Representation Theory as Gauge Theory

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Themes

I. Harmonic analysis as the exploitation of symmetry[1]

II. Commutative algebra signals geometry

III. Topology provides commutativity

IV. Gauge theory bridges topology and representation theory

[1] Mackey, Bull. AMS 1980

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Outline

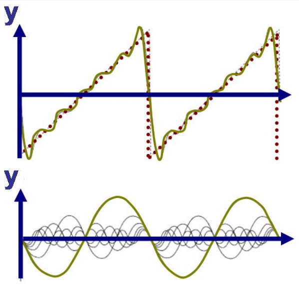
Fourier Series

$G = U(1)$ acts on S^1, hence on $L^2(S^1)$:

Fourier series

$$L^2(S^1) \simeq \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C} e^{2\pi i n\theta}$$

joint eigenspaces of rotation operators



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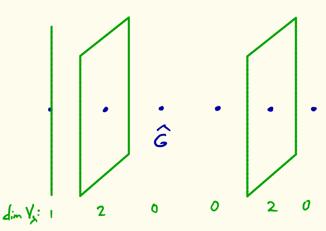
The dual

Basic object in representation theory:
describe the dual of a group G:

$\widehat{G} = \{$ irreducible (unitary) representations of $G\}$

e.g. $\widehat{U(1)} \simeq \mathbb{Z}$

$G \circlearrowright V$ representation $\Longrightarrow$
spectral decomposition of V as family of vector spaces over dual

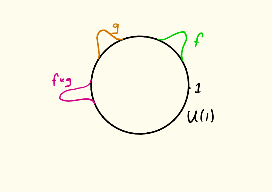


Group algebras

How to access $\widehat{G}$?

Linear combinations of group elements form ring $\mathbb{C}G$ acting on any representation V

e.g., $C(U(1)) \circlearrowright L^2(S^1)$ (convolution)



Pontrjagin duality

G abelian

$\Rightarrow \mathbb{C}G$ is commutative (ring or C^* algebra)

$\Rightarrow \widehat{G}$ underlies space: its spectrum

II. Commutative algebra signals geometry.

Spectral decomposition: $\mathbb{C}G$-modules give
sheaves / vector bundles / projection valued measures
on $\widehat{G}$

Fourier transform

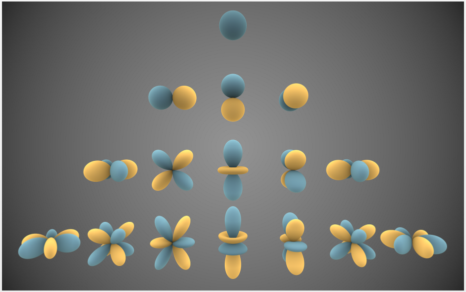
G	$\widehat{G}$
Representation theory	Geometry
$U(1)$ (compact)	$\mathbb{Z}$ (discrete)
$\mathbb{R}$	$\mathbb{R}$
Irreducible reps $\mathbb{C}e^{2\pi ixt}$	Points $t \in \mathbb{Z}$ or $\mathbb{R}$
$L^2(G)$ basis of irreducibles	$L^2(\widehat{G})$ (Fourier transform)
Translation (group) Differentiation (enveloping algebra) Convolution (group algebra)	Multiplication (simultaneous diagonalization)
Representation	Family of vector spaces (spectral decomposition)

Example: Spherical Harmonics

$G = SO(3)$ acts on S^2, hence on $L^2(S^2)$

$$L^2(S^2) \simeq \widehat{\bigoplus_{l \in \mathbb{Z}_+}} V_l$$

irreducible representations of $SO(3)$ (of dim $2l+1$)



Example: spherical harmonics

$SO(3) \circlearrowright L^2(S^2)$ commutes with spherical Laplacian Δ:

Irreducible summands are precisely Δ-eigenspaces

Δ comes from quadratic Casimir $C = \frac{1}{2}(\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2)$ in center of enveloping algebra of $SO(3)$, gives function on unitary dual

$$\widehat{SO(3)} \simeq \mathbb{Z}_+, \qquad C|_{V_l} = l(l+1)$$

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Nonabelian Duality

How to access $\widehat{G}$ for G nonabelian?
seek commutative algebra.

