Lie Groups

Note Title 8/22/2009

Lie groups are the symmetry groups of manifolds

<u>Definition</u> A Lie group is a manifold G & a group in a compatible way:

$1 \in G$, $\underbrace{m : G \times G \longrightarrow G, \quad i : G \to G}_{\text{continuous maps}}$

Satisfying group axioms

The most important example:

$GL_n\mathbb{R}$ = $n \times n$ invertible matrices

$1 = Id$, m = matrix multiplication, i = inverse matrix.

Manifold structure: $GL_n\mathbb{R} \subset M_n$ all $n \times n$ matrices.

$M_n \cong \mathbb{R}^{n^2}$ has obvious manifold structure.

$\det : M_n \longrightarrow \mathbb{R}$ continuous (in fact C^∞) function

$GL_n\mathbb{R} = \det^{-1}(\mathbb{R} \setminus 0)$ is <u>open</u> subset

$\Rightarrow$ natural manifold structure.

Usual formulas for m, i show they are continuous (in fact C^∞) maps,

so $GL_n\mathbb{R}$ is a Lie group, n^2 dimensional

Examples: $GL_1\mathbb{R} = \underset{1\text{-dim}}{\mathbb{R}^*} = \mathbb{R} \setminus 0 \cong \mathbb{R}^+ \amalg \mathbb{R}^-$

$\mathbb{R}^+$ is a subgroup & a closed submanifold

<u>Def</u> A Lie subgroup $H \subset G$ is a closed submanifold which is a subgroup

<u>Def</u> A homomorphism of Lie groups $\varphi : H \to G$ is a continuous map which is a group homomorphism.

$\det : GL_n\mathbb{R} \longrightarrow GL_1\mathbb{R} = \mathbb{R}^*$ is a homomorphism of Lie groups

$GL_n\mathbb{R}^+ = \det^{-1}(\mathbb{R}^+)$ is a Lie subgroup : open & closed, and preserved by multiplication.

<u>Def</u> A <u>matrix group</u> (or linear group) is a Lie subgroup of $GL_n\mathbb{R}$ for some $\mathbb{R}$.

(mostly ignore closedness / submanifoldness for now: talk of subgroups of $GL_n\mathbb{R}$, with the induced topology).

<u>Def</u> $GL_n\mathbb{C} \subset M_n(\mathbb{C})$ invertible $n \times n$ <u>complex</u> matrices

- again $\det : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$, $GL_n\mathbb{C} = \det^{-1}(\mathbb{C}^*)$ open, natural manifold structure.

From $\mathbb{C}$ to $\mathbb{R}$: Write $\mathbb{C} = \mathbb{R}1 \oplus \mathbb{R}i$

$z = x+iy \in \mathbb{C} \Rightarrow$ multiplication by z gives a 2×2 real matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$(x+iy)(u+iv) = (xu - yv) + i(yu + xv)$

So $M_1\mathbb{C} \subset M_2\mathbb{R}$ closed submanifold

$M_n\mathbb{C} \subset M_{2n}\mathbb{R}$ " ", dim $2n^2$

$GL_n\mathbb{C} \subset GL_{2n}\mathbb{R}$ " " dim $2n^2$

eg $GL_1\mathbb{C} = \mathbb{C}^* \simeq \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x^2+y^2 \neq 0 \right\}$
$\subset GL_2\mathbb{R}$

2-dimensional Lie group

Def $O_n = \{ g \in GL_n\mathbb{R} : g^t = g^{-1} \}$

orthogonal group.

Prop $O_n = \{ g \in GL_n\mathbb{R} : \langle u,v\rangle = \langle gu, gv\rangle \; \forall\, u,v \in \mathbb{R}^n \}$

- preserve standard inner product (dot product)

Proof: write u,v as column vectors

$\langle u,v\rangle = (-u-)\begin{pmatrix} | \\ v \\ | \end{pmatrix} = u^t v \in M_1$

$\langle gu, gv\rangle = (gu)^t(gv) = u^t g^t g v$

So if $g \in O_n \Rightarrow g^t g = 1$, ✓

if $g^t g \neq 1$, $g^t g v \neq v$ for some v

$\Rightarrow \langle -, g^t g v \rangle$ different linear functional

(nondegeneracy of $\cdot$ product) ∎

<u>Example</u> $O_1 = \pm 1 \simeq \mathbb{Z}/2$

Note $\det : O_n \longrightarrow \pm 1 \subset \mathbb{R}^* = GL_1 \mathbb{R}$

since $(\det(g))^2 = \det g^t \det g$

$= \det g^t g = 1$.

<u>Example</u> O_2: orthogonal transformations of the plane $= \underbrace{\{\text{rotations}\}}_{\substack{\text{orientation preserving} \\ \text{- subgroup } SO_2 \subset O_2}} \amalg \{\text{reflections}\}$

$\underset{\text{abelian Lie group}}{SO_2 \simeq S^1} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\; \theta \in \mathbb{R}/2\pi i \mathbb{Z} \right\}$

Topologically, $O_2 = SO_2 \amalg \underbrace{\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}}_{\text{or any reflection}} SO_2$

$\simeq S^1 \times \mathbb{Z}/2$. But NOT as groups: reflections & rotations don't commute!

But SO_2 is normal, & $O_2/SO_2 \simeq \mathbb{Z}/2$

<u>Prop</u> For any Lie group, G^0 = path component of Id is a Lie subgroup & normal. Its cosets are the components of G

<u>Corollary</u> G/G^0 is a (discrete) group, the group of connected components, $\pi_0(G)$.

<u>Proof</u> $g, h \in G^0$ take paths

$p_g : [0,1] \longrightarrow G$, $p_g(0) = 1$, $p_g(1) = g$

p_h : .... , — must stay in G^0

Define $p_{gh} : [0,1] \longrightarrow G$

$p_{gh}(t) = p_g(t) \cdot p_h(t)$. : continuous

$p_{gh}(0) = 1$, $p_{gh}(1) = g \cdot h \Rightarrow gh \in G^0$

Define $p_{g^{-1}}(t) = p_g(t)^{-1}$, so $g^{-1} \in G^0$

G^0 subgroup, & closed.

For component $C \subset G$, fix $g \in C$

Any $h \in C$ can be connected to g via
$\gamma : [0,1] \longrightarrow C$ $\gamma(0) = g$ $\gamma(1) = h$

Consider $g^{-1}\gamma : [0,1] \to G$,

$g^{-1}\gamma(0) = 1$, $g^{-1}\gamma(1) = g^{-1}h \in G_0$

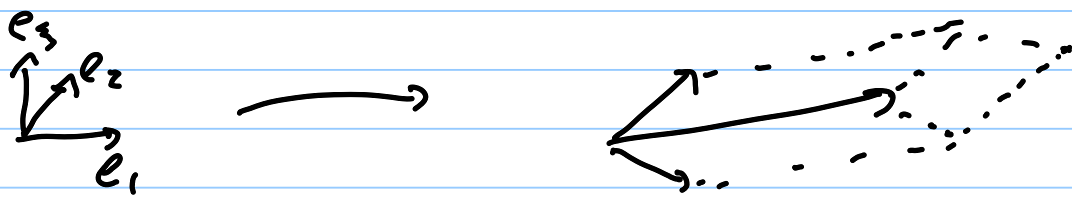
so $h \in gG_0$ left coset, converse clear.. ∎

<u>Def</u> $SL_n\mathbb{R}$, $SL_n\mathbb{C}$ n×n real/complex matrices with determinant 1.
("unimodular")

- clearly subgroups of $GL_n\mathbb{R}$, $GL_n\mathbb{C}$.

eg $SL_1\mathbb{R} \cong SL_1\mathbb{C} \cong \{1\}$. trivial group

Recall: determinant of a matrix is the volume of a parallelpiped (oriented)



So $SL_n\mathbb{R}$ are matrices that preserve <u>volume</u> in $\mathbb{R}^n$.

