Tony Pantev - Abelianized Hecke Correspondences

Note Title 1/12/2007

Describe Hecke correspondences on $Higgs_{{}^LG}$

$\tilde{x} \in \tilde{C} \longrightarrow C$ cameral cover, $\mu \in char^+ G = cochar^+ {}^LG$

Abelianized Hecke functors: [D. Arinkin]

$H_{ab}^{\mu,\tilde{x}} : D^b(Higgs_{{}^LG}, \tilde{C}, \mathcal{O})^{\mathcal{O}}$ (Hitchin fiber over $\tilde{C}$)

is just pullback by the automorphism of the Hitchin fiber

$\mathcal{L}, \dots \longmapsto \mathcal{L} \otimes \mathcal{S}^{\mu,\tilde{x}}$

where $\mathcal{S}^{\mu,\tilde{x}}$ is the W-equivariant ${}^LT$ bundle given by $\bigotimes_{w \in W} w \cdot \mu(\mathcal{O}(w \cdot \tilde{x}))$

(μ cocharacter $\Longrightarrow$ get ${}^LT$ bundle from line bundle $\mathcal{O}(w\cdot\tilde{x})$)

<u>Remark</u>: $H_{ab}^{\mu,\tilde{x}} = H_{ab}^{w\cdot\mu,\, w\cdot\tilde{x}} \quad \forall w \in W$ [outside branch pt]

so naturally labelled by pairs

$x \in C \quad \tilde{\mu} \in \tilde{C}_x \times_W cochar\, {}^LG$ twist

So outside of Branch $(\tilde{C} \to C)$
we have abelianized Heckes labelled by

$$\tilde{C} \times_W \mathrm{Char}(T) = \mathrm{Char}(\mathcal{T}|_{\tilde{C} \setminus \mathrm{Branch}})$$

Recall $\bar{\bar{\mathcal{T}}} = p_*(\mathrm{cochar}\, T \otimes \mathcal{O}^*)^W$

$$\bar{\bar{\mathcal{T}}} \supset \bar{\mathcal{T}} \supset \mathcal{T}^0$$

$\mathcal{T}^0$ = connected component

$\bar{\mathcal{T}} = \{\text{sections } s \in \bar{\bar{\mathcal{T}}} : \alpha(s|_{D^\alpha}) = 1 \quad \forall \text{ root } \alpha\}$

D^α = α-component of branch divisor

All three the same except for types B, C.

$SO(2n+1)$ $\quad \mathcal{T}, \bar{\mathcal{T}}$ differ by $\mathbb{Z}/2$ at branchpts

$Sp(2n)$ $\quad \mathcal{T}^0, \mathcal{T}$ " " " " " .

<u>Theorem</u> (Donagi–Gaitsgory)

$\mathrm{Higgs}_{G,\tilde{C}}$ = torsor over $\mathrm{Tors}(\mathcal{T})$

+ for curves, due to Hitchin section,
$= \mathrm{Tors}(\mathcal{T})$

(depends on choice of θ-characteristic)

So outside of $\mathrm{Branch}(\tilde{C})$, abelianized Hecke operators are parametrized by $\tilde{\mu} \in \hat{T}_x$, Cartan dual of T_x

There is a collection of operators

${}^L A^{\rho x}$ labelled by $\rho \in \mathrm{Rep}({}^L G)$, $x \in C$

which are convolution functors (i.e. not coming from maps of the bases)

generating the same algebra as the ${}^L H^{\tilde{\mu}, x}_{ab}$

that should be the classical limit.

Outside of the branch locus: (outside discriminant)

Let $v: B \longrightarrow \mathrm{Higgs}_G$

${}^L v: B \longrightarrow \mathrm{Higgs}_{{}^L G}$

$$v(\tilde{C}) = (V, \varphi: T \hookrightarrow V \times_{Ad} G,$$
$$\theta: \tilde{C} \to \mathfrak{t} \otimes \Omega^1_C)$$

(φ identifies $Z(\theta)$ with T)

$$ {}^L\!A^{\rho,x} = \bigoplus_{\tilde{\mu} \in \mathrm{Char}(\mathcal{T}_x)} \rho(V_x)_{\tilde{\mu}} \otimes {}^L\!H_{as}^{\tilde{\mu},x} $$

$\tilde{\mu}$ isotypic component of ρ

Want to define this
geometrically, so it's defined across branch locus

Consider $\hat{\mathcal{T}}$ = Cartier dual of $\mathcal{T}$
= group int scheme representing
$\mathrm{Hom}_C(\mathcal{T}, \mathbb{G}_m)$
... generically family of lattices,
get formal groups at branch locus
($\mathcal{T}$ gets additive parts... family of regular centralizers!)

