E. Witten - Gauge Theory & The Geometric Langlands Program 6/28/06

1) Introduction to electric-magnetic duality
2) Overview of Kapustin-Witten
3) Hecke & 't Hooft operators
4) Ramified Case

Classical E & E

1. E electric B magnetic
start as "vectors" on $\mathbb{R}^3$.
Use metric $\rightsquigarrow$ one-forms on $\mathbb{R}^3 \times \underset{\text{time}}{\mathbb{R}^1} = \mathbb{R}^{3,1}$

Relativistically E, B combine to
$F = dt \wedge E + (*)_3 B$ 2-form
3-dim Hodge $*$

Maxwell's eqns in vacuum:
$0 = dF = d(*F)$ here $*$ = 4-dim Hodge $*$

19th century: symmetry $F \to *F$
$*F \to -F$
most elementary form of electric-magnetic duality.
In vacuum case still exists quantum mechanically but much more subtle.

<u>Quantum Case</u> Need Lagrangian / action $\Rightarrow$ must break E-M symmetry.
Introduce line bundle $\mathcal{L} \to M_4$
M_4 = 4 dim spacetime, e.g. $\mathbb{R}^{3,1}$
$(\mathcal{L}, \nabla)$ connection $\nabla = d + A$ in local triv
A = connection form = vector potential
$F = \nabla^2$ = curvature

Thus $dF = 0$ is the Bianchi identity.
The other half $d * F = 0$ must come from a Lagrangian. Here $F/*F$ symmetry broken by writing F in terms of A

<u>Action</u> $$I = \frac{1}{4e^2}\int_{M_4} F \wedge * F + \frac{i\theta}{(2\pi)^2}\int_{M_4} F \wedge F$$

e = charge of electron $\in \mathbb{R}$

Euler-Lagrange equation: $\delta I = 0$ ... critical point of action.
Want $\delta I = 0 \iff d * F = 0$
Unaffected by adding locally constant function (topological term) ... hence can add $i\theta \int_{M_4} c_1(\mathcal{L})^2$

Formally (Lagrangian quantum mechanics) imagine integral

$$\int_{\substack{\text{all connections}\\ \text{on line bundles}\\ \text{mod gauge transformations}}} (\mathcal{D}A)' \exp(-I(A))$$

including sum on $\mathcal{L}$

Action quadratic in F so can actually make sense of this integral via $\int dx_1 \ldots dx_n \exp -Q(x)$

Q quadratic $$= \frac{\pi^{n/2}}{\sqrt{\det Q}}$$

Quantum theory depends on e, θ nontrivially.
Let $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \in \mathbb{H}$ upper half plane

Elementary "classical" symmetry (topological):
$\tau \longmapsto \tau + 1 \iff \theta \longmapsto \theta + 2\pi$

Non-classical symmetry comes from Fourier transform on space of connections
$\tau \longmapsto -1/\tau$

Recall $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{ixy} e^{-\lambda y^2/2} = e^{-x^2/2\lambda^{-1}}$:

Fourier of Gaussian is Gaussian

$\tau \to \tau + 1$ & $\tau \to -1/\tau$ give $SL_2\mathbb{Z}$ symmetry.

E - M duality

<u>Wilson & 't Hooft operators</u>

L oriented loop in M_4. Two resulting constructions:

M_4

I. Pick $n \in \mathbb{Z} \Longleftrightarrow$ rep of $G = U(1)$

take holonomy around L, $\mathrm{Hol}(L, n) = \exp in \int_L A$

in representation determined by n

(Wilson operator)

To quantize this classical operator insert it in path integral:

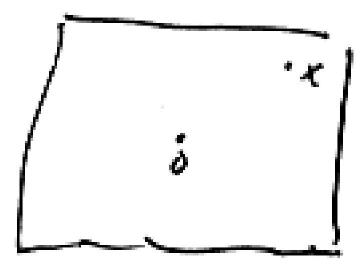
$$\int (DA)' \exp(-I(A)) \prod_{i=1}^{s} \mathrm{Hol}(L_i, n_i)$$

... <u>order</u> <u>operators</u> in statistical mechanics.

Duality often relates <u>order</u> & <u>disorder</u> operators.

II. Modify path integral by changing space of fields on which we integrate to fields with prescribed singularity (point magnetic monopole).

Standard solution: pick point $0 \in \mathbb{R}^3$



$F = \frac{1}{2} * d\frac{1}{|x|}$

Obeys Maxwell's eqns on $\mathbb{R}^3 \setminus 0$

$d * F = d * d\frac{1}{|x|} = \delta_0$, vanishes on $\mathbb{R}^3 \setminus 0$

$F = \nabla^2$ is curvature of line bundle $(\mathcal{L}, \nabla)$ on $\mathbb{R}^3 \setminus 0$. Need $\int_S \frac{F}{2\pi} = 1$ integral on sphere around 0.

Could take m^{th} power of $\mathcal{L} \longleftrightarrow F = \frac{m}{2} * d\frac{1}{|x|}$.

So given loop labeled by integer m,

can look at $(\mathcal{L}, \nabla)$ with singularity along

loop L of prescribed singularity

& integrate over all such.

... 't Hooft 1979 : dual to including holonomy in path integral under E-M duality.

(have homomorphism $m : U(1) \longrightarrow G = U(1)$ giving local singularity).

Back to Maxwell in vacuum: have complete symmetry $0 = dF = d * F$.

Case I (Wilson) : still have $0 = dF$ (Bianchi)

But equation $d * F = 0$ came from making I stationary, & now we have extra operators in integrand....

So Euler-Lagrange eqn becomes $d * F = n\delta_L$.

Case II ('t Hooft) : integrand didn't change, so we still get $d * F = 0$.

But have singularity in F giving $dF = m\delta_L$.

G = gauge group

I (Wilson) Pick R = Rep of G,

get Wilson operator from holonomy $\mathrm{Tr}(\mathrm{Hol}(A; L, R))$

if M has boundary get holonomy <u>operator</u>.

II ('t Hooft) need codimension 3 singularity.

Consider $\rho : U(1) \longrightarrow G$ & ask singularity looks like ρ(basic $U(1)$ singularity) ... pick such ρ up to conjugation.

Goddard - Nuyts - Olive (1976) find correspondence between groups G, ${}^LG$ that exchanges I, II.

I : R has a highest weight $w: T \to U(1)$
T = max torus of G

II $\quad p: U(1) \to T \xrightarrow{i} G$ mod Weyl

$$\mathrm{Hom}(U(1), T_{{}^LG}) \simeq \mathrm{Hom}(T_G, U(1)).$$

More systematic approach to singularity: it's a conical singularity in $\mathbb{R}^3$ obeying Yang-Mills eqns $\rightsquigarrow$ soln of Y-M on $S^2 = \mathbb{CP}^1 = \{\mathbb{R}^3 - 0\}/\mathbb{R}_+$

Why? in QM equations are not always obeyed, but our singularities should obey classical equations so we can integrate fluctuations around them...