Center of $\mathbb{C}G$: commute with all $g \in G$
$\Rightarrow$ operators in G averaged over conjugacy classes:

class functions $\mathbb{C}G/G$

Schur's lemma:
Center acts by scalar on any irreducible representation:
give functions on $\widehat{G}$

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Dual and center

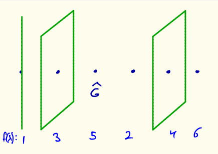
Conversely:

Functions on $\widehat{G}$ give symmetries of arbitrary representations of G

– act as scalars on fibers of spectral decomposition

$\rightsquigarrow$ operators commuting with G (and G-maps):

Bernstein center

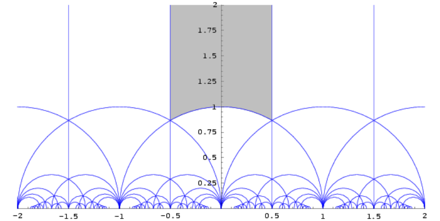


Automorphic Forms

Triumph of exploitation of symmetry (I): the Langlands program.

Problem: spectral decomposition of $L^2(X)$ for X moduli space of elliptic curves

$$SL_2\mathbb{Z}\backslash\{\text{upper half plane}\} \simeq SL_2\mathbb{Z}\backslash SL_2\mathbb{R}/SO_2$$



more generally arithmetic locally symmetric spaces

$$X = \Gamma\backslash G_{\mathbb{R}}/K$$

$G_{\mathbb{R}}$ real reductive group, $K \subset G_{\mathbb{R}}$ compact, $\Gamma \subset G_{\mathbb{R}}$ arithmetic ($\leftrightarrow$ a number field F, and "level structure" at some primes)

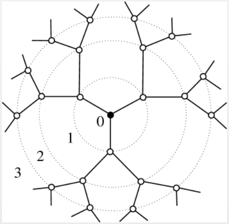
Hecke operators

What acts?

- Laplacian Δ_X (and higher Casimirs): center of $U\mathfrak{g}$

- For every prime p, algebra of Hecke operators:
"higher discrete Laplacians"

e.g. average functions over elliptic curves "differing only at p"
(by p-isogenies)



Holy grail of representation theory

All Hecke operators at almost all p commute

$\rightsquigarrow$

Langlands program:

1. Study spectral decomposition of $L^2(X)$ under Hecke operators
2. Identify joint spectrum with a space of Galois representations
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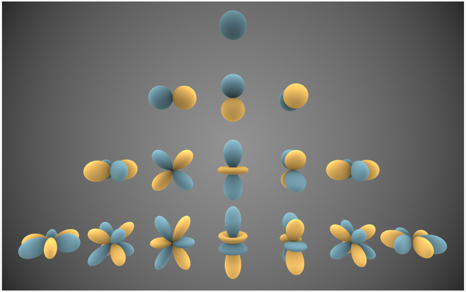
Outline

The Gruppenpest



$L^2(X)$: Hilbert space of a quantum particle on X
$\frac{1}{2}\Delta_X$: Hamiltonian

$G \circlearrowright X \implies G \circlearrowright L^2(X)$:
Symmetries of quantum system.

Exploitation of symmetry (I): find spectrum!
$\rightsquigarrow$ The plague of groups.

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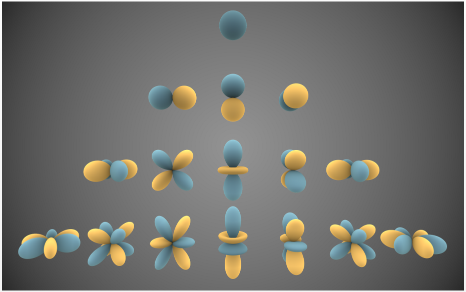

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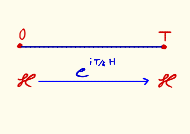
Quantum mechanics: States

Quantum mechanics attaches

- Hilbert space $\mathcal{H}$



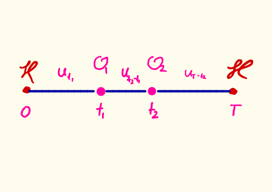
- Time evolution: group generated by Hamiltonian H

$$U_T = e^{iTH/\hbar} : \mathcal{H} \to \mathcal{H}$$

Quantum Mechanics: Operators

- Observables: (self-adjoint) operators on $\mathcal{H}$.