Most Lie groups are naturally describe as <u>symmetries</u>: transformations preserving some structure / some geometry.

Klein's Erlanger Program:
conversely geometry is <u>defined</u> by the group of symmetries that preserve it...

$SL_n\mathbb{R}$ again: $GL_n\mathbb{R}$ acts on $\mathbb{R}^n$, get action on $\Lambda^k\mathbb{R}^n$: $v_1\wedge v_2\wedge\dots\wedge v_k$
$\mapsto gv_1\wedge gv_2\wedge\dots\wedge gv_k$ $\quad (GL_n\mathbb{R}\to GL_{\binom{n}{k}}\mathbb{R})$

$\Lambda^n \mathbb{R}^n$ 1-dimensional, $c e_1 \wedge e_2 \wedge \dots \wedge e_n$

$SL_n\mathbb{R}$ = matrices which give the <u>identity</u> when acting via Λ^n on $\mathbb{R}^n$...

<u>Example</u> $SO_n = O_n \cap SL_n\mathbb{R}$ unimodular orthogonal matrices

<u>Def</u> $U_n = \{ g \in GL_n\mathbb{C} : \bar{g}^t = g^{-1} \}$

$SU_n = U_n \cap SL_n\mathbb{C}$

<u>Prop</u> $U_n = \{ g \in GL_n\mathbb{C} : \langle u,v\rangle = \langle gu, gv\rangle \}$ for standard Hermitian inner product on $\mathbb{C}^n$

$\langle u,v\rangle = \bar{u}^t v = \sum \bar{u}_i v_i$

- proof same as for O_n.

<u>Example</u> $U_1 \subset GL_1\mathbb{C} = \mathbb{C}^*$:

z st. $\bar{z} = z^{-1}$, $z\bar{z} = \|z\|^2 = 1$

$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ s.t. $x^2+y^2=1$ ...

ie $U_1 \simeq SO_2$

This group is so basic it gets its own letter

$\mathbb{T} = U_1 \simeq SO_2 \simeq S^1$.

<u>Example</u> $\mathbb{R}$ is a Lie group!
How to realize as a matrix group?
many ways!

eg $\exp : \mathbb{R} \xrightarrow{\sim} \mathbb{R}^+ \subset GL_1\mathbb{R}$
$\{(e^x), x \in \mathbb{R}\}$

or $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\} \subset GL_2\mathbb{R}$

$\mathbb{R}$ & $\mathbb{T}$ exhaust all possible 1d Lie groups which are <u>connected</u>:
can always take product with finite group & sometimes interesting twisted products like O_2 .. we'll mostly focus on connected Lie groups

<u>Exercise</u> G, H Lie groups $\Rightarrow$ so is $G \times H$

Two-dimensional connected groups:
can take products $\mathbb{R} \times \mathbb{T}$, $\mathbb{R}^2$, $\mathbb{T}^2$,
all abelian. One nonabelian example:

$$\mathrm{Aff}_1 = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\} \subset GL_2\mathbb{R}$$

Aff_1 is the affine (or Galileo) group in one dimension:
it can be identified with affine symmetries of $\mathbb{A}^1 \simeq \mathbb{R}^1$, ie translations & rescalings

$x \in \mathbb{R} \longmapsto ax + b$: check composition is same as product of matrices in $GL_2\mathbb{R}$

Three dimensional Lie groups

Won't enumerate all, stick to key examples.

- The Heisenberg group:

$N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3\mathbb{R}$: topologically

$N \simeq \mathbb{R}^3$ but not abelian

$T_a M_b \left\{ \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right.$

$M_b T_a \left\{ \begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right.$

Exercise: $U_c = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$ is central

This group appears naturally in Fourier analysis & quantum mechanics..

Consider functions (eg C^∞) on $\mathbb{R}$

& the operators $(T_a f)(x) = f(x-a)$

$(M_b f)(x) = f(x)\, e^{2\pi i x b}$

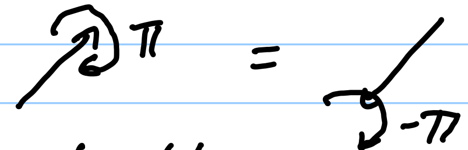
$(U_c f)(x) = f(x) = e^{2\pi i c}$

Exercise T, M, U satisfy the relations of the Heisenberg group

$\underline{O_3}$ Two components: rotations (SO_3) & reflections in a plane ($\det = -1$)

How to describe SO_3? Rotations $\longleftrightarrow$ axis and angle of rotation between $-\pi$ & π

$\rightsquigarrow$ Looks like closed ball of radius π in $\mathbb{R}^3$

– but  so

identify antipodal points on the surface of sphere: glue D^3 & $\mathbb{RP}^2$.

Quaternions $\mathbb{H} \simeq \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$

associative algebra $\boxed{i^2 = j^2 = k^2 = ijk = -1}$

$\Rightarrow$ $ij = k \quad jk = i \quad ki = j$ (cyclic)

– think of as $\underset{1\ i}{\mathbb{C}} \oplus \underset{j\ k}{\mathbb{C}}$, $ij = -ji$ rest determined

$\rightsquigarrow$ convenient to represent as $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

$1 \leftrightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad i \longleftrightarrow \begin{pmatrix} i & \\ & -i \end{pmatrix}$ (right multiplication on above)

$j \longleftrightarrow \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\quad k \longleftrightarrow \begin{pmatrix} & i \\ i & \end{pmatrix}$

Norm $\|q\|^2 = qq^*$ $1^* = 1$ $i^* = -i$ $j^* = -j$ $k^* = -k$

$= \det(q \text{ as } 2\times 2 \text{ matrix})$

$=$ sum of squares of coefficients.

$q^{-1} = \dfrac{q^*}{\|q\|}$, so for unit $q^{-1} = q^*$

Lemma The group of unit quaternions $\cong SU_2$:

Check $\begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix} \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = Id$

Corollary $SU_2 \underset{\text{diffeo}}{\cong} S^3 \subset \mathbb{H}$ unit sphere

From $SU_2 \longrightarrow SO_3$:

$\mathbb{R}^3 \subset \mathbb{H}$ imaginary quaternions

$g \in SU_2$ unit quaternion, $v \in \mathbb{R}^3 \longmapsto gvg^{-1} \in \mathbb{R}^3$

preserves norm.

