SYLLABUS:
CATEGORICAL REPRESENTATION THEORY

1. The discrete setting

1.1. **Representations of finite groups: Iordan Ganev.** Morita equivalence for matrix algebras. Convolution product and group algebras. Relative version: groupoid algebras, Morita equivalence to quotient space. Hecke algebras: definition, realization as endomorphisms of induced representations or of functor of invariants. Example: the finite Hecke algebra. Philosophy of describing group representations via Hecke algebras.

Structure of the group algebra: Class functions as center and cocenter/universal trace of the group algebra. idempotents, characters, Wedderburn theorem. Example: Frobenius character formula.

1.2. **Categorical representations of finite groups: Evan Jenkins, Chris Policastro.** Monoidal categories, dualizability (rigidity), pivotal structure. module categories. Drinfeld center. Morita equivalences. all illustrated through Vect(G).

2. Categorical Machinery

2.1. **The Monadicity Theorem: Ben Elias, Alexander Ellis.** Limits and colimits. Adjoint functors: definition, properties (commutation with limits/colimits, adjoint functor theorem). Lots of examples (see wiki page!). Barr-Beck theorem: how to describe a category using an adjunction? Monads/comonads. Examples: categories as modules for an algebra, Morita theory (functors as bimodules), Tannakian duality (tensor categories as representations), descent, sheaves on an affine variety/morphism.

2.2. **Introduction to ∞-categories: Philsang Yoo.** Models as quasicategories, topological or simplicial categories. Source: Dwyer-Kan simplicial localization. limits and colimits, adjunctions, Barr-Beck-Lurie. Some expositions (beyond [Jo] and [L1]) include [Gr, Ber, T]

2.3. **Stable ∞-categories: David Platt.** Relation to dg/A_∞/triangulated categories. Small and big (presentable) categories, Ind-categories. [**?**, G:DG, Ke]

2.4. **Monoidal ∞-categories: Patrick Schultz.** Associative monoids in the oo-setting. Monoidal oo-categories, algebras and modules. The monoidal category of stable presentable oo-categories.

3. Topological field theory

3.1. **TFT basics: Zsuzsanna Dancso, Jaimal Thind.** Definition of a topological field theory, 2d TFT and commutative Frobenius algebras, extended 2d TFT and Frobenius algebras 2d TFT associated to a finite group - in particular center = abelianization = class functions. [F1, FHLT] Fusion categories and extended 3d TFTs.

3.2. **The cobordism hypothesis: Pavel Safronov, Lee Cohn.** Dualizability and dimension. 2-dualizability, Hochschild homology and cohomology, CY algebras. TFT from commutative algebras (baby factorization homology). [**?**, F2]

3.3. **Hochschild theory: Shilin Yu.** Dualizability and dimensions. Intro to Hochschild homology/abelianization and Hochschild cohomology/center. Bar complex. Commutative case: HKR theorem. Relation to loop space (Jones). [Deligne conjecture, cyclic homology] [Lo, BFN]

3.4. **Characters: David Jordan.** Construction of characters in Hochschild homology. Examples: group characters, Chern characters, categorified characters. Functoriality of characters, Riemann-Roch and character formulas. Frobenius setting: small modules determined by their characters.

4. Algebraic G-categories

4.1. **Quasicoherent sheaves: Justin Hilburn.** dg categories of quasicoherent sheaves: perfects=dualizables=compacts. Perfect stacks. [BFN, Section 3.1], [G:DG], [**?**].

Example of Barr-Beck-Lurie: Koszul duality.

4.2. **Algebra with quasicoherent sheaves: Chris Elliott.** Identifying integral transforms, functors and tensor products [BFN, Section 4.2]. Hochschild homology and cohomology for $QC(X)$. Topological field theory from commutative algebras/symmetric categories as functions/sheaves on derived mapping spaces. [BFN]

5. Smooth G-categories

5.1. **Basics of D-modules: Hendrik Orem.** definitions and motivation, functoriality, coherent and holonomic. [Time allowing: formulation of "theories of sheaves" as functors out of varieties with correspondences [G:Ind, **?**]]

Categorical properties of $\mathcal{D}$-modules. integral transforms for $\mathcal{D}$-modules (relation to classical integral transforms) [BN1, Section 3] [**?**, Section 1.4], [GR]

5.2. **Beilinson-Bernstein localization: Ian Le, Chunyi Li.** State the Beilinson-Bernstein localization theorem for g-modules and its refinement for (g,K)-modules. Minimal discussion of twisted D-modules/monodromic D-modules on G/N. [BD, Hecke patterns] [G:O, Sections 6,8], [HTT]

Describe proof of Beilinson-Bernstein theorem in families following [BN2], and "scattering theory" picture via the wonderful compactification.

5.3. **Smooth G-categories: Travis Schedler.** definition. basic examples: reps of Lie algebra, D-modules on G-varieties, related by BB localization. [**?**, Section 20]

Failure of Morita equivalence for smooth G-categories - G-equivariant sheaves on a point. "Hecke patterns": $(\mathfrak{g}, K)$-module categories as Hecke modules. [BD]

5.4. **The Hecke category: Rahbar Virk.** equivariant, monodromic, category O. Relation to Kazhdan-Lusztig theory, role as endofunctors of D(G/B). Koszul duality of [BGS, BG, S]

5.5. **Character sheaves: Laura Rider.** The Harish Chandra system and characters for HC modules. Geometric definition via characteristic variety. Constructive definition via the horocycle correspondence. Characteristic p: relation with characters of finite groups of Lie type. [MV, McG]

5.6. **The character theory: Dario Beraldo, Michael Groechening.** The Hecke category is a fully dualizable 2CY category. Character sheaves form its Hochschild (co)homology (hence appear as categorical characters). Example: HC system/Springer sheaf as G-character of D(G/B).

5.7. **Character theory on surfaces: Sam Gunningham.** Tell us what is known and what is (optimistically) expected for the character theory on surfaces, possible relations to Hausel-Letellier- Rodriguez-Villegas. Will likely require a review of Springer theory.

6. Further directions

6.1. **Geometric theory of characters: BZ.** Atiyah-Bott-Frobenius character formula via derived loop spaces (character of G acting on X is given by map $L(X/G) \to L(pt/G) = G/G$). geometric construction of Harish Chandra characters for admissible representations.

6.2. **Character theory and Langlands duality: BZ.** Character theory as a dimensional reduction of geometric Langlands. The affine character theory, geometric Langlands for elliptic curves and the geometric trace formula. Towards the Soergel conjecture.

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