Measurement $\leftrightarrow$ diagonalization



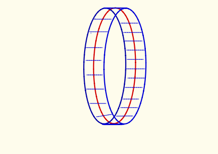
Noncommutative algebra (order in time matters!)

$\rightsquigarrow$ can't diagonalize both position (x) and momentum ($-i\hbar\frac{\partial}{\partial x}$): Heisenberg uncertainty

Sketch of Quantum Field Theory, after Segal

An n-dimensional[2] quantum field theory $\mathcal{Z}$ gives rise to:

- "States": A vector space $\mathcal{Z}(M)$ assigned to [3] $(n-1)$-dimensional Riemannian manifold M



Disjoint unions $\mapsto$ tensor products

[2] (Euclidean)

[3] (n-dimensional collars around)

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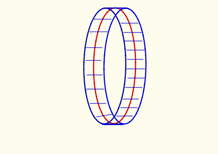


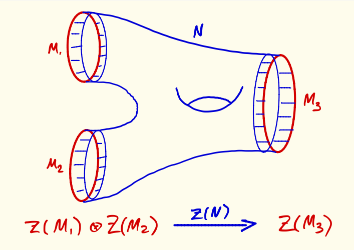
Disjoint unions $\mapsto$ tensor products

[2](Euclidean)
[3](n-dimensional collars around)

Cobordisms

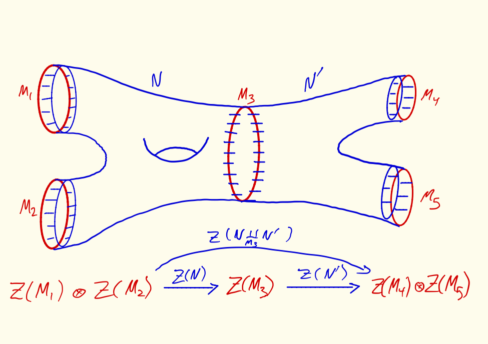
- "Time evolution": An n-dimensional cobordism $N : M_{in} \to M_{out}$ defines a linear map

$$\mathcal{Z}(N) : \mathcal{Z}(M_{in}) \to \mathcal{Z}(M_{out}),$$



Composition

- Time evolution: Maps compose under gluing.



Topological Field Theory

Radical simplification: make $\mathcal{Z}$ depend only on topology of spacetime M

Mechanism: Supersymmetry / Hodge theory

e.g. in quantum mechanics ($n = 1$):
pass from functions $L^2(X)$ to forms $\Omega^\bullet(X)$,

$$\Delta = dd^* + d^*d$$

$\Rightarrow H = 0$ on cohomology:

Time evolution disappears, only space of states left:

Topological Quantum Mechanics $\leftrightarrow$ a vector space

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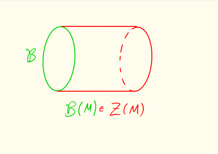
Extended locality

States $\mathcal{Z}(M)$ in TFT attached to compact $(n-1)$-manifolds - complicated!