In fact for $u \in \mathbb{R}^3$ unit vector

Exercise: $g = \cos\theta + u\sin\theta$ is rotation around axis u through angle 2θ.

So $SU_2 \twoheadrightarrow SO_3$ double cover,
kernel is $\mathbb{Z}/2 = \pm Id$

__Corollary__ $SO_3 \cong \mathbb{RP}^3 = S^3/\pm 1$.
- nontrivial double cover!

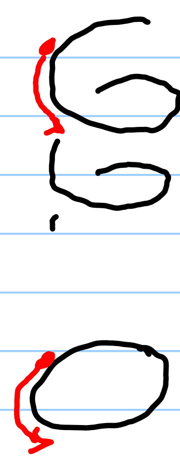
Covering Spaces & Lie Groups

(connected spaces)

Recall $X \xrightarrow{\pi} Y$ is a covering space if locally on opens $U \subset Y$
$\pi^{-1}(U)$ is a disjoint union $\coprod V_i$
with $\pi|_{V_i} : V_i \xrightarrow{\sim} U$ homeomorphism

- eg $\mathbb{R} \longrightarrow S^1$ or $S^n \longrightarrow \mathbb{RP}^n$
or ∂Möbius $\longrightarrow S^1$ or $\mathbb{R}^2 \to \mathbb{R} \times S^1 \to S^1 \times S^1$
etc.

Path lifting property:
given path γ in Y starting at y & a choice of $x \in \pi^{-1}(y)$
$\exists!$ path in X covering γ



Universal cover: cover $\tilde{Y}$ of Y which is simply connected
- unique up to homeo. Has $\pi_1(Y, y_0)$ as __covering group__ - symmetries of $\tilde{Y}$ preserving projection to Y

$\{\text{Covering spaces}\}_{/\sim} \longleftrightarrow$ subgroups of $\pi_1(Y, y_0)$

$\Gamma \subset \pi_1 \longleftrightarrow \tilde{Y}/\Gamma$ orbits of covering group.

<u>Theorem</u> a. H Lie group, $\Gamma \subset Z(H)$ discrete subgroup of the center.
$\Rightarrow \exists!$ Lie group structure on $G = H/\Gamma$ s.t. $H \to G$ is a map of Lie groups

b. G a Lie group, $H \xrightarrow{\varphi} G$ a connected covering space, & $e \in H$ an element with $\varphi(e) = 1$.
$\Rightarrow \exists!$ Lie group structure on H with e as identity s.t. φ is a homomorphism.

Cf: $\Gamma = \mathrm{Ker}(\varphi)$ is discrete & central. For H simply connected, $\Gamma \simeq \pi_1(G, 1)$. In particular π_1 of a Lie group is abelian!

<u>Proof</u> a. Pick U neighborhood of $e \in H$ s.t. U & all its Γ translates are disjoint
$\Rightarrow$ chart near $1 \in G = H/\Gamma$, now move it around to get charts everywhere.

b. By a, suffices to prove for H: universal cover of G: Can represent pts of H as follows:

$$H \simeq \{(g,\gamma) : g\in G,\ \gamma:\overset{1}{[0},\overset{g}{1]}\to G\}/\text{homotopy}$$

(= cord + twisting of belt!)

Define $(g_1,\gamma_1)\cdot(g_2,\gamma_2) = (g_1 g_2, \gamma_{12})$

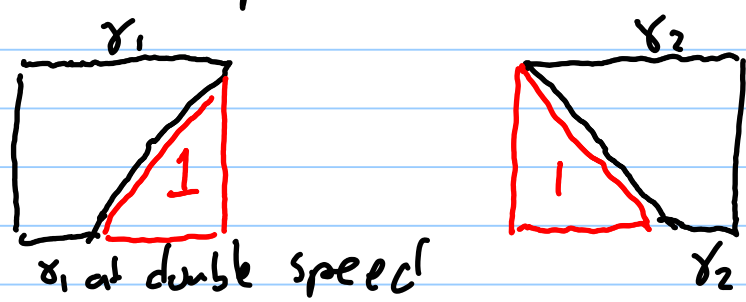
$\gamma_{12}(t) = \gamma_1(t)\cdot\gamma_2(t)$. Well defined up to homotopy. Unique since continuity forces paths in G to multiply as above, & path lifting property forces this rule for H.

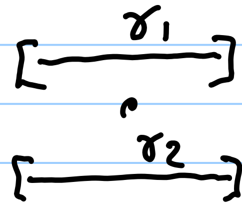
The kernel $\Gamma = \varphi^{-1}(1)$ is clearly discrete.

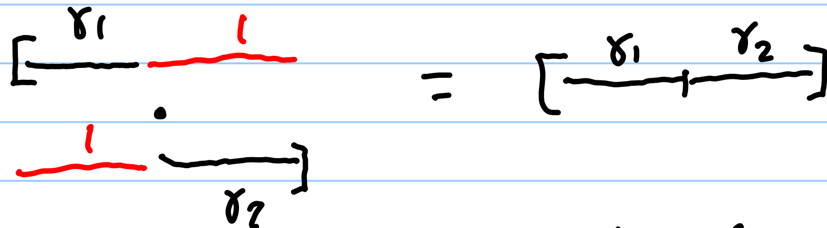
<u>Claim</u> $\Gamma \simeq \pi_1(G,1)$.

Given $(1,\gamma_1), (1,\gamma_2) \in \Gamma$, their composition in H is

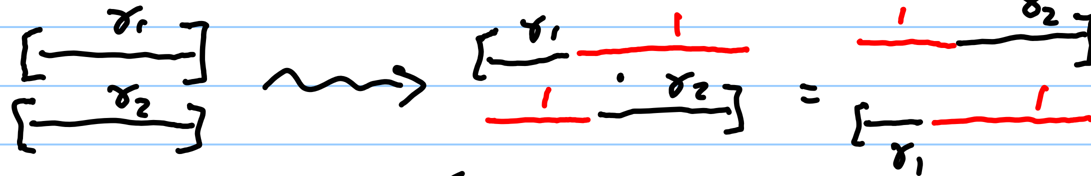
Consider deforming γ_1, γ_2 by having a constant path at the identity for some time:

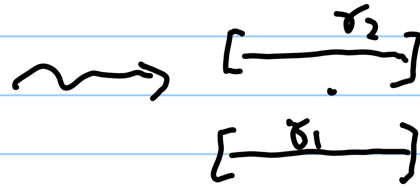


This deforms  to



To see centrality: $(1,\gamma_1)\cdot(g,\gamma_2)$





$\blacksquare$

So π_0 of a Lie group is a group & π_1 is abelian! interesting topological restrictions. Note in particular that discrete normal subgroups (for G connected) are central!

<u>Def</u> Two Lie groups G, H are said to be isogenous, or locally isomorphic, if their universal covers are isomorphic ... equivalence relation generated by <u>isogenies</u> $\varphi: H \to G$ covering map.