Atiyah-Bott: solutions of YM on a Riemann surface $\longleftrightarrow$ G-bundles on C.
Here $C = S^2 = \mathbb{P}^1$.
G-bundles over C $\iff$ $\rho: \mathbb{C}^\times \to G_{\mathbb{C}}$
$\longleftrightarrow U(1) \to G$.

Montonen - Olive (1977) : $\exists$ a 4-dim gauge theory

$$I = \frac{1}{4e^2}\int \mathrm{Tr}\, F \wedge * F + \frac{i\theta}{4\pi^2}\int \mathrm{Tr}\, F\wedge F + \cdots$$

s.t. have symmetry
$\tau \longleftrightarrow -\frac{1}{\tau}$
$G \longleftrightarrow {}^LG$
Wilson $\longleftrightarrow$ 't Hooft
't Hooft $\longleftrightarrow$ Wilson

${}^LG$, τ	G, $-\frac{1}{\tau}$
R rep of ${}^LG$	$\rho: U(1) \to T \to G$
$\mathrm{Tr}\,\mathrm{Hol}(A, L, R)$	$\rho \longleftrightarrow R$ of ${}^LG$.
	't Hooft operator

2) • Duality is not understood in the nonabelian case, related to a web of other conjectures.

• Gauge theory in 4d only makes sense for compact gauge group! G & ^LG are compact.

Maximal supersymmetry, $N=4$.

Analogy with Hodge theory:

Physicists always add a dimension to a geometric question, there's always time.

e.g. G group, R rep, get by quantizing G/T (or $G_{\mathbb{C}}/B$). For a physicist get this from maps 1-manifold $\longrightarrow G/T$ & quantizing.

diff. forms on manifold $X \Leftrightarrow$ SUSY theory of Φ : 1-manifold $\longrightarrow X$

So Hodge theory $\leadsto$ 1-dim QFT

X complex Kahler: have $Spin(3) = SL_2$ action on cohomology of X. $3 = \dim_{\mathbb{R}} \mathbb{C} + 1$

$\partial, \bar\partial, d = \partial + \bar\partial, \partial^*, \bar\partial^*$ $\Delta = d^*d + dd^*$

Hamiltonian commutes with Spin 3 action $\Longrightarrow$ get it on cohomology.

For hyperkähler X get Spin 5 action ($Sp\,4$ or $Sp\,2$).

$5 = \dim_{\mathbb{R}} \mathbb{H} + 1$

Similarly Spin 9 action on cohomology of an octonionic manifold.... $9 = \dim_{\mathbb{R}} \mathbb{O} + 1$

— NONE SUCH.

However for every G there's a canonical quantum mechanics problem with Spin 9 symmetry.

... this was all 1-dim QFT.

Each of these theories can be lifted to higher dim quantum mechanics problems.

Try to see how high we can lift!

find theory on $\mathbb{R}^{1,n-1} = \mathbb{R}^1 \times \mathbb{R}^{n-1}$

such that when fields are independent of the $\mathbb{R}^{n-1}$ get above theory.

Maximal dim (classically)

Kahler	4	$\Rightarrow$ symmetry	Spin 3
hyperkahler	6		Spin 5
octonionic	10		Spin 9

Octonionic case, understood as a gauge theory, the maximal quantum dim in which it's defined is 4

(can't quantize in range 5-10).

To go up to 6 or 10 need to leave gauge theory, in 10 need full string theory.

4-dim: N=4 Super Yang Mills we'll reduce to lower dimensions quantum mechanically

Classically start on $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$.

take fields independent of $\mathbb{R}^6$ directions, get Spin 6 symmetry

Spin 6 $\cong$ SU(4) R-symmetry.

In 10 dim bosonic field is a connection on a G-bundle $E \to \mathbb{R}^{1,9}$

Fermions are sections $\psi \in \Gamma(\mathbb{R}^{1,9}, S_+ \otimes \mathrm{ad}E)$

where S_+, S_- are spin bundles.

$$\mathcal{L} = \frac{1}{4e^2} \int \mathrm{Tr}(F \wedge * F + \psi i \not{D} \psi)$$

Supersymmetry of this depends sensitively on dimension.

Reduce to $D=4$, $\mathbb{R}^{1,3}$:

Bosons: gauge field A + 6 scalars $\phi_1, \dots \phi_6 \in \Gamma(\mathbb{R}^{1,3}, \mathrm{ad}E)$ in $\underline{6}$ of Spin 6.

1-dim problem: has 9 bosons in $\underline{9}$ of Spin 9,
get 1-dim gauge field & a bunch of fermions
(SUSY: automorphisms of $\mathbb{R}^{1,9|16}$ act on 10-dim theory)

Now pass to $\mathbb{R}^4$, Euclidean signature, &
construct a "twist topological field theory"
... several inequivalent versions ...
"Twisting" modifies the action of Aut $(\mathbb{R}^4)$
Bosonic symmetry group is Aut $\mathbb{R}^4$ × Spin 6

$$0 \to \mathbb{R}^4 \to P \to \text{Spin } 4 \to 0$$

... twist by taking a homomorphism of Spin 4 $\xrightarrow{\lambda}$ Spin 6,
get new group Spin' 4 $\hookrightarrow$ Spin 4 × Spin 6
as image of $(1\times\lambda)(\text{Spin } 4)$.
We want Spin' 4 to have an invariant vector in the
rep S_+ of Spin 4 × Spin 6 $\hookrightarrow$ Spin 10.
3 such up to equivalence, two give Donaldson-type,
one gives geometric Langlands.

$$\begin{array}{c} \text{Spin } 4 \times \text{Spin } 2 \hookrightarrow \text{Spin } 6 \\ \lambda \nearrow \\ \text{Spin } 4 \end{array}$$

Why? want to go from $\mathbb{R}^4$ to general 4-manifold
and keep some supersymmetries, ie spinor fields
in S_+ invariant under symmetries.
... ie connection <u>with torsion</u> on tangent bundle
might have covariantly constant spinors.
If Q is a Spin' 4-invariant odd symmetry $\Rightarrow \underline{Q^2=0}$
$\Rightarrow$ Q cohomology is a topological field theory.
Here the space of Spin' 4 invariants is 2-dim,
with basis Q_+, Q_- $\Rightarrow$ $\mathbb{P}^1$ family
of TQFTs $Q = uQ_+ + vQ_-$, $t = v/u \in \mathbb{P}^1$

~~Good p~~

Good parameter: $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \frac{t - t^{-1}}{t + t^{-1}} .$$

$SL_2\mathbb{Z}$: $\quad \Psi \longmapsto \frac{a\Psi + b}{c\Psi + d}$.

Geometric Langlands: $\Psi = \infty \iff \Psi = 0$.

Now compactify to two dimensions:

$M_4 = \Sigma \times C$ product of two Riemann surfaces

Think of diam $C <<$ diam Σ

$\rightsquigarrow$ effective field theory on Σ

which is a σ-model on Σ of maps

$\Sigma \longrightarrow M_H$, the hyperkähler moduli of Higgs bundles on C

... so we add two dimensions & look at maps

$\Phi : \Sigma \to M_H(G, C)$.

A. Kapustin Gauge Theory & Geometric Langlands 6/28/06

S-duality conjecture

$N=4$ SYM theory with gauge group G simple

$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ $G \longleftrightarrow {}^LG$

$\tau \longleftrightarrow {}^L\tau = -\frac{1}{\tau n_g}$ $n_g = 1,2,3$

Supercharges Q^i_α, $\bar{Q}_{\dot\alpha, i}$ $\alpha, \dot\alpha = 1,2$,

$i = 1 \ldots 4$ acted on by Spin 4 × Spin 6

$\{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\} = i\,\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^i_j$

Under S-duality $Q \longrightarrow e^{i\phi/2} Q$

$\bar{Q} \longrightarrow e^{-i\phi/2} \bar{Q}$

where $e^{i\phi} = \frac{\tau}{|\tau|}$

Topological twist: Q_l, Q_r survive (two supersymmetries)

with $\{Q_-, Q_-\} = 0$

$Q_{BRST} = u Q_l + v Q_r$ $t = \frac{v}{u} \in \mathbb{P}^1$

Theory depends on t, e^2, θ.

S-duality: $t \longrightarrow \frac{\tau}{|\tau|} t$ [Note: stress-energy tensor preserved by S-duality]

The theory is topological!

$S = \frac{i\psi}{4\pi}\int \mathrm{Tr}\, F\wedge F + \{Q_{BRST}, V\}$ all metric dependence here.

Field content: A connection, $\phi \in \Omega^1_M(\mathrm{ad}\,E)$ from $\phi_1, \ldots \phi_4$

& $\phi_s, \phi_r \in \Omega^0_M(\mathrm{ad}\,E)$

$\psi, \tilde\psi \in \Omega^1_M(\mathrm{ad}\,E)$, $\eta, \tilde\eta \in \Omega^0(\mathrm{ad}\,E)$, $\chi \in \Omega^2_M(\mathrm{ad}\,E)$

Path integral localizes on solutions to some PDEs from Q (applied to fermions) precisely, not just asymptotically.

BPS eqns: $(F - \phi\wedge\phi + t D\phi)^{+} = 0 \quad (F - \phi\wedge\phi - t^{-1} D\phi)^{-} = 0$
& $D{*}\phi = 0$.

Special case $\underline{t = i}$ eqns become
$[\mathcal{A} = A + i\phi \quad \mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}]$
$\mathcal{F} = 0$ & $D{*}\phi = 0$:
i.e flat connections with auxiliary ϕ which can be taken as promoting the gauge group to its complexification.

<u>S-duality</u> $(\theta = 0)$ $t = i \rightsquigarrow t = 1$
where we get Bogomolny type eqns
$\begin{cases} F - \phi\wedge\phi + {*}D\phi = 0 \\ D{*}\phi = 0 \end{cases}$

<u>Compactification</u> $M = \underset{\text{large}}{\Sigma} \times \underset{\text{small}}{C}$

2d BPS eqns: $F - \phi\wedge\phi = 0$, $D{*}\phi = 0$.

Take $\Sigma = \mathbb{R}^2$. What are vacua?

solutions with zero energy are
$\Phi : \mathbb{R}^2 \longrightarrow \mathcal{M}_{H,t}(G, C)$
fields of supersymmetric σ-model.
Can actually define more twists when looking at special M of product form, not 4d TQFT but a SUSY QFT..

X Kähler with $c_1(X) = 0$ have two different attached TQFTs : A-model (depends only on symplectic structure, counting holomorphic curves in X)
B-model (complex geometry, purely classical at tree level)
appear as Hochschild cohomologies of corresponding categories : Fukaya, $D^b(\mathrm{Coh}\, X)$.
Mirrors : $A(X_1) \simeq B(X_2)$: i.e categories exchanged.

S-duality implies $\mathcal{M}_{Hit}(G,C)$ is mirror to $\mathcal{M}_{Hit}({}^LG,C)$: $\mathcal{M}_{Hit}(G,C)$ ($t=1$, A-model with ω_K symplectic form) mirror to
$\mathcal{M}_{Hit}({}^LG,C)_J$ ($t=i$, B-model in J complex structure.

I, J, K complex structures:

$$\mathcal{M}_{Hit}(G,C)_I = \mathcal{M}^{stable}_{Higgs}(G_{\mathbb{C}}, C)$$

indep. of complex structure on C
$$\begin{cases} \mathcal{M}_{Hit}(G,C)_J = \mathcal{M}_{flat\ conns}(G_{\mathbb{C}}, C) & \text{(irreducible)} \\ \omega_K = \int_C \mathrm{Tr}(\delta\phi \wedge \delta A) & \text{symplectic form} \end{cases}$$

Hitchin map:
$$\begin{array}{c} \mathcal{M}_{Hit}(G,C) \\ \downarrow \\ V \quad \text{affine space} \end{array}$$

generic fiber is a Lagrangian torus for ω_K.

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- In $N=4$ SYM have complete symmetry between G & ${}^L G$ ... but symmetry is broken by choice of question in geometric Langlands

- Localization of functional integrals:
$$\int (\mathcal{D}A \ldots)' \exp(-I) \,(\text{operators})$$
over infinite dimensional space of fields $\mathcal{A}$
with odd symmetry Q, $Q^2=0$.
Q is an odd vector field on $\mathcal{A}$.
If this vector field has no zeros, divide by Q-orbits:
$$\mathbb{C}^{0|1} \longrightarrow \mathcal{A}$$
"fibers"
$$\downarrow$$
$$\mathcal{A}_0$$
integrate first along the fibers:
$\int d\theta\, f = 0$: $\frac{\partial}{\partial\theta} f(\theta) = 0$ since f indep. of Q
so fermionic integral vanishes ...
so zeros of Q critical ... & this is a finite dimensional variety by ellipticity.
$$Q = (F - \varphi\wedge\varphi)\frac{\partial}{\partial\chi} + (D * \varphi)\frac{\partial}{\partial\eta} + \ldots$$
$\chi, \eta, \ldots$ fermions.
Vanishing of $Q \Rightarrow$ solution of all of these equations.
Solution set for us:
$$(F - \phi\wedge\phi + t D\phi)^+ = 0$$
$$(F - \phi\wedge\phi - t^{-1} D\phi)^- = 0$$
$$D * \phi = 0$$
$(*)$ universal properties & vanishing
elliptic for t real

$t \to 0$, eqns have good limits, get $\mathbb{RP}^1$ family of elliptic equations ...

** for 4-manifolds each individual one may not be interesting but whole family should be!

$M_4 = \Sigma_{\text{big}} \times C_{\text{small}}$ (small $\Rightarrow$ can assume <u>classical</u> eqns of motion on C but must treat Σ quantum mechanically

$\Rightarrow$ SUSY σ-model of maps

$\Phi : \Sigma \longrightarrow \mathcal{M}_{Hit}$

$\mathcal{M}_H$ = space of solutions of $(*)$ equations, pulled back from C. This space of solutions is independent of t (all other solutions correspond to <u>singularities</u> of $\mathcal{M}_H$, where σ-model breaks down).

Equations of Hitchin:

$$0 = F - \phi \wedge \phi$$
$$0 = D\phi = D{*}\phi$$

$\mathcal{M}_H$ is hyperkähler $\Rightarrow$ family of kähler structures parametrized by $\mathbb{P}^1$.

I : moduli of stable Higgs bundles

(E, φ) E = G-bundle $\to C$, C holomorphic.

$\varphi \in H^0(C, K_C \otimes \mathrm{ad}\, E)$ $\varphi = (\phi)^{1,0}$

* Pair being stable does <u>not</u> imply underlying bundle E is stable $\Longrightarrow$ source of singularities of geometric Langlands $\mathcal{D}$-modules!

$\mathcal{A} = A + i\phi$ connection for $G_{\mathbb{C}}$ — moment map equation

which is <u>flat</u>. Equations $D{*}\phi = 0$ + curv$(\mathcal{A}) = 0$ mod G-valued gauge transformations is roughly equivalent to looking at flat $G_{\mathbb{C}}$-connections up to $G_{\mathbb{C}}$-gauge transformations.

$\Rightarrow (\mathcal{M}_H)^J$ = moduli of stable local system i.e. $\rho: \pi_1(C) \to G_{\mathbb{C}}$ which are completely reducible. $a = A + i\phi$ flat

- Complex structure K: $a = A + i * \phi$ is flat ... isomorphic to J structure.

More generally have $\mathbb{C}^*$ symmetry of hyperkahler $\mathbb{P}^1$ w/ I & $-I$ as fixed points.

$\Longrightarrow \mathbb{C}^*$ acts holomorphically in structure I (and $-I$)

$(E, \varphi) \rightsquigarrow (E, \lambda\varphi)$

Start with local system : E $G_{\mathbb{C}}$-bundle, associated $G_{\mathbb{C}}/G$ bundle is contractible & Corlette-Donaldson prove it has a unique harmonic section.

Viewed as space of Higgs bundles (complex structure I) $\Rightarrow$ completely integrable Hamiltonian system. $\mathcal{M}_H \underset{\text{roughly}}{\cong} T^*\mathcal{M}$, $\mathcal{M}$ = moduli of stable $G_{\mathbb{C}}$-bundles.

(really $\mathcal{M}_H$ is a symplectic partial completion ...)

Hitchin hamiltonians: for $G = SU(N)$,

$(E, \varphi) \mapsto \det(\lambda - \varphi) \in \bigoplus_2 H^0(C, K_C^\ell)$

or $\quad \mathrm{Tr}\, \varphi^\ell \quad \ell = 2, \dots N$

$\in H^0(C, K_C^\ell)$

$$\begin{array}{c} \mathcal{M}_H \\ \pi \downarrow \\ B \end{array}$$

B = Hitchin base, a vector space, pullback of functions Poisson commute.

$\dashrightarrow$ smooth fibers are abelian variety

Hitchin's fibration is holomorphic wrt complex structure I.

The S-duality acts by T-duality on the Hitchin fibers

..... fiberwise Fourier-Mukai transform on fibers.

$\tau \longleftrightarrow -\frac{1}{\tau}$

$I_A \longleftrightarrow I_A$ Fourier-Mukai for Fukaya category

$J_B \longleftrightarrow J_B$ Fourier Mukai for coherent sheaves

$J_A \longleftrightarrow K_B$

$\boxed{J_B \longleftrightarrow K_A}$ Geometric Langlands :

$K_A \longleftrightarrow J_B$

$K_B \longleftrightarrow J_A$

$\psi = \infty$	$\psi = 0$
J_B	K_A
${}^LG$	G

Mirror symmetry comes from a symmetric situation: ~~A-model~~ SUSY σ-model. But we break this symmetry by choosing A-model on one side & B-model on other – some asymmetry arises in geometric Langlands.

Branes We take M_4 (microscopic) or Σ ($M_4 = \Sigma \cdot C$) to have a boundary.

For an elliptic PDE we need to choose a local elliptic boundary condition.

In gauge theory could ask

eg $F|_{\partial M} = 0$

or $*F|_{\partial M} = 0$

For real-valued field ϕ have

Dirichlet : $\varphi|_{\partial M} = f$ fixed

Neuman $\underbrace{(n\cdot\nabla)}_{\text{normal derivative}} \varphi|_{\partial M} = 0$

In our case $\Phi : \Sigma \longrightarrow X$ σ-model can combine Dirichlet, Neuman in interesting ways:

can say $\phi|_{\partial\Sigma} : \Sigma \longrightarrow Y$ $Y \subset X$ some submanifold

If $0 < \dim Y < \dim X$ this gives mixed Dirichlet-Neuman boundary conditions ... to make system elliptic control normal derivatives as well as position.

More general boundary conditions: $V \to Y$ vector bundle with connection C.

fields: $\Phi : \Sigma \to X$ + fermions

I = σ-model action

$$\int_{\Phi : \partial\Sigma \to Y} (D\Phi \ldots) \exp(-I) \ (\text{operators}) \cdot \left(\mathrm{Tr}\ \mathrm{Hol}(\partial\Sigma, \Phi^* C)\right)$$

insert trace of holonomy of Chan-Paton bundle pulled back to $S^1 = \partial\Sigma$.

Recall we started from twisted topological theory on M_4 with Q_1, Q_2 & $Q = uQ_1 + vQ_2$.

Narrow this depending on what supersymmetry we want:

In a B-model $Y \subset X$ holomorphic & $V \to Y$ is hol.

$(Y, V) \rightsquigarrow$ object in "derived category of coherent sheaves"

... in physical theory have C^∞ connections but B-model observables only care about holomorphic structure.

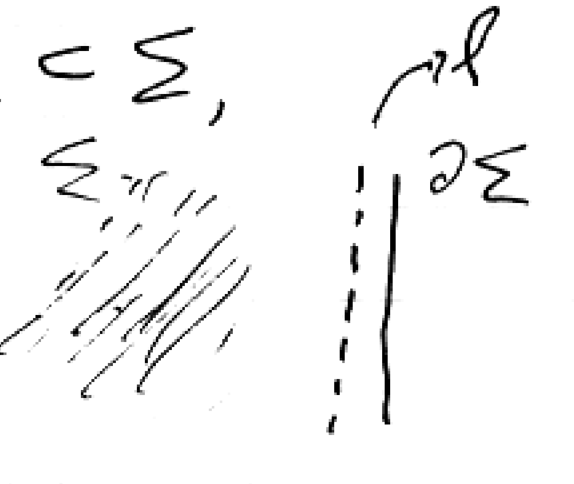
In an A-model $\Rightarrow$ Floer-Fukaya category.

Obvious objects: Y Lagrangian, V flat.

Remember: gauge theory has Wilson + 't Hooft operators.

Loop operators: $L = \ell \times p$, ℓ = line $\subset \Sigma$, $p \in C$ point.

We'll take ℓ to run near $\partial\Sigma$:



Take limit as $\ell \rightsquigarrow \partial\Sigma$:

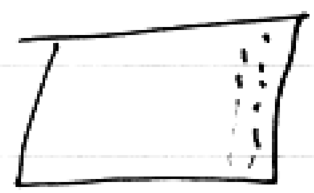
well, we're in a topological situation

Composite of L + brane gives a new brane!
Physical picture: send wave off to boundary.
see a composite of brane + line operator collectively as a new brane.

So can look at eigenbranes: B brane, L operator $\rightsquigarrow$ LB new brane, can look to see if $LB = B \otimes W$.

$p \in C$, R rep of G or ${}^LG$ $\Rightarrow$ line operator $L(p)B$. depends flatly on p by topological property: $L(p)B = B \otimes W_p$.
W_p will form a local system.

Because of 4-dimensionality, can move lines around each other & get commutativity!



$L'LB \simeq LL'B$ & no braiding, like you would get in 3 dimensions.

Electric eigenbrane: joint eigenbrane for all Wilson operators.
Magnetic eigenbrane: joint eigenbrane for all 't Hooft operators.

Let's find how the Wilson operators act on branes.

Universal Higgs bundle $(\mathcal{E}, \phi)$, $\mathcal{E} \to C \times \mathcal{M}_H$ over Hitchin moduli space (doesn't quite exist)
... let's pretend $G = E_8$: simply connected, adjoint, no center!

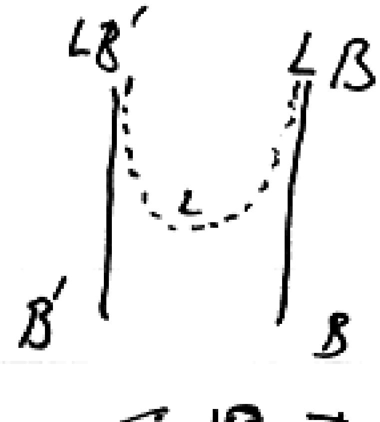
For any rep R of G & $p \in C$ get associated vector bundle $E_R(p) = \mathcal{E}|_{p \times \mathcal{M}_H} \otimes_{G_\mathbb{C}} R$ on $\mathcal{M}_H$.

Wilson operator $P\exp \oint A|_R$ approaching boundary
→ goes over to holonomy of universal bundle $\mathcal{E}_R(p)$ (connection over $\mathcal{M}_H$)
If $\mathcal{B}$ is defined by $V \to Y$, Wilson operator acts by $V \rightsquigarrow V \otimes \mathcal{E}_R(p)$.

$\mathcal{E}$ has hyperkähler connection: simultaneously holomorphic in all 3 complex structure, completely canonical.

Electric eigenbranes can arise if have $Y \subset \mathcal{M}_H$ such that $\mathcal{E}_R(p)|_Y = W$ a fixed vector space.
Obvious choice: $Y =$ a point in $\mathcal{M}_H \Rightarrow$ in J structure we get an electric eigenbrane for any complex structure

Action of line operators as functors:



$\Sigma = \mathbb{R} \times I$

Magnetic eigenbranes

Dual Hitchin fibers are dual abelian varieties, get a Fourier Mukai transform from electric to magnetic

$$\mathcal{M}_{H,G} \xrightarrow{\pi} \mathcal{B} \xleftarrow{\pi} \mathcal{M}_{H,{}^LG}$$

$\mathcal{B}$ Hitchin base
Same for both

$I \quad J \quad K$

Electric branes at points have form (B, B, B)
Local system on Hitchin fibers has type (B, A, A)
... holomorphic in I, Lagrangian for J, K.

For point in $\mathcal{M}_{loc}({}^L G)$ we get a magnetic eigenbrane in $\mathcal{M}_H(G)$ (wrt K complex structure).
... ie object of Fukaya category with 't Hooft eigenproperty.

To make contact w/ geometric Langlands need to relate A-branes of type K on $\mathcal{M}_H$ to (twisted) D-modules on Bun_G $\mathcal{M} = \mathcal{M}(G, C)$.
$\mathcal{M}_H \underset{birat}{\simeq} T^* \mathcal{M}$.
Trick: special "big" A-brane:
Usual A-branes are flat unitary bundles on Lagrangians
Kapustin-Orlov noticed that there are more A-branes:
supported on coisotropic submanifolds - (ie submanifolds locally defined by Poisson commuting functions).
-- there don't exist such A-branes for generic CY3s.
But here there is a canonical big coisotropic A-brane, with $Y = X$ $(= \mathcal{M}_H)$
.... not understood in rank > 1.
Rank 1: $(\mathcal{L}, \nabla) \longrightarrow Y = X$ s.t.
$F = \nabla^2$ is nondegenerate as a 2-form
& $(\omega^{-1} F)^2 = -1$ $\quad (\omega^{-1} F \in End\, TX)$
so $\omega^{-1} F$ is an almost complex structure which is <u>integrable</u> (Kapustin-Orlov)

(If $\mathcal{L}$ were trivial would set Neumann conditions to get brane... for A-SUSY need these conditions)

If X is hyperkähler, $\omega = \omega_K$, $F = \omega_J$
& $I = \omega^{-1} F = \omega_K^{-1} \omega_J$
but need integrality of ω_J to quantize to $(\mathcal{L}, \nabla)$
So works for any hyperkähler manifold with ω_J integral.
e.g. cotangent bundle ... For us
$[\omega_J] = [\omega_K] = 0$ in $H^2(\mathcal{M}_H, \mathbb{R})$
so $\mathcal{L}$'s exist & are topologically trivial, & have canonical choice of 1-form λ_J with $d\lambda_J = \omega_J$.

This big brane $\mathcal{B}$ turns A-branes into $\mathcal{D}$-modules
.... There are no instanton corrections here! so can calculate
End $(\mathcal{B})$ in A-brane category

algebra structure :
associativity comes from topological structure.

B-model has no instanton corrections $\Rightarrow$ can always sheafify things, calculate everything locally.
Usually can't sheafify A-model because of holomorphic maps $\phi : \Sigma \to X$; instantons.

However, if a component of $\partial\Sigma$ is labeled by $\mathcal{B}$ = big brane
$\Rightarrow$ no instantons, not allowed by $(\mathcal{L}, \nabla)$.
So $\mathcal{B}, \mathcal{B}$ strings give a sheaf of algebras.

Describing B-B strings additively get $H^0(\mathcal{M}_H(I), \mathcal{O})$,
I enters since $I = \omega_K^{-1} F$ $(F = \omega_J)$
... see holomorphic I-structure as byproduct of structure of coisotropic brane.

Additively, can sheafify over $\mathcal{M}_H$.

<u>Multiplicatively</u>, only over $\mathcal{M}$: to describe as algebra must restrict to opens of the form
$U = \pi^{-1}(V)$ $\quad V$ open in $\mathcal{M}$
..... like Douglas et al emergence of NC Yang-Mills .....

How to calculate fns? first look for $\operatorname{Im}\tau \gg 0$
$\left(\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right)$.

g, f hol fns on $\mathcal{M}_H(I)$, $\hat{g}, \hat{f}$ corresponding ops on strings. Find

$$\hat{g}\hat{f} = \hat{f}\hat{g} + \frac{1}{\operatorname{Im}\tau}\widehat{\{f,g\}} + \mathcal{O}\left(\frac{1}{(\operatorname{Im}\tau)^2}\right)$$

This won't converge for τ ~~to~~ on arbitrary opens of the cotangent, but on conical opens
$\mathbb{C}^*$ degree calculations --- find asymptotic expansion becomes only finitely many terms.

So get a sheaf which must be $\mathcal{D}(\mathcal{L})$ for some line bundle $\mathcal{L}$.

Which $\mathcal{L}$? use time reversal symmetry:
$\mathcal{D}(\mathcal{L}) \longleftrightarrow \mathcal{D}(K \otimes_{\mathcal{M}} \mathcal{L}^{-1})$ Serre dual.
But iff $\theta = 0$ do we have a symmetry $D \longleftrightarrow D^{\text{transpose}}$

So if $\theta = 0 \Rightarrow$ B-B strings are differential operators on $K_M^{1/2}$.

For any θ ($t=1$), $\Psi = \theta/2\pi$, find
$K_M^{1/2} \rightsquigarrow K_M^{1/2} \otimes \mathcal{L}^{\Psi}$ at parameter Ψ
where $\mathcal{L}$ is the determinant line bundle

So $\Psi = 0$ Local system for $^L G \rightsquigarrow$ A-brane F
$\rightsquigarrow \mathrm{Hom}_{A\text{-branes}}(\mathcal{B}, F) \circlearrowleft \mathrm{End}(\mathcal{B})$

$\rightsquigarrow$ D-module (twisted by $K_M^{1/2}$).

Example: F supported on zero-section $M \subset T^*M$
$\rightsquigarrow K_M^{1/2}$ itself as $D(K_M^{1/2})$-module.

Type K A-brane $\Rightarrow$ module for diff ops on $K_M^{1/2} \otimes \mathcal{L}^{\Psi}$ ..
L $\Psi \longrightarrow \dfrac{a\Psi + b}{c\Psi + d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL_2\mathbb{R}$ duality group
.... $\Psi \to \Psi + 1$ symmetry is obvious.

(point like)
Electric eigenbranes : (B, B, B) find branes via Fourier-Mukai
(corresponding) Magnetic eigenbrane (B, A, A) transform: take Hitchin fiber & interpret it in complex structure J.

Canonical coisotropic : type (A, B, A)
... we only used so far that it has K-type A,
But in fact it is of type (A, B, A).

Brane supported on opers (holomorphic in complex structure J) is Lagrangian wrt I, K. Conjecture this (A,B,A) brane is the dual of the canonical coisotropic brane. ...... would give physical picture of Beilinson-Drinfeld.

Suspicion: opers form a nonholomorphic section of Hitchin's fibration.

Why is it important that $\mathcal{M}_H$ is hyperkähler?
-or- Geometric Langlands in 3 easy steps.

1. Start with a local system on C $\rightsquigarrow$ map to a Higgs bundle $(E, \varphi) \in \mathcal{M}_H$ (stable Higgs bundles)
2. Fourier-Mukai transform on $\pi: \mathcal{M}_H \to B$ maps pt $\in \mathcal{M}_H \longrightarrow$ "Higgs bundle over $\mathcal{M}$"
3. Higgs bundle over $\mathcal{M}$ $\rightsquigarrow$ D-module on $\mathcal{M}$.

1. Solve PDE $\mathcal{F} = 0$, find harmonic section of $\mathcal{E} \times_{G_{\mathbb{C}}} G_{\mathbb{C}}/G$ ($\mathcal{E}, \nabla$ flat bundle, $\nabla^2 = \mathcal{F}$)

$\Rightarrow E, \varphi$ Higgs, $E \not\cong \mathcal{E}$, new bundle.

* (analog. of Bezrukavnikov-Braverman three steps)

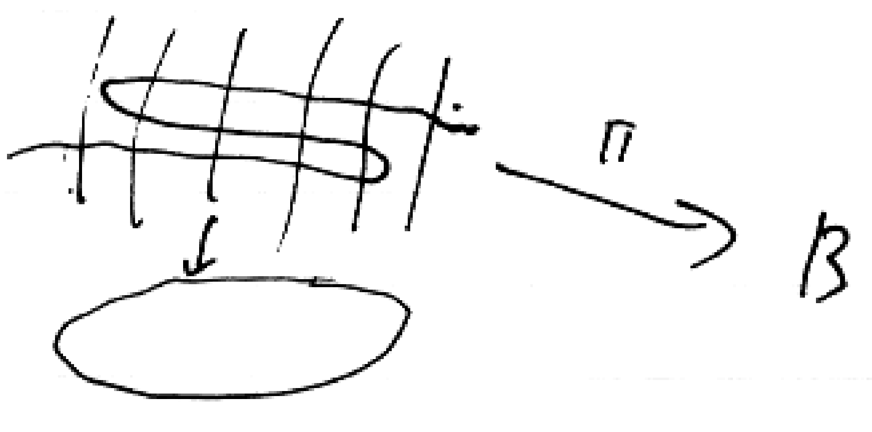
2. Fourier-Mukai: on Hitchin fibration
... supported on a fiber of $\pi: \mathcal{M}_H \to B$ with $\mathcal{L}$ flat unitary bundle over F.

We want to interpret this on $\mathcal{M}$, following Simpson:
.... works only on good part of $\mathcal{M}$
(very stable bundles) !

get $(\mathcal{E}, \varphi)$: $\mathcal{E} \to \mathcal{M}$ holomorphic bundle,
$\varphi : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,0}(M)_K \Longrightarrow \varphi \in H^0(M, T^*M \otimes \mathrm{End}(E))$
& $\varphi \wedge \varphi = 0$.
$K_M^{\frac{1}{2}} \otimes \mathcal{E} = \pi_*(\mathcal{I} \otimes K_F^{\frac{1}{2}})$

We want to interpret this Lagrangian in M_H
as a Higgs bundle on $\mathcal{M}$ --- however
we've destroyed some compactness: we've
thrown out part of $\mathcal{M}_H$ to get to $T^*\mathcal{M}$
--- fine on very stable locus, but
we're producing some singularities in our D-module.

3. Simpson's nonabelian Hodge theory produces a
local system from a Higgs bundle [in compact
settings ...]



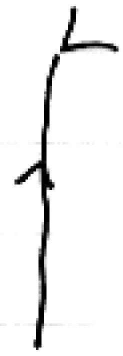
F brane on Hitchin fiber, supposed to be Hecke eigenbrane. How to see this?

Let T = 't Hooft loop of rep. ${}^L R$ of ${}^L G$

Want to check $TF = F \otimes W$.

F is type (B,A,A) ... we care about A-model of type K BUT can do calculations for B-model of type I! F is a brane (B,A,A) supersymmetry & our operators preserve all three properties ..... so can do the "classical Langlands" calculations of Donagi-Pantev etc ... to see Hecke eigenproperty. For general branes things will be more complicated, but the branes we most care about can use this enhanced symmetry

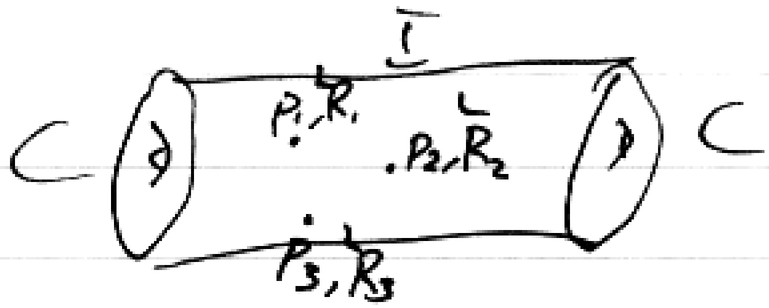
't Hooft & Hecke Operators



Study codimension 3 singularities

... singularity given by $\rho: U(1) \to G$.

Take $M_4 = \mathbb{R} \times \underbrace{I \times C}_{M_3}$, $L = \mathbb{R} \times p$ $p \in M_3$



Solve eqns with given singularities .... $t=1$ time-independent version of $(F - \varphi \wedge \varphi + t D\varphi)^+ = 0$ etc!

Describe by correspondence....

Let's further assume $\varphi = 0$ on C:

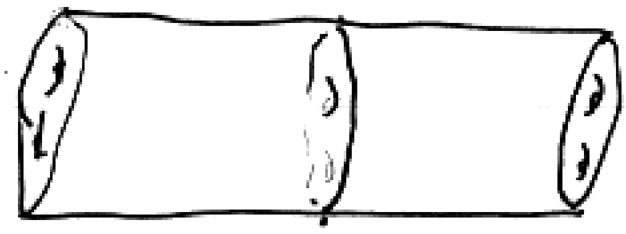
apply 't Hooft operators to bundles rather than Higgs bundles

... equations then specialize to Bogomolny equations

for $E \longrightarrow M_3$ with connection A

and field $\phi_0 \in \Omega^0(M_3, \mathrm{ad}\, E)$:

$$F_A = * D\phi_0 .$$



Y

E_y is holomorphic bundle of C. The holomorphic type of this bundle is <u>constant</u> in y if we don't insert 't Hooft singularities

... variation of holomorphic type of E is given using Bogomolny by $F_{y\bar z} = -i D_{\bar z} \phi_0$: ie ϕ_0 gives a gauge transformation killing variation.

With singularities holomorphic type is canonically trivialized outside singularity point.

Case of $G = U(1)$: $E = \mathcal{L}$ is a line bundle,

't Hooft operator is just twist by $\mathcal{O}(q)^m$,

$\mathcal{L} \to \mathcal{L}(m \cdot q)$. $\quad m = \int_{S^2} c_1(\mathcal{L})$

Nonabelian case: 't Hooft operators lie in torus... so fix torus reduction, $E = \bigoplus_i \mathcal{L}_i$ near singularity, & 't Hooft operator is $\mathcal{L}_i \rightsquigarrow \mathcal{L}_i(m_i q)$

Most powerful view: moment map POV. $W = (A, \phi_0)$

W has a complex structure with holomorphic structure $\bar{D} = \frac{\partial}{\partial \bar{z}} + A_{\bar{z}}$ declared to vary holomorphically.

equations $F = * D\phi_0$

1. holomorphic eqn $\left[\bar{D}, \frac{D}{Dy} + i\phi_0\right] = 0$
2. Moment map for gauge actn
$$\mu(\varepsilon) = \int_{C\times I} \mathrm{Tr}\left(\varepsilon\left(F_{z\bar{z}} - D_y\phi_0\right)\right) + \int_{\partial(C\times I)} \left(\mathrm{Tr}(\varepsilon\phi_0)\right)\omega$$

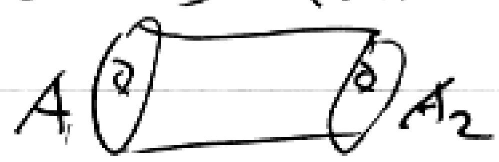
If boundary term $= 0 \Rightarrow$ Bogomolny eqns are

1. holomorphic & 2. $\mu = 0$.

Holomorphic data are Hecke modifications, this describes them as hamiltonian reduction from fields satisfying the holomorphic eqns.

Bun_G = "a stack" $\longleftrightarrow$ $\mathcal{A}$ = all connections (all determine holomorphic bundles) with gauge symmetry.

Fiber of Hecke correspondence: fix actual bundles ($\longleftrightarrow$ actual connections) on two ends (rather than up to isomorphism/gauge).



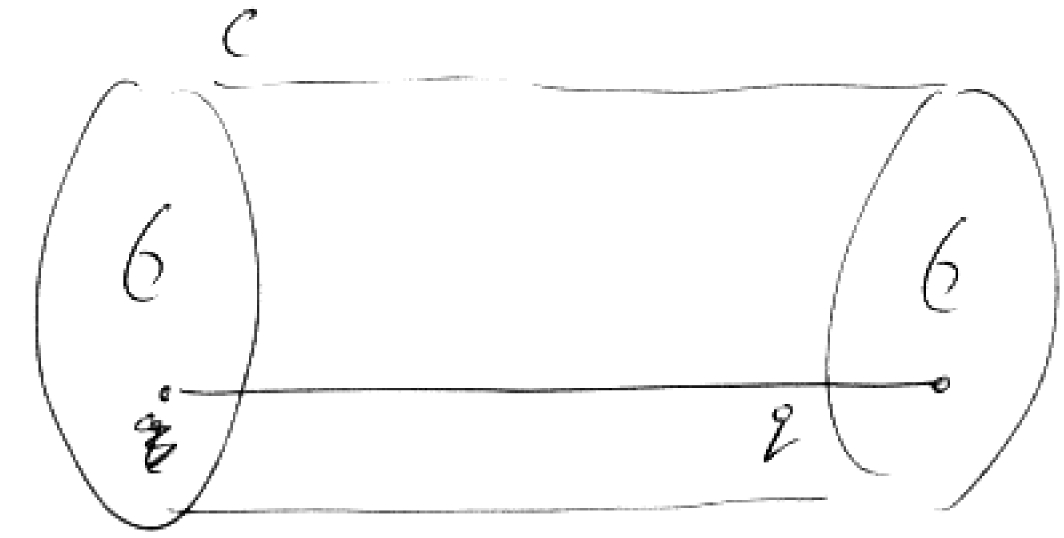
virtual dim $= -\dim \mathcal{M} + \sum_{\text{singularities}} \dim \mathcal{O}_{\lambda}$ corresponding orbit

• • • •

Describe singularities of closures of orbits in Grassmannian: find hyperkahler model by changing POV: cutting up

Ramified case (Gukov)

Bad point



excluding time:

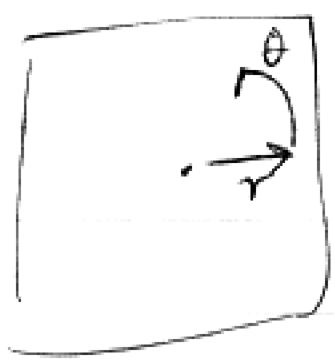
bad line in $I \times C$

(surface in M_4)

$q \in C$ codim $= 2$

What sing in codim $=2$ can we have in gauge theory?

A only (no ϕ)



plane $\perp$ to sing

$A = \alpha \, d\theta$, $\alpha \in \mathfrak{g} = \text{Lie alg}(G)$

$\phi = \gamma \, d\theta + \beta * d\theta$

Higgs field

Want scale-inv sing to be comp. with symmetries

$\Rightarrow [\alpha, \beta] = [\beta, \gamma] = [\alpha, \gamma] = 0 \quad \Rightarrow \alpha, \beta, \gamma \in \mathfrak{t}$

Local system $\mathcal{A} = A + i\phi$: monodromy has eigenvalues $\exp 2\pi i(\alpha + i\gamma)$

β is mysterious, but Simpson explains it in terms of "filtered" local systems.

Can consider this as defining a Higgs bundle on $\mathbb{R}^2$ − origin. From this POV
$\varphi \sim \frac{dz}{z}(\beta + i\gamma)$, α more analogous ... Simpson interprets via parabolic structures.

Simpson constructs hyperkähler moduli space for this ramified situation, Biquard extends to higher order singularities.... whole story carries over!

4-dim picture:  M_4 4-manifold, $\cup$ S surface,

normal plane to S has above kind of singularity.

1. Classical fact $\beta, \gamma \in \mathfrak{t}$, $\alpha \in \mathfrak{t}/\Lambda$ due to gauge transformations shifting α by lattice $\Longrightarrow$ $\alpha \in T$ = torus of G.
   β, γ invariant under Langlands: determine pole in Higgs field & Hitchin fibration is invariant under duality.

2. Quantum theory also needs a θ-angle η: Along S we've reduced structure group to a $\underline{T\text{-bundle}}$ U, which has a Chern class $c_1(U) \in H^2(S, \Lambda)$ $\qquad \Lambda$ lattice

→ can add to action term $I \rightsquigarrow I + i\int_S \langle \eta, c_1(U) \rangle$

Parabolic structure: have not just flag, have weights of pieces in flag, part of definition of stable pairs. Need these weights ~~ ~~ given by α on Higgs part or β on local systems side, needed in Simpson's isomorphism of corresponding moduli spaces..

Θ angle:
$$\eta : \{ Bun_T(S) \longrightarrow U(1) \}$$
$$\| $$
$$Hom(\, Hom(U(1), T), U(1)) \quad \text{i.e. } \eta \in {}^L T \text{ dual torus}$$

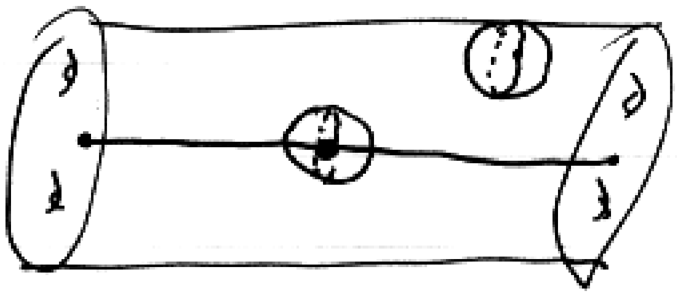
Proposal (Gukov-W.) S-duality takes $(\beta,\gamma) \to (\beta,\gamma)$ & $(\alpha, \eta) \longrightarrow (\eta, -\alpha)$ (more generally full S-duality group acts)
... translation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ performs $(\alpha,\eta) \rightsquigarrow (\alpha+\eta, \eta)$

[abelian cases: electric & magnetic fluxes on S]

Ramified geometric Langlands: B-branes $\longleftrightarrow$ local systems with singularities, controlled by (α,γ).
Dual: A-brane : $\mathcal{D}$-module on $Bun_{parabolic}(C)$ depending on ~~ ~~ (η,γ)
... η breaks time reversal symmetry just like Θ did before.: get $\mathcal{D}$-modules twisted by $K_M^{\frac{1}{2}} \otimes \mathcal{L}^{\eta + i\gamma}$ $\quad \mathcal{L}$ line bundle (picture for SL_2)
--- line bundle from $\mathcal{O}(1)$ on G/B

Role of γ: for $\gamma \neq 0$ Hitchin space is a <u>twisted</u> cotangent bundle, generically breaking symmetry.

Hecke operators in ramified case:
consider bad line with bad point on it



bad points = 't Hooft operators
bad lines = ramification

α, β, γ •  • α, β, γ On $\mathbb{P}^1$ get Higgs bundle with singularities given by α, β, γ at the two bad points.
... Higgs singularity $\longleftrightarrow$ $\beta + i\gamma$, parabolic structure $\longleftrightarrow$ α.

1. Bad case: if local system has unipotent monodromy [tamely ramified case] $\Rightarrow$ G-bundle on $\mathbb{CP}^1$ with parabolic structure at 2 points.
2. Regular semisimple monodromy: get only a T-bundle with no ramification ... reduction from regular semisimple Higgs field.

Can get picture from σ-model to hyperkähler moduli space studied by Kronheimer & depending on α, β, γ.