Avatars of states attached to small patches of space:
local boundary conditions

$(n-1)$-dimensional field theories $\mathcal{B}$
which output states on compact M



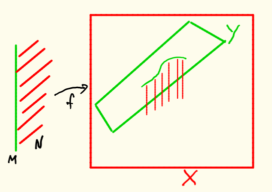
Example: σ-models

σ-model: Quantum theory of maps of space-time to target X

($n = 1$: quantum particle on X)

Local boundary conditions ($n = 2$):
quantum mechanical systems "over X", e.g.:

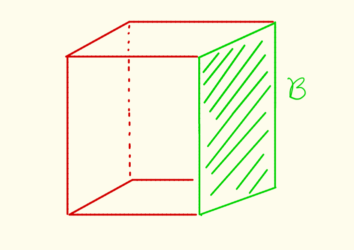
- quantum particle on a space $Y \longrightarrow X$
- family $\mathcal{H}$ of vector spaces over X



Produce cohomology classes on X ($\sim \mathcal{Z}(S^1)$)
(fundamental class / Chern character)

Algebra of boundary conditions

What do boundary conditions form?



Boundary conditions can have interfaces (morphisms), and interfaces between interfaces (2-morphisms), and . . . :

(n-1)-category

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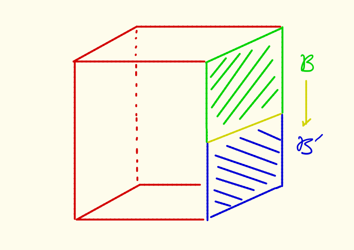

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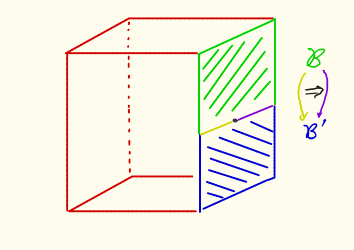

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(n-1)-category

How do you construct a TFT?

Topological QM determined by its states (vector space $\mathcal{H}$)

Topological QFT determined by its local boundary conditions:

Theorem (Lurie)

Baez-Dolan Cobordism Hypothesis: An n-dimensional topological field theory $\mathcal{Z}$ is uniquely determined by its higher category of boundary conditions.

How do you construct a TFT?

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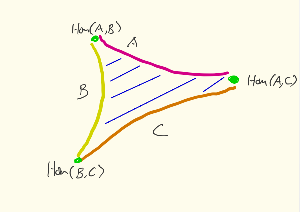
Cobordism Hypothesis

Cobordism Hypothesis:

- input (suitably finite) higher category
- output topological field theory

e.g.,

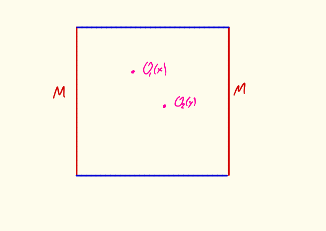
Categories ↔ 2-dimensional TFTs

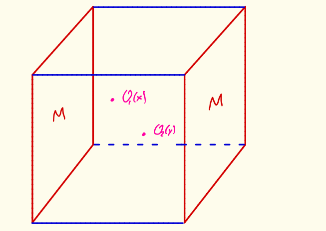


deep geometric perspective on categories

Observables

Key structure: local operators.
Measurements at points of space-time





Operators in TFT $\leftrightarrow$ states on a small linking sphere, $\mathcal{Z}(S^{n-1})$.

Act on states, and on boundary conditions

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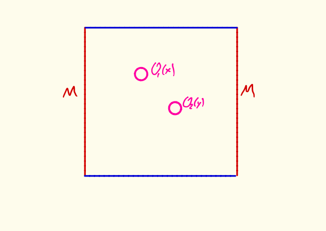

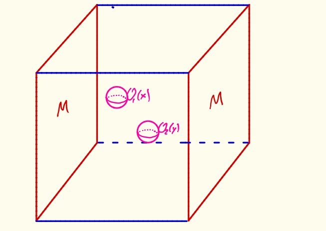

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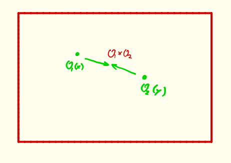
Act on states, and on boundary conditions

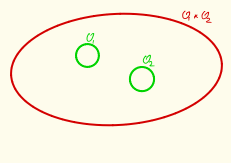
Operator product expansion

Topological pictures of composition of local operators:

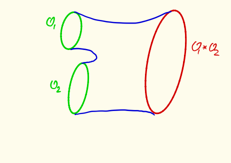
$$\mathcal{Z}(S^{n-1}) \otimes \mathcal{Z}(S^{n-1}) \longrightarrow \mathcal{Z}(S^{n-1})$$



OPE



Little Discs



Pair of Pants

For $n > 1$, can move operators around each other:
local operators in TFT commute!

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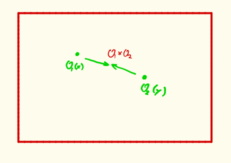

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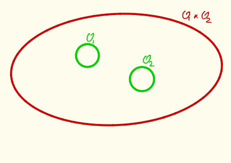

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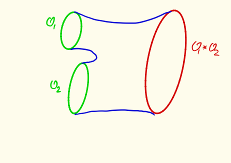

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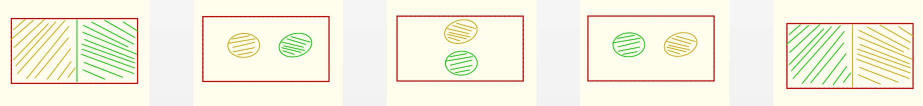
Local operators commute

Commutativity of local operators:
same as commutativity of π_n $(n \geq 2)$.
Picture of based maps to a pointed target $(X, *)$:



more room to move around!

Noncommutativity of quantum mechanics $\rightsquigarrow$ commutativity in TFT

III. Topology provides commutativity

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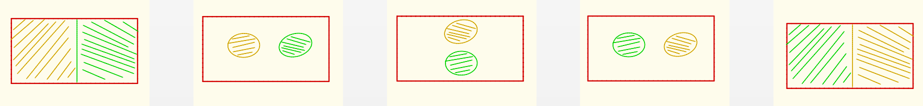

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III. Topology provides commutativity

Fourier Transform for TFT

II. Commutative algebra signals geometry

Local operators in $\mathcal{Z} \Longrightarrow$ moduli space of vacua $\mathfrak{M}_{\mathcal{Z}}$

$\mathcal{Z}$	$\mathfrak{M}_{\mathcal{Z}}$
TFT	Geometry σ-model low-energy effective theory
Local operators	Functions vacuum expectation values
OPE	Multiplication
Boundary conditions (action of local operators)	Families over $\mathfrak{M}_{\mathcal{Z}}$ (sheaves)
2d TFT	$\mathfrak{M}_{\mathcal{Z}}$ discrete

Fourier Transform for TFT

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Outline

Gauge theory

G compact Lie group

Gauge theory: Quantum theory of G-bundles with connection (up to gauge transformations) on space-time

Tremendously influential in low dimensional topology:

invariants of manifolds M from cohomology of moduli spaces of G-bundles on M.

Revenge of the Gruppenpest

Local point of view:

Locally connections are trivial, but carry point-wise G symmetry

$\Longrightarrow$ boundary conditions: $(n-1)$-dim QFTs with G symmetry:

representations of G on field theories.

Cobordism hypothesis: this determines the topological theory.

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Quantum mechanics utilizes representation theory

Quantum field theory can encode representation theory

IV. Gauge theory bridges topology and representation theory

2d Yang-Mills theory

2d Yang-Mills theory:

"Counts" G-bundles with connection on surfaces
(literally for G finite)

Boundary conditions: quantum mechanics with G-symmetry:

e.g. particle on a G-space X: $G \circlearrowright (L^2(X), \Delta_X)$:

Topological Yang-Mills: can define theory via

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"Counts" G-bundles with connection on surfaces
(literally for G finite)

Boundary conditions: quantum mechanics with G-symmetry:

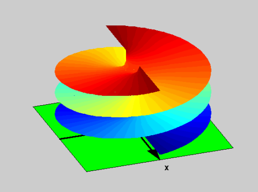
e.g. particle on a G-space X: $G \circlearrowright (L^2(X), \Delta_X)$:

Topological Yang-Mills: can define theory via

boundary conditions $\leftrightarrow$ representations of G

Local operators

Local operators: functions on G-connections on S^1,
i.e. functions of monodromy, conjugacy class $[g] \in G/G$

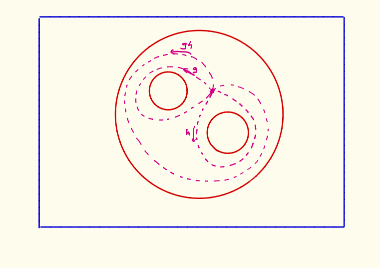


Local operators are class functions

$$\mathcal{Z}(S^1) = \mathbb{C}G/G$$

Local operators are Bernstein center

OPE $\leftrightarrow$ convolution of class functions:



Local operators
$\leftrightarrow$ center of group algebra $\mathbb{C}G$
$\leftrightarrow$ Bernstein center of boundary conditions
$\leftrightarrow$ functions on dual $\widehat{G}$

The dual $\widehat{G}$ is the moduli space of vacua $\mathfrak{M}_{\mathcal{Z}}$

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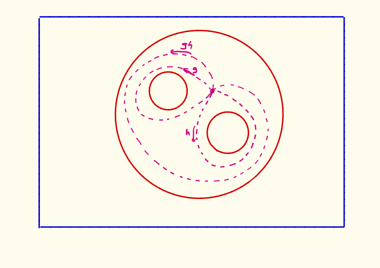

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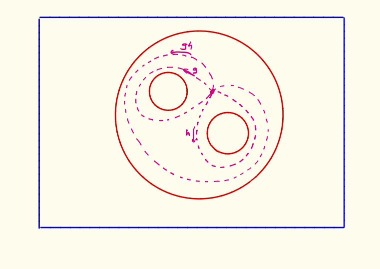

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Propaganda for three dimensions

3d gauge theories are precisely
modern geometric setting for representation theory

• Seiberg-Witten Geometry: Moduli spaces $\mathfrak{M}_{\mathcal{Z}}$ symplectic, carry canonical integrable systems, quantizations
(cf. Braverman's talk Monday)

• Teleman: powerful exploitation of symmetry in symplectic geometry (Gromov-Witten theory)
(talk Friday)

• BZ-Gunningham-Nadler: new commutative symmetries and spectral decomposition in classical representation theory
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Geometric Langlands Program

G complex reductive ($GL_n\mathbb{C}$, $SL_n\mathbb{C}$, $SO_n\mathbb{C}$, $Sp_n\mathbb{C}$,...)

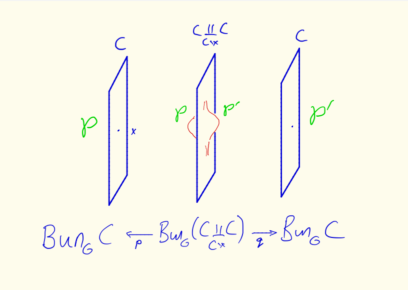
Langlands	Geometric Langlands
Number field	Riemann surface C
Locally symmetric space $X_\Gamma = \Gamma\backslash G_{\mathbb{R}}/K$	Moduli of G-bundles on C $Bun_G(C)$
Hilbert space $L^2(X_\Gamma)$	"Orbifold" Cohomology of $Bun_G(C)$
Hecke operator at prime p	Hecke operator at point $x \in C$
modify elliptic curve by p-isogeny	modify G-bundle at x

Modifying bundles

Hecke operators:
Convolution operators on $Bun_G(C)$:

take weighted average of a cohomology class
over bundles differing only by modification at $x \in C$

($\leftrightarrow \mathcal{P}, \mathcal{P}'$ glue to bundle on C with c doubled)



Hecke modifications as magnetic monopoles

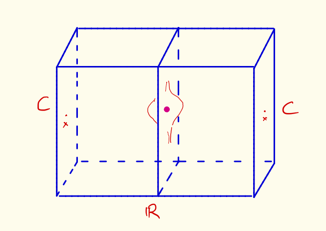
Kapustin-Witten: interpretation via (maximally supersymmetric) gauge theory.

Hecke modification is creation of magnetic monopole in 3d theory on $C \times \mathbb{R}$

- prescribed singularity in gauge fields at point of monopole

– i.e., 't Hooft operator in gauge theory.

Acts as operator on states on C.



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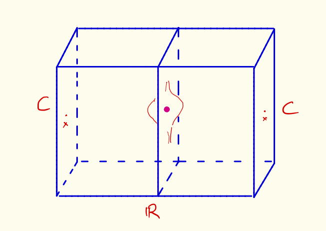
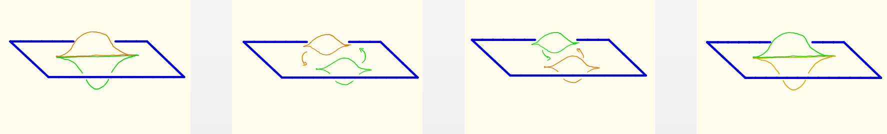


Why do Hecke operators commute?

Inspired by operator products in QFT, Beilinson-Drinfeld discovered source of commutativity:

to compose Hecke modifications, we can stack or slide them around the surface:



Commutativity of Hecke operators is that of local operators and of homotopy groups (III)

Langlands Duality

Commutativity of Hecke operators signals geometry (II)
$\Longrightarrow$ calculate spectrum / dual / moduli space:

Geometric Satake Theorem[4]:
Hecke operators $\leftrightarrow$ representations of
the Langlands dual group $G^\vee$

$(GL_n \leftrightarrow GL_n,\ SL_n \leftrightarrow PGL_n,\ SO_{2n+1} \leftrightarrow Sp_{2n}, \ldots)$

$\Longrightarrow$ spectrum is character variety
$Loc_{G^\vee}(C) = \{\pi_1(C) \to G^\vee\}$

(geometric analog of Galois representations)

[4] Lusztig, Ginzburg, Mirkovic-Vilonen

Duality in physics

Kapustin-Witten: realize as electric-magnetic duality: nonabelian, SUSY version[5] of symmetry of Maxwell's equations under exchange of electric and magnetic fields

G	$G^\vee$
Magnetic Automorphic	Electric Spectral
$Bun_G(C)$	$G^\vee$ Character variety $\{\pi_1(C) \to G^\vee\}$
Creation of monopole ('t Hooft operator) (Hecke operator)	Measurement of field (Wilson operator) (measure holonomy)
Representation theory	Geometry

[5] Montonen-Olive S-duality

Factorization algebras

Beilinson-Drinfeld:

Factorization algebra:
algebraic structure encoded in collisions of points

$\rightsquigarrow$ geometric theory of operators in

1. conformal field theory (vertex algebras),
2. topological field theory[6], and
3. quantum field theory in general[7]

[6] Lurie, Francis-Ayala
[7] Costello-Gwilliam

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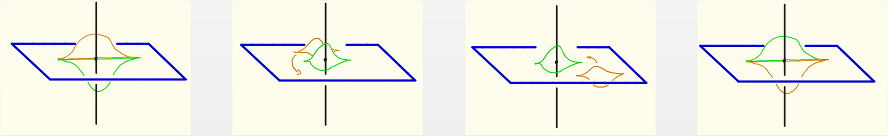
Bernstein center

Ramification / level structure: allow singularities at points of C (physically: solenoids)[8]

Gaitsgory: unramified Hecke operators can slide around, act centrally on "pinned" ramified Hecke operators

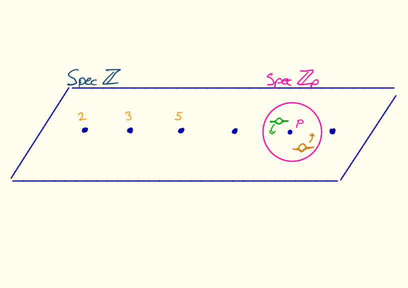


[8] Gukov-Witten

From Physics to Number Theory

Scholze magically ported factorization to number theory:

geometric source of commutativity of classical Hecke operators



⇝ Fargues' conjecture: a radically new geometric understanding of the classical (local) Langlands program. cf. Fargues' talk Friday

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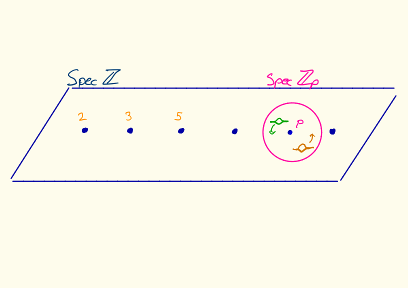

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Where next?