Example Another 3d Lie group:

$SL_2\mathbb{R} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^4 : ad-bc = 1 \}$

Exercise $Z(GL_n\mathbb{R}) \cong \mathbb{R}^* \cdot Id$
$Z(GL_n\mathbb{C}) \cong \mathbb{C}^* \cdot Id$ } scalar matrices

$Z(SL_2\mathbb{R}) = \pm Id = Z(SL_2\mathbb{C})$

Def $PSL_2\mathbb{R} = SL_2\mathbb{R} / \pm Id$
$PSL_2\mathbb{C} = SL_2\mathbb{C} / \pm Id.$

Role: symmetries of $\mathbb{CP}^1$ Riemann sphere
$= \mathbb{C} \cup \{\infty\}$:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{C} \rightsquigarrow \quad z \longmapsto \frac{az+b}{cz+d}$
Möbius transformation

$\pm Id$ act trivially $\Rightarrow$ action of $PSL_2\mathbb{C}$ on $\mathbb{P}^1$

[**Def** An <u>action</u> of a Lie group G on a manifold M is an action $G \times M \to M$ of the group G on the set M which is <u>continuous</u>.]

Origin: $GL_2\mathbb{C} \circlearrowright \mathbb{C}^2$, takes lines to lines $\quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

$z = \frac{x}{y} \longmapsto \frac{az+b}{cz+d}$ action on slopes of lines

$\{\text{lines in } \mathbb{C}^2\} \longleftrightarrow \mathbb{CP}^1$

$SL_2\mathbb{R} \circlearrowright \mathbb{C}^2$ preserving $\mathbb{R}^2$

$\Rightarrow$ acts on $\mathbb{CP}^1$ preserving $\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1$

In fact preserves the upper half plane:

$$\operatorname{Im} \frac{ai+b}{ci+d} = \frac{1}{2}\left(\frac{ai+b}{ci+d} + \frac{b-ai}{d-ci}\right)$$

$$= \frac{1}{2}\left(\frac{b^2+a^2}{d^2+c^2}\right) > 0 .$$

__Lemma__ $PSL_2\mathbb{R} \simeq \mathbb{H} \times \mathbb{RP}^1 \simeq D^2 \times S^1$

solid torus

__Proof__ Take $g \in PSL_2\mathbb{R}$ to $(g \cdot i, g \cdot 0)$

- determines g uniquely since $g \cdot -i = \overline{g \cdot i}$ & the following

__Prop__ A Möbius transformation is uniquely determined by the images of any three distinct points, which can be any three distinct points ["simply 3-transitive"]

__Proof__ - suffices to check we can send $0, 1, \infty$ anywhere

- solve $g(0) = \frac{b}{d}$ $g(\infty) = \frac{a}{c}$ $g(1) = \frac{a+b}{c+d}$

uniquely up to overall scalar. $\boxtimes$

<u>Corollary</u> $\pi_1(PSL_2\mathbb{R}) \simeq \mathbb{Z}$
Same for $\pi_1(SL_2\mathbb{R})$.

<u>Counterexample</u> Universal cover $\widetilde{SL_2\mathbb{R}} \simeq D^2 \times \mathbb{R}$ is <u>NOT</u> a matrix group: can't be embedded into any $GL_n\mathbb{R}$ .....

Lie algebras

Definition A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear operation
$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ (ie $[cx,y] = c[x,y] = [x,cy]$) $c \in \mathbb{R}$
satisfying
- Skew symmetry: $[x,y] = -[y,x]$
- Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Example
- any vector space $\mathfrak{g}$ with $[\,,\,] = 0$ ("abelian" Lie algebra)

$\mathfrak{so}_3$
- $\mathbb{R}^3$ with $[a,b] = a \times b$ cross product.
$a \times b = -b \times a$. Not associative, but
$(a \times b) \times c = (a \cdot c) b - (b \cdot c) a$
Check equivalent to Jacobi identity!
$i \times j = k$, $j \times k = i$ $k \times i = j$

- $M_n = n \times n$ matrices, $[A,B] = AB - BA$
– clearly skew
Exercise: check Jacobi identity

[• More generally, ~~A~~ any associative algebra $[a,b] = ab - ba$ is a Lie bracket
Check!]

Where does Jacobi identity come from?
let's rewrite it:
$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

or adopt notation $[x,-] = \partial_x$,

$\Rightarrow \quad \partial_x [y,z] = [\partial_x y, z] + [y, \partial_x z]$

– looks like Leibniz rule $(fg)' = f'g + fg'$ ...

Vect($\mathbb{R}^n$)

Consider vector fields on $\mathbb{R}^n$, $\{\xi = \sum \xi_i \frac{\partial}{\partial x_i}\}$

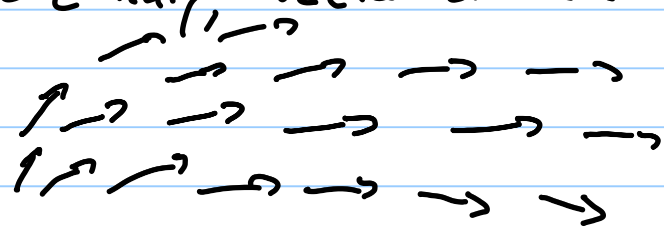
$(\xi_i \in C^\infty(\mathbb{R}^n))$ – all possible first order derivatives.

– can consider as special operators on C^∞ functions

$f \mapsto \xi \cdot f = \sum \xi_i \frac{\partial f}{\partial x_i}$, satisfying Leibniz rule

$\xi \cdot (fg) = (\xi \cdot f) g + f(\xi \cdot g)$: <u>derivations</u>

– geometrically, vector at each point of $\mathbb{R}^n$



Can't multiply vector fields: $\left(\frac{\partial}{\partial x_i}\right)^2$ not a vector field!

but key rule : <u>mixed partials commute</u>

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}$$

<u>Def</u> The commutator of two vector fields ξ, η is $[\xi, \eta] = \xi\eta - \eta\xi$ – commutator as linear operators on functions

Prop The commutator of two vector fields is a vector field, hence $\mathrm{Vect}(\mathbb{R}^n), [,]$ is a Lie algebra.

e.g. $[\partial_{x_1}, \partial_{x_2}] = 0$ i.e. $\partial_{x_1}\partial_{x_2} = \partial_{x_2}\partial_{x_1}$

but $[\partial x_1, x_1\partial x_2] = \partial_{x_1}(x_1\partial x_2) - x_1\partial_{x_2}\partial_{x_1}$

$= x_1\partial_{x_1}\partial_{x_2} + \partial_{x_2} - x_1\partial_{x_1}\partial_{x_2} = \partial_{x_2}$

- quadratic terms always cancel!

How to get <u>finite dimensional</u> Lie algebras from this? e.g. CONSTANT vector fields $\{\sum c_i \partial_{x_i}\} \simeq (\mathbb{R}^n, [,]=0)$ abelian Lie algebra

Or look at $\mathrm{Span}\{\underbrace{\frac{\partial}{\partial x}}_{e}, \underbrace{-2x\frac{\partial}{\partial x}}_{h}, \underbrace{-x^2\frac{\partial}{\partial x}}_{f}\} \subset \mathrm{Vect}\,\mathbb{R}$

(normalization will be motivated later!)

$\mathfrak{sl}_2\mathbb{R}$ $\left[\begin{array}{l} [e,f] = h \\ [h,e] = 2e \\ [h,f] = -2f \end{array}\right\}$ defines $\mathbb{R}^3, [,]$ Lie bracket, $\neq \mathbb{R}^3, \times$.