Isotypic pieces above are fibers of Fourier transforms:

$\mathcal{T}$ acts on $\rho(V)$ (vector bundle on curve) from left
via the map φ

Now Fourier-Mukai: $\rho(V) \Longrightarrow \widehat{\rho(V)} \in D^b(\hat{\mathcal{T}})$
quasi coherent sheaf on $\hat{\mathcal{T}}$
- in fact it's a coherent, supported on finite subsets
sheaf /C.

The stalk $\widehat{\rho(V)}\big|_{\tilde\mu} = \rho(V_x)_{\tilde\mu}$ isotypic component

The assignment $(\tilde\mu, x) \longmapsto {}^L H_{ab}^{\tilde\mu, x}$

gives a homomorphism of ind-groups

$$\hat{\tilde{C}} \xrightarrow{\kappa} \mathrm{Aut}_{/C}\left(\mathrm{Higgs}_{{}^LG,\tilde{C}} \times C\right)$$

as sheaves over C away from branch points..

If we think of $\mathrm{Tors}({}^LT)$ as automorphisms of $\mathrm{Higgs}_{{}^LG,\tilde{C}}$ $\Longrightarrow$ can rewrite

$$\begin{tikzcd} \hat{\tilde{C}} \arrow[rr, "\kappa"] \arrow[dr] & & \mathrm{Tors}({}^LT) \arrow[dl] \\ & C & \end{tikzcd}, \quad \tilde\mu \longmapsto {}^L H_{ab}^{\tilde\mu, x}\left({}^L\underbrace{\nu(\tilde{C})}_{\text{Hitchin section}}\right)$$

Now rewrite ${}^L A^{\mu,x}$ geometrically:

$$(1) \qquad \begin{tikzcd} & \mathrm{Tors}({}^LT) \times \mathrm{Higgs}_{{}^LG,\tilde{C}} \arrow[dl, "p_2"', "\text{project}"] \arrow[dr, "a", "\text{action}"'] & \\ \mathrm{Higgs}_{{}^LG,\tilde{C}} & & \mathrm{Higgs}_{{}^LG,\tilde{C}} \end{tikzcd}$$

$$\rho(\nu(\tilde{\tau})) = (\rho(V), \rho(\theta))$$

vector bundle on C — Ω^1_C valued section on $\mathrm{End}\,\rho(V)$

-- Higgs on C with action of T

$$\widehat{\rho(\nu(\tilde{\tau}))} \text{ — } p_C^*\Omega^1_C\text{-valued}$$

Higgs sheaf on $\hat{\tau}$

$$\kappa : \hat{\tau} \longrightarrow \mathrm{Tors}({}^L T)$$

so now take $\kappa_* \widehat{\rho(V(\tilde{\tau}))}$

Higgs sheaf on $\mathrm{Tors}({}^L T) \times C$ with $p_C^* \Omega^1_C$

$\Longleftrightarrow$ coherent sheaf on $\mathrm{Tors}({}^L T) \times T^*C$

(2)
$$\begin{tikzcd}
 & \mathrm{Tors}({}^L T) \times \mathrm{Higgs}_{{}^L G, \tilde{\tau}} \times T^*C \arrow[dl, "p_2"'] \arrow[dr, "a \circ p_3"] & \\
\mathrm{Higgs}_{{}^L G, \tilde{\tau}} & & \mathrm{Higgs}_{{}^L G, C} \times T^*C
\end{tikzcd}$$

<u>Claim</u> If ${}^L\mathcal{A}^\rho = \coprod_{x \in C} {}^L\mathcal{A}^{\rho,x}$

$\mathcal{M} \in D^b(\mathrm{Higgs}_{{}^LG,C}, \mathcal{G})$

$$\Rightarrow {}^L\mathcal{A}^\rho(\mathcal{M}) =$$

$$(a \times p_3)^* (p_{13}^* \, \mathcal{K}_* \rho(\widehat{V(\tilde{C})}) \otimes p_2^* \mathcal{M})$$

$$\in D^b(\mathrm{Higgs}_{{}^LG,C} \times T^*\tilde{C})$$

So we rewrite ${}^L\mathcal{A}^\rho$ as a F.M

... but we need for this
$\mathcal{K}$ everywhere, including at
branch pts ....

This is local questn at branch pts,
on formal discs

$\Rightarrow$ need Carlson-Carrère for nonsplit
groups ${}^LT$: become split tori on finite
cyclic cover, & must apply Galois cohomology
to show descends ....