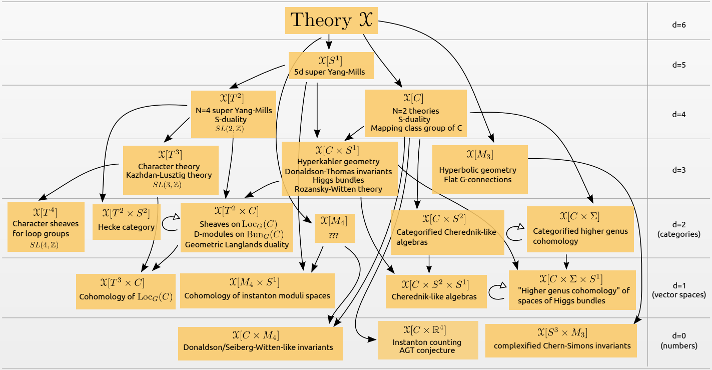
Onward to six dimensions:

Topic of great current interest -
a mysterious six-dimensional QFT, Theory $\mathfrak{X}$

Encodes electric-magnetic duality,
Seiberg-Witten theory, much else in purely geometric terms.

Suggests countless conjectures and inspirations for future work.

The End?


Theory $\mathcal{X}$
$\mathcal{X}[S^1]$
5d super Yang-Mills
$\mathcal{X}[T^2]$
N=4 super Yang-Mills
S-duality
$SL(2,\mathbb{Z})$
$\mathcal{X}[C]$
N=2 theories
S-duality
Mapping class group of C
$\mathcal{X}[T^3]$
Character theory
Kazhdan-Lusztig theory
$SL(3,\mathbb{Z})$
$\mathcal{X}[C \times S^1]$
Hyperkahler geometry
Donaldson-Thomas invariants
Higgs bundles
Rozansky-Witten theory
$\mathcal{X}[M_3]$
Hyperbolic geometry
Flat G-connections
$\mathcal{X}[T^4]$
Character sheaves
for loop groups
$SL(4,\mathbb{Z})$
$\mathcal{X}[T^2 \times S^2]$
Hecke category
$\mathcal{X}[T^2 \times C]$
Sheaves on $\mathrm{Loc}_G(C)$
D-modules on $\mathrm{Bun}_G(C)$
Geometric Langlands duality
$\mathcal{X}[M_4]$
???
$\mathcal{X}[C \times S^2]$
Categorified Cherednik-like
algebras
$\mathcal{X}[C \times \Sigma]$
Categorified higher genus
cohomology
$\mathcal{X}[T^3 \times C]$
Cohomology of $\mathrm{Loc}_G(C)$
$\mathcal{X}[M_4 \times S^1]$
Cohomology of instanton moduli spaces
$\mathcal{X}[C \times S^2 \times S^1]$
Cherednik-like algebras
$\mathcal{X}[C \times \Sigma \times S^1]$
"Higher genus cohomology" of
spaces of Higgs bundles
$\mathcal{X}[C \times M_4]$
Donaldson/Seiberg-Witten-like invariants
$\mathcal{X}[C \times \mathbb{R}^4]$
Instanton counting
AGT conjecture
$\mathcal{X}[S^3 \times M_3]$
complexified Chern-Simons invariants
d=6
d=5
d=4
d=3
d=2
(categories)
d=1
(vector spaces)
d=0
(numbers)

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Schrödinger with Cat:

http://www.shardcore.org/shardpress/2011/08/05/schrodinger-and-the-cat-2011/

Logarithm:

http://ocw.mit.edu/courses/mathematics/18-04-complex-variables-with-applications-fall-1999/study-materials/riemann-surfaces-f-z-log-z2-1/