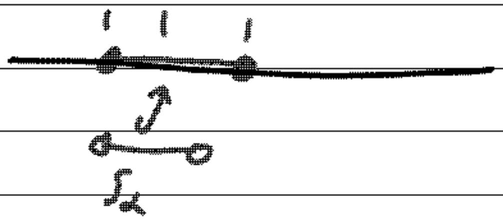
Fix stratification $\mathcal{S}$

"Basis" for $D_{\mathcal{S}}(X)$ given by standards

(another by costandards)

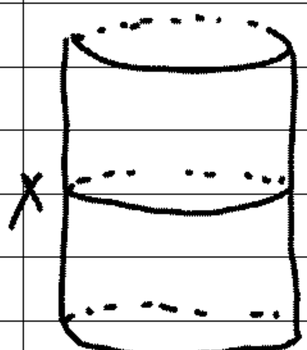
$i_\alpha : S_\alpha \hookrightarrow X$, $\quad i_{\alpha *} \mathbb{C}_{S_\alpha}$

corresponding constructible fn:



costandards

$i_{\alpha !} S_\alpha$



T^*X : choice of zero section is important piece of structure: equivalent to choice of θ with $d\theta = \omega$: $\mathrm{Fuk}(T^*X)$ indep of choice but the isomorphism with $D_c(X)$ depends

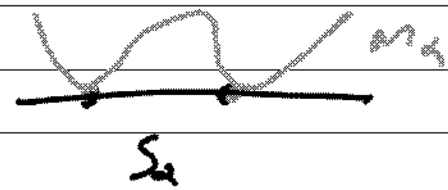
μ on standards: if $S_\alpha \hookrightarrow X$ is closed $\Rightarrow$ just take conormal

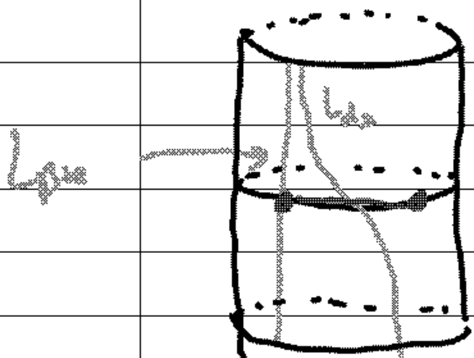
$S_\alpha = \bullet$ $\qquad \mathbb{C}_{S_\alpha x} = i_{\alpha *}\mathbb{C}_{S_x}$

$L_{\alpha x} = \mu(\mathbb{C}_{S_\alpha x})$

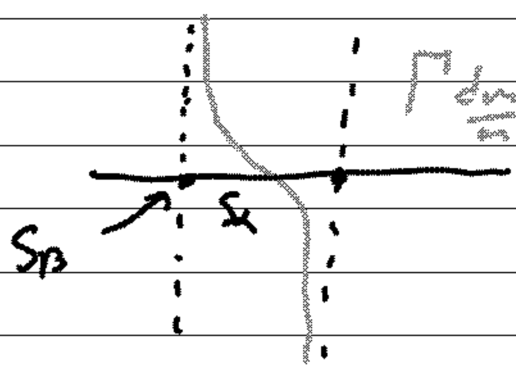
In general choose $m: X \to \mathbb{R}_+$
$m=0$ precisely on ∂S_α (only care about $m_\alpha|_{S_\alpha}$)

eg

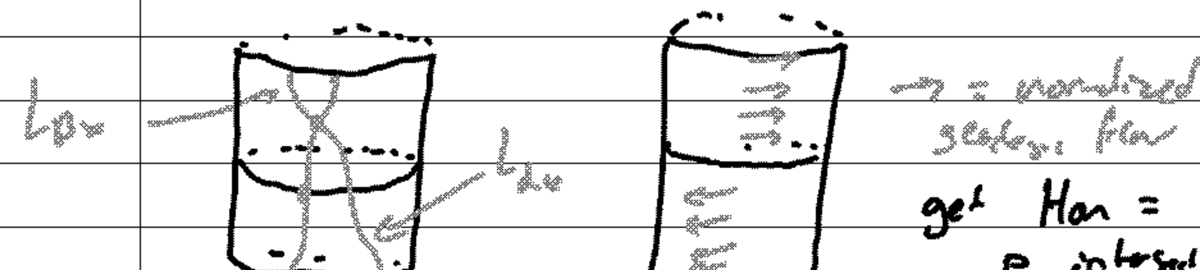




Define $L_{\alpha x} = T^*_{S_\alpha}X + \Gamma_{\frac{dm_\alpha}{m_\alpha}}$



To calculate hom$(L_{\alpha x}, L_{\beta x})$:
perturb by geodesic flow : $\delta_\alpha < \delta_\beta$ small

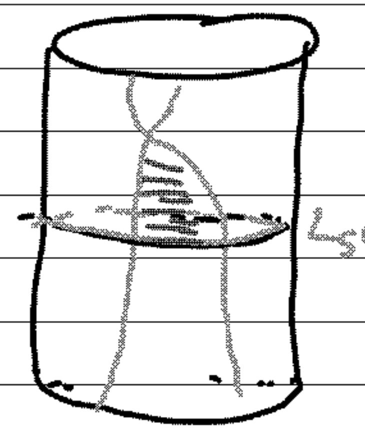


get Hom $= \mathbb{C}\cdot p$
p intersection pt

get nontrivial hom $(L_{\alpha *}, L_{\beta *}) = C_p$
but no hom the other way $\qquad (S_\beta \subset \bar{S}_\alpha)$

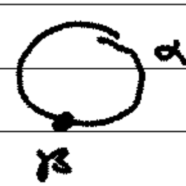
Example of a composition :

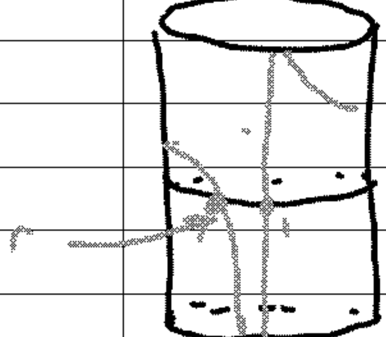
$$\begin{tikzcd} L_{\alpha *} \arrow[r, "p"] & L_{\beta *} \\ L_{S^1} \arrow[u] \arrow[ur] & \end{tikzcd}$$



Map p is seen in global sections (ie maps from L_{S^1}) $\Rightarrow$ get disc  measuring above composition

Example of a standard triangle : $\mathbb{C}_{\beta *}[-1] \xrightarrow{i} \mathbb{C}_{S^1} \xrightarrow{r} \mathbb{C}_{\alpha *} \to$
case of restriction
$\mathbb{C}_{S^1} \xrightarrow[r = res]{} \mathbb{C}_{\alpha *}$

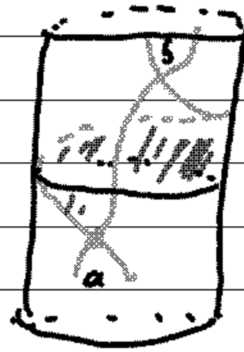




Let's see composition : $\cdot \,\check{}\, \wedge \,|\, = 0$
after perturbation $\quad$ hom $(\mathbb{C}_\beta[-1], \mathbb{C}_\alpha) = 0$

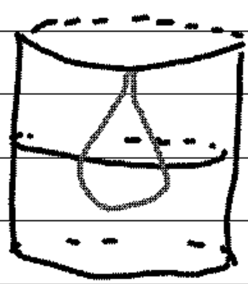
But hom $(\mathbb{C}_{\Delta x}, \mathbb{C}_{psx}[-1]) = \mathbb{C}^2$ in degree 0:

perturb in other direction



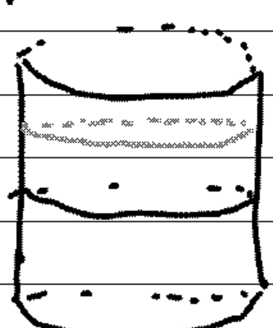
Differential ∂
in dame triangle
is b-a : $\partial \circ r = 0$;

Example of branes isomorphic to zero:



L=0 in $DF(T^*X)$: linked of by fact that when we dilate it down to get conical cycle get something =0.

Dependence on 0 section:



$L_{S^1 \times r}$ translate of 0-section: not exact (corresponds to generator of $H^1(S^1)$) -- but has Maslov class=0

This is a nonhamiltonian deformation of the zero section .... to such deformations assign
<u>twist</u> of local system on the base:
deform by nonexact one form $\Longleftrightarrow$
add this 1-form to the local system:

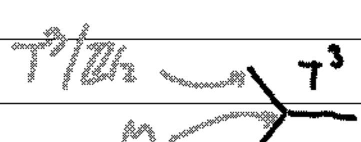
Prop $L_{S^1, r} \;\cong\; L_{S^1}$ with connection $\nabla = d + r\,d\theta$
nonunitary connection

[Note: Lagrangian  is terrible: nonzero Maslov class, twists around twice $\Rightarrow$
can fill in with discs... don't want these in our category]

[Comment: Perverse sheaves $\Longleftrightarrow$ holomorphic branes]

Panteev: example with no instanton corrections!
D_4 dihedral grp of order 8 (Heisenberg $/\mathbb{Z}/2$)
$M = T^3/D_4$ free action. T^*M Hitchin map

$$\begin{array}{c} T^*M \\ \downarrow \\ B \subset R^4 \end{array}$$



Hyperkähler case: no instanton corrections.
but T^*S^3 eg do have them...

F-k (E) elliptic curve: get infinitely many discs, can start adding them, get ∞ combinations, sometimes can sum them up to θ-fns etc but in general should keep them around as formal sums, as elts in a Novikov ring. Such problems will arise in dealing with noncompact Lagrangians...

Proof of embedding: write constructible sheaves using Morse theory, then use Fukaya / Morse dictionary,

To prove equivalence harder:
Natural functor back: use Yoneda embedding into A_∞ functors which describe sheaves

Proof uses formalism of Lagrangian correspondences of Wehrheim - Woodward, Khovanov - Rozansky:
map not Riemann surfaces but foams of R.S. glued along edges  $\longleftrightarrow$ Lagrangian in 3-fold products.

"quilted Riemann surfaces" 

put Lagrangian boundary conditions on intersections.

Need to know things like diagonal = identity, fusion

Proof: realize identity by the diagonal Lagrangian correspondence, which is pr of the diagonal Δ in $X \times X$

But Δ can be deformed to a series of $\boxtimes$ & get motion from the identity to a bunch of projections.