- eg $[h,-]$ has eigenvalues $-2, 0, 2$

while $i\times\_$ in $\mathbb{R}^3$ has no eigenvectors $/\mathbb{R}$!

(over $\mathbb{C}$ in fact they <u>are</u> isomorphic.....)

One more example

$\mathfrak{gl}_n\mathbb{R} \hookrightarrow \mathrm{Vect}\,\mathbb{R}^n$:

$$\overset{\cup}{M} \longmapsto \{\, x \longmapsto M\cdot x \in \mathbb{R}^n = \text{vectors of } x \,\}$$

get vector fields satisfying $\xi|_{cv} = c\,\xi|_v$,

$\xi|_{v+w} = \xi|_v + \xi|_w$.

check Lie algebra homomorphism,
ie brackets go to brackets

Tangent spaces & vector fields on manifolds

$x \in X \subset \mathbb{R}^N$ submanifold $\Rightarrow$ tangent space

$T_x X$ = Image of linear map $\mathbb{R}^n \to \mathbb{R}^N$ given by derivative of a local parametrization $0 \in U \xrightarrow{\sim} V \ni x$, $U \subset \mathbb{R}^n$, $V \subset X \subset \mathbb{R}^N$

$= \{ \gamma'(0) : \gamma : (-\varepsilon, \varepsilon) \to X, \ \gamma(0) = x \}$, $(-\varepsilon,\varepsilon) \subset \mathbb{R}$

More abstractly: For any diffeomorphism $\varphi : U_1 \xrightarrow{\sim} V_2$, $U_1 \subset \mathbb{R}^n$, $V_2 \subset \mathbb{R}^n$ & $x_1 \in U_1$ we get an isomorphism $D\varphi_{x_1} : \mathbb{R}^n \to \mathbb{R}^n$ of vector spaces.

Given $x \in X$ define $T_x X$ to be $T_{\varphi(x)} \mathbb{R}^n$ for any local coordinate φ near x

- given any other coordinate φ' identify $T_{\varphi(x)} \mathbb{R}^n \xrightarrow[D(\varphi' \varphi^{-1})_{\varphi(x)}]{\sim} T_{\varphi'(x)} \mathbb{R}^n$

- any notion on $\mathbb{R}^n$ that's invariant under diffeomorphisms makes sense on any n-dim manifold.

Tangent bundle = pairs (x,v) $x \in X$ $v \in T_x X$

– locally looks like $\mathbb{R}^n_x \times \mathbb{R}^n_v$, glue via $(x,v) \mapsto (\varphi(x), D\varphi_x(v))$.

Vector field: smooth section
$$\begin{tikzcd} TX \arrow[d] \\ X \arrow[u, bend right, "\xi"'] \end{tikzcd}$$

G a Lie group

Def $\xi \in \mathrm{Vect}\, G$ is a <u>left invariant vector field</u> if $\forall g \in G$, $l_g : G \longrightarrow G$, $h \longmapsto g \cdot h$

we have $Dg_* \xi = \xi$, i.e.

$\forall h \in G \qquad \xi(g \cdot h) = Dg|_h(\xi(g))$.

Likewise for right invariant: $r_g : G \longrightarrow G$, $h \longmapsto h \cdot g$

Prop The vector space of left invariant (resp. right invariant) vector fields is isomorphic to $\mathfrak{g} = T_1 G$ via $\xi \longmapsto \xi(1)$.

Proof given $\xi_1 \in \mathfrak{g}$ let $\xi \in \mathrm{Vect}\, G$ be

$\xi(g) = Dl_g|_1(\xi_1) \in T_g G$.

– well defined smooth vector field

$l_{gh} = l_g l_h \implies \xi$ is left invariant! $\blacksquare$

Corollary The tangent bundle of a Lie group is trivial: $\exists$ canonical isomorphism

$TG \simeq G \times \mathfrak{g}$ (in fact two, l & r)

So a basis of $\mathfrak{g}$ gives an isomorphism

$TG \simeq G \times \mathbb{R}^n$.

e.g., Surface of genus $\neq 1$ cannot be made a Lie group: can't comb a peach

Hopf index theorem:

Euler characteristic of X can be calculated from the index of a vector field - ie from its zeros, counted carefully....

In fact S^n $(n \neq 1,3)$ can't be made a Lie group!

eg $GL_n\mathbb{R} \subset M_n$ open $\Rightarrow$ $T_g GL_n\mathbb{R} = M_n$ $\forall g$

- explicit trivialization of tangent bundle!

Recall a one-parameter subgroup is a homomorphism $\rho: \mathbb{R} \longrightarrow G$.

$$\{\text{one-parameter subgroups}\} \longrightarrow \mathfrak{g} = T_1 G$$

$$\rho \longmapsto \rho'(0)$$

Theorem This map is a bijection!

.... recall $T_x M =$ arcs $\begin{array}{c}(-\varepsilon,\varepsilon) \to M \\ 0 \mapsto x\end{array}$

up to equivalence, identify two which are tangent at x.

In a Lie group have a canonical way to find an arc in every direction through 1.

We'll discuss the Theorem first for matrix groups & then in general.

Fundamental problem: integrating vector fields/ solving first order ODE:

<u>Def</u> ξ vector field on M. An <u>integral curve</u> γ of ξ through $x \in M$ is a smooth map $\gamma : \begin{array}{c}(-\varepsilon,\varepsilon) \to M \\ 0 \mapsto x\end{array}$, $\gamma'(t) = \xi(\gamma(t))$

For vector fields ξ_A on $\mathbb{R}^n$ given by matrices $A \in \mathfrak{gl}_n \mathbb{R}$, this gives

$$\gamma'(t) = A \cdot \gamma(t), \quad \gamma(0) = x.$$

Can solve by a power series:

$$\gamma(t) = \sum t^k \gamma_i \qquad \gamma_0 = x$$

Ignoring convergence find $\gamma_{k+1} = \frac{1}{k+1} A \gamma_k$

$$\Rightarrow \gamma_k = \frac{1}{k!} A^k \gamma_0,$$

$$\gamma(t) = \left(\sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k\right)\gamma_0 = (\exp tA)\cdot x$$

$$\exp A = \sum_0^\infty \frac{1}{k!} A^k \quad \text{matrix exponential.}$$

<u>Props</u> (Rossmann's book) $\exp tA$ converges in norm for all t

(hence term-by-term differentiation allowed)

1. $\frac{d}{dt}\exp tA = A\exp tA = (\exp tA)\cdot A$

   & $a(t) = \exp tA$ is unique solution of
   $a'(t) = A\,a(t)$, $a(0) = 1$.

2. $\exp X \exp Y = \exp(X+Y)$ $\boxed{\text{for } X, Y \text{ commuting}}$

3. $(\exp A)^{-1} = \exp(-A)$

4. $\exp(sA + tA) = \exp(sX)\exp(tX)$

   & $\exp tA$ is unique solution of
   $a(s+t) = a(s)\,a(t)$, $a(0) = 1$, $a'(0) = A$.

<u>Corollary.</u> One parameter subgroups in $GL_n\mathbb{R}$ are precisely $\exp(tA)$, $A \in \mathfrak{gl}_n\mathbb{R}$.

<u>Proofs</u> 1. $\frac{d}{dt}\exp tA = \sum_1^\infty \frac{t^{k-1}}{(k-1)!}A^k = A\exp tA$
$= \exp tA\; A$

Check $\frac{d}{dt}\left(\exp(-tA)\,a(t)\right) = 0$ & $= 1$ at $t = 0$

$\rightsquigarrow \quad \alpha(t) = \exp(-tA)^{-1}$ unique solution

$\Rightarrow \exp(-tA) = \exp(tA)^{-1}$.

2. $\exp X \exp Y = \sum_{j,k=0}^{\infty} \frac{X^j Y^k}{j!\,k!}$

$$= \sum \frac{1}{m!} \Big(\sum_{j+k=m} \frac{m!}{j!\,k!} X^j Y^k \Big)$$

$$\underset{X,Y \text{ commute}}{=} \sum \frac{1}{m!} (X+Y)^m = \exp(X+Y)$$

3, 4 Special cases of 2 $\blacksquare$

Relation to left invariant vector fields

$g \in GL_n\mathbb{R} \qquad l_g : GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$

$$\begin{array}{ccc} Dl_g : T_1 GL_n\mathbb{R} & \longrightarrow & T_g GL_n\mathbb{R} \\ \| & & \| \\ M_n & \xrightarrow{g\cdot} & M_n \end{array}$$

$$\frac{d}{dt}\big(g\cdot(\mathrm{Id} + tA)\big)\Big|_{t=0} = g\cdot A.$$

So left invariant vector fields are of form

$\xi_A : g \longmapsto g\cdot A$. right invariants: $g \longmapsto A\cdot g$.

One parameter subgroups are integral curves of <u>either</u> through $1 \in G$!

<u>General case</u> 1 param subgroups $\longleftrightarrow T_1 G$:

Follows from general $\exists$ & ! theorem for ODEs:

Theorem Suppose ξ smooth vector field on $0 \in V \subset \mathbb{R}^n \Rightarrow \exists\, \varepsilon > 0$ & $\exists!\ f : (-\varepsilon, \varepsilon) \to U$ smooth s.t.

- $f(0) = 0$
- $f'(t) = \xi(f(t)) \ \forall t \in (-\varepsilon, \varepsilon)$

... more generally, f depends smoothly on parameters: if $\xi(v)$ is a smooth family of vector field on U depending on $v \in V \subset \mathbb{R}^m$

$\Rightarrow \forall v_0 \ \exists\ v_0 \in V_0 \subset V$ & $\varepsilon > 0$ s.t.

$\exists!\ f : V_0 \times (-\varepsilon, \varepsilon) \to V_0 \times \mathbb{R}^n$

$f(0, v) = 0$ & $\left.\frac{df}{dt}\right|_{t_0} = \xi(f(t_0, v)) \ \forall v, t.$

<u>Uniqueness</u> $\xi_1 \in \mathfrak{g}$, suppose $\xi_1 = \varphi'(0)$
$\varphi : \mathbb{R} \to G$ one parameter subgroup.

Extend ξ_1 to a left invariant vector field ξ on G

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\varphi} & G \\ {\scriptstyle l_t}\downarrow & & \downarrow{\scriptstyle l_{\varphi(t)}} \\ \mathbb{R} & \xrightarrow{\varphi} & G \end{array}$$

commutes $\Rightarrow$

$\varphi'(t) = \xi(\varphi(t))$

$\rightsquigarrow \varphi$ is an integral curve for ξ through I

$\rightsquigarrow$ unique in some neighborhood of every point of $\varphi(\mathbb{R}) \Rightarrow$ unique.

<u>Existence</u> Know we have an integral curve for ξ locally, $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$.

Note that $\varphi(s+t)$ & $\varphi(s)\varphi(t)$ <u>both</u> solutions for s fixed & wherever defined $\Rightarrow$ must be equal.

To extend φ to all of $\mathbb{R}$:

For fixed $t \in \mathbb{R}$, pick N s.t. $\frac{t}{N} < \frac{\varepsilon}{2}$

& set $\varphi(t) = \varphi\left(\left(\frac{t}{N}\right)\right)^N$

– well defined since for another N',

$\varphi\left(\frac{t}{NN'}\right)^N = \varphi\left(\frac{t}{N'}\right) \Rightarrow \varphi\left(\frac{t}{N}\right)^N = \varphi\left(\frac{t}{NN'}\right)^{NN'}$

$= \varphi\left(\frac{t}{N'}\right)^{N'}$

etc $\Rightarrow \varphi$ is a one parameter subgroup! $\blacksquare$

$\Rightarrow$ <u>Def</u> The <u>exponential map</u>

$\exp : \mathfrak{g} \longrightarrow G$ sends ξ to $\varphi(1)$

or $t\xi$ to $\varphi(t)$ where $\xi \leftrightarrow \varphi$ by above theorem.

Prop exp is smooth

Proof Fix $v_0 \in V \subset \mathfrak{g} \Rightarrow$ solution of integral curve problem depends smoothly on v_0 as maps $(-\varepsilon, \varepsilon) \to G$ & so does extension $\varphi(\frac{t}{N})^N$... ∎

Note $\exp' : T_1 G \to T_1 G$ is the identity...
$\Rightarrow$ exp is a diffeomorphism from some neighborhood of $0 \in \mathfrak{g}$ to some neighborhood of $1 \in G$. Call inverse <u>log</u>.

Prop exp is functorial / natural: given $\psi : G \to H$ homomorphism, we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\psi'"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

Proof: follows from observation that $\mathbb{R} \xrightarrow[\varphi]{} G \xrightarrow[\psi]{} H$ is a 1-parameter subgroup of H ∎

Prop. Any neighborhood $1 \in U \subset G^0$ generates G^0 as a group.

Proof: $\langle U \rangle \subset G^0$ is <u>open</u> (any $x \in \langle U \rangle$ sits in $Ux \subset \langle U \rangle$)

$\Rightarrow$ any coset $\langle U \rangle x \subset G^\circ$ is open $\Rightarrow$ $G^\circ \setminus \langle U \rangle$ open

$\Rightarrow$ $\langle U \rangle$ also closed ! ∎

<u>Theorem</u> G connected $\Rightarrow$ any homomorphism of Lie groups $\psi : G \longrightarrow H$ is determined by $\psi' : \mathfrak{g} \longrightarrow \mathfrak{h}$ linear map.

<u>Proof</u> :

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\psi'"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

$\Rightarrow$ ψ determined by ψ' in some neighborhood of 1

$\Rightarrow$ on all of $G^\circ = G$. ∎

<u>Key question</u> How to rewrite multiplication on G (near identity at least) in terms of $\mathfrak{g}$? ie describe map

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "\exp \times \exp"] \arrow[d, "*"'] & G \times G \arrow[d, "m"] \\ \mathfrak{g} & G \arrow[l, "\log"'] \end{tikzcd}$$

(only makes sense on some small open $0 \in V \subset \mathfrak{g}$)

Zeroth order : $x * y = x + y + o(r)$

- this uses only that exp is a local coordinate on G & μ is smooth : (r = distance to 0 of (x,y) in any metric)

$x * y$ must have form $a + bx + cy + o(r)$

plus in $x=0, y=0$ in turn to get result.

We'll study $*$ much more closely soon...
but first :

<u>Theorem</u> A connected, abelian Lie group has the form $\mathbb{T}^a \times \mathbb{R}^b$.

<u>Proof</u> First claim $\mathfrak{g} \xrightarrow{\exp} G$ is a <u>homomorphism</u> ($\mathfrak{g} \cong \mathbb{R}^n$ vector space)

$$\exp s \exp t = \left(\exp \tfrac{s}{N}\right)^N \left(\exp \tfrac{t}{N}\right)^N$$

$$= \left(\exp \tfrac{s}{N} \exp \tfrac{t}{N}\right)^N$$

$$= \left(\exp \left(\tfrac{s}{N} + \tfrac{t}{N} + o\left(\tfrac{1}{N}\right)\right)\right)^N$$

$$= \underbrace{\exp(s + t + o(1))}_{\text{as function of } N} = \exp(s+t) \checkmark$$

$\dots >$ in particular exp is <u>onto</u>

So $G \simeq \mathbb{R}^n / \mathrm{Ker}(\exp)$

discrete subgroup, must be $\simeq \mathbb{Z}^a$

$\Rightarrow G \simeq \mathbb{T}^a \times \mathbb{R}^{n-a}$. ∎

<u>First order</u>: Write $X * Y = X + Y + \frac{1}{2}[X,Y] + O(r^2)$

<u>Theorem</u> $X,Y \mapsto [X,Y]$ makes $\mathfrak{g} : T_1 G$
into a Lie algebra, called $\mathrm{Lie}(G)$.

We'll return to this soon...

Theorem Any closed subgroup H of a Lie group G is a Lie subgroup

Proof in 4 steps:

1. If $H \subset G$ is both a subgroup & a submanifold $\Rightarrow$ H is a Lie group

Pf: $H \times H \subset G \times G$ submanifold, smooth maps $G \to G$ & $G \times G \to G$ respecting H must be smooth on H. $\square$

2. Pick norm on $\mathfrak{g}$. Suppose $\{h_n\} \in \mathfrak{g} \setminus 0$ sequence with $h_n \to 0$, $\frac{h_n}{|h_n|} \to v \in \mathfrak{g}$ & $\exp h_n \in H$.

Then $\exp tv \in H \quad \forall t \in \mathbb{R}$

Proof $\frac{t}{|h_n|} h_n \to tv$ & $|h_n| \to 0$

$\rightsquigarrow$ can find integers m_n s.t. $m_n |h_n| \to t$

so $\exp m_n h_n \to \exp tv$

$\quad (\exp h_n)^{m_n} \in H$ (with $\exp m_n h_n = (\exp h_n)^{m_n}$) & H closed

$\Rightarrow \exp tv \in H$ $\blacksquare$

3. Let W = all vectors tv for v as above.
Then W is a vector subspace of H.

<u>Pf</u> Given $w_1, w_2 \in W$, $w_1 + w_2 \neq 0$, show $w_1 + w_2 \in W$:

$\exp(tw_1)\exp(tw_2) \in H$. For t small write

$$\exp(tw_1)\exp(tw_2) = \exp(f(t))$$

$f(t)$ smooth curve in $\mathfrak{g}$, $f(0) = 0$.

$\frac{1}{t} f(t) \longrightarrow w_1 + w_2$ as $t \to 0$ since
$\exp tw_1 \exp tw_2 = \exp t(w_1 + w_2) + o(1)$

$\Rightarrow v = \frac{w_1 + w_2}{|w_1 + w_2|} \in W$ $\left(h_n = f\left(\frac{1}{n}\right)\right)$. ∎

4. $\exp W$ is a neighborhood of 1 in H.

<u>Pf</u> Write $\mathfrak{g} = W \oplus W^\perp$

Let $\varphi : \mathfrak{g} \to G$ be $\varphi(w, w^\perp) = \exp w \exp w^\perp$.

Its derivative at 0 is Id on W & on $W^\perp$ & is linear so is Id $\Rightarrow$
φ is a local diffeomorphism.

Assume $\exists$ sequence $(w_n, w_n^\perp)$ with
$\varphi(w_n, w_n^\perp) \in H$, $\varphi(w_n, w_n^\perp) \longrightarrow 1$

& $w_n^\perp \neq 0 \implies \exp w_n^\perp = \varphi(w_n, w_n^\perp) \cdot \exp w_n^{-1} \in H$

$\implies$ pass to subsequence of $w_n^\perp$'s so $\frac{w_n^\perp}{|w_n^\perp|} \longrightarrow w^\perp \in W^\perp$

$\implies w^\perp \in W$, contradiction. $\blacksquare$

$\implies \exp \mathfrak{h} := W$ gives good chart around $1 \in H \implies$ get charts everywhere! $\blacksquare$

<u>Corollary</u> $SL_n\mathbb{R}$, $SO_n\mathbb{R}$, $O_n\mathbb{R}$, U_n, SU_n etc are all Lie groups: clearly closed subsets (defined by equations!) & subgroups.

$T_1 SL_n\mathbb{R}$: subspace of matrices A st $\exp tA$ has determinant 1, $\forall t$.

Exercise $\det(\mathrm{Id} + tA + o(t))$

$= 1 + t \operatorname{tr} A + o(t)$: write A in Jordan form $\begin{pmatrix} \lambda_1 \ddots & 0 \\ 0 & \square \ddots \square \end{pmatrix}$ & calculate

$\implies T_1 SL_n\mathbb{R} = \mathfrak{sl}_n\mathbb{R}$, traceless matrices.

$T_1 O_n$: $(1 + tA + \dots)(1 + tA^t + \dots) = 1$

$\Rightarrow \quad A + A^t = 0$:

$\mathfrak{so}_n$, skew symmetric matrices

$T_1 U_n$: $A + \bar{A}^t = 0 \Rightarrow \mathfrak{u}_n$,

skew-hermitian matrices.

Another example of a Lie group:

- $GL_n(\mathbb{H})$ $n \times n$ matrices of quaternions : eg $GL_1(\mathbb{H}) = \mathbb{R}^4 \setminus 0$ like $GL_1\mathbb{R} = \mathbb{R}^\times$, $GL_1\mathbb{C} = \mathbb{C}^\times$ $= \mathbb{H}^\times$
- $Sp(n) \subset GL_n \mathbb{H}$ matrices with $g\bar{g}^t = Id$: quaternionic analog of $O(n)$, $U(n)$ ...

Exercise : • $Sp(n)$ is a Lie group

• $T_1 Sp(n) = \mathfrak{sp}_n := \{A + \bar{A}^t = 0\} \subset \mathfrak{gl}_n \mathbb{H}$.

• Example : $Sp(1) = SU_2 \xrightarrow{2:1} SO_3$

The groups GL, SL, SO, SU, Sp & variants are known collectively as the classical groups.

Homogeneous Spaces

Def A homogeneous space for a Lie group G is a manifold X with a transitive G-action $G \times X \to X$.

Fix $x \in X$, let $H \subset G$ be the stabilizer of X
$\Rightarrow$ as a set, $X \simeq G/H$ set of right cosets gH with its left G action

Def G a Lie group, $H \subset G$ Lie subgroup. The quotient space G/H is the set of cosets gH with the quotient topology for $p: G \to G/H$

Proposition G/H has a manifold structure s.t. p is smooth & $f: G/H \to M$ smooth iff $G \xrightarrow{fp} M$ is smooth

Proof $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{h}'$ some splitting
Let $0 \in U \subset \mathfrak{h}'$ neighborhood

$$\pi: U \xrightarrow[\exp]{} G \xrightarrow[p]{} G/H$$

$\Rightarrow$ π is a homeomorphism near eH ... $\blacksquare$

Lie Groups & Lie Algebras

- Case of matrices:

$$\exp(x)\exp(y) = \left(1 + x + \frac{x^2}{2} + \dots\right)\left(1 + y + \frac{y^2}{2} + \dots\right)$$

$$= 1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \dots$$

$$\exp(x+y) = 1 + x + y + \frac{1}{2}\left(x^2 + xy + yx + y^2\right) + \dots$$

Correct so as to get right quadratic term:

$$\exp\left(x + y + \tfrac{1}{2}(xy - yx)\right) =$$

$$1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \dots \quad \checkmark$$

so $\quad \log(\exp x \exp y) = x + y + \frac{1}{2}[x,y] + \dots$

General G

Theorem $\mathfrak{g} := T_1 G$ has a natural structure of Lie algebra

Proof/Construction $\mathfrak{g} \simeq$ left invariant vector fields on G

ξ, η left invariant, $[\xi, \eta] \in \mathrm{Vect}\, G$

Claim $[\xi, \eta]$ left invariant also

– follows from naturality / coord. independence of bracket:

for diffeomorphism θ & any vector fields ξ, η we have

$$\theta_* [\xi, \eta] = [\theta_* \xi, \theta_* \eta]$$

– apply to $\theta = l_g$ left translation

More concretely:

– recall $[\xi, \eta]$ is the vector field which acts on smooth functions as $\xi\eta - \eta\xi$.

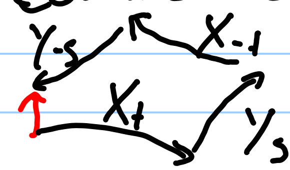
after translation get $l_g(\xi\eta - \eta\xi)$

$$= l_g(\xi\eta) - l_g(\eta\xi) = l_g(\xi) l_g(\eta) - l_g(\eta) l_g(\xi)$$

$$= [\xi, \eta].$$

$\blacksquare$

Geometric picture for the commutator of two vector fields:



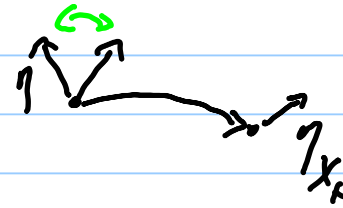
A vector field ξ defines a flow:

a one-parameter subgroup of local diffeomorphisms

X_t : take each point x & follow ξ from x to time t. (inverse: flow for time $-t$).

Can compare η after time t with η now using backwards flow X_t:

$$[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} dX_{-t}(\eta_{X_t})$$



$$= \lim_{t \to 0} \frac{dX_{-t}\,\eta_{X_t} - \eta}{t}$$

[Proof this agrees with our other formula:
Warner, Foundations of Diff. Manifolds & Lie Groups, p.70.]

[Or can let Y_s be flow along η

$$\Rightarrow \quad [\xi, \eta](x) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} X_t Y_s X_{-t} Y_{-s}(x)$$

... Case of matrices:

$$\exp x \, \exp y \, \exp(-x) \, \exp(-y)$$
$$= 1 + (xy - yx) + (\text{cubic \& higher}) \quad]$$

Proposition The one parameter group / flow associated to the left invariant vector field ξ on G is right multiplication $r_{\exp t\xi}$.

... note that left & right multiplications commute:

$$l_g r_h = g(\ )h) = (g(\ ))h = r_h l_g$$

– ie right multiplications are left invariant!

Proof Claim The integral curve of ξ through g is $l_g(\exp t\xi) = g \cdot \exp t\xi$

exercise ... follows from left invariance

$\Rightarrow$ the flow X_t sends $g \mapsto g \exp t\xi$ as desired. $\blacksquare$

Corollary Lie algebra of $[\,,\,]$ given by

$$[x,y] = \frac{d}{dt}\Big|_{t=0} d\, r_{\exp -tx}\left(Y_{\exp tx}\right)$$

$$\left[= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \exp tx \exp sy \exp -tx \exp -sy \cdot 1 \right]$$

.... follows from noting $r_{\exp tx} \cdot 1 = l_{\exp tx} \cdot 1 = \exp tx.$

Another point of view: the adjoint representation

G acts on itself by conjugation

$$a_g : h \longmapsto g h g^{-1}$$

& $a_g(h_1 h_2) = a_g(h_1)\, a_g(h_2)$,

$\Rightarrow$ $a : G \longrightarrow \operatorname{Aut} G$ group of all invertible homomorphisms $G \to G$

This action is functorial, in the sense that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "a_g"'] & H \arrow[d, "a_{\varphi(g)}"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes

$a_g(1) = 1 \ \forall g$: action preserves 1,

so can differentiate,

$$\operatorname{Ad} g := Da_g\big|_1 : \mathfrak{g} \to \mathfrak{g}$$

$$\operatorname{Ad} : G \longrightarrow \operatorname{Aut} \mathfrak{g}$$ homomorphism

<u>Def</u> A representation of a Lie group G on a vector space V is a homomorphism

$$G \longrightarrow GL(V) = \operatorname{Aut} V$$

Ad is the adjoint representation of G: canonical rep of any Lie group.

Ad is functorial:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "Ad(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{h} \arrow[d, "Ad(\varphi(g))"] \\ \mathfrak{g} \arrow[r, "\varphi'"'] & \mathfrak{h} \end{tikzcd}$$

Concretely, $Ad\, g\,(y) = \frac{d}{dt}\Big|_{t=0} g \exp ty\, g^{-1}$

· infinitesimal form of conjugation. — in fact we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad\, g"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "a_g"] & G \end{tikzcd}$$

e.g for matrix groups $Ad\, g\,(A) =$

$$\frac{d}{dt}\Big|_{t=0} g\,(1 + tA + t^2(\ldots\ldots)g^{-1}) = gAg^{-1}$$

... action of invertible matrices on all matrices!

Next, differentiate at identity $\quad Ad : G \longrightarrow Aut\,\mathfrak{g} = GL(\mathfrak{g})$

$\Rightarrow$ linear map $ad : \mathfrak{g} \longrightarrow End\,\mathfrak{g}$ called the adjoint action or representation of the Lie algebra.

Thus we have (derivative of a homomorphism of Lie groups)

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "ad"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & End\,\mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "Ad"] & Aut\,\mathfrak{g} \end{tikzcd}$$

<u>Proposition</u> $ad_X(Y) = [X,Y]$, ie $ad : X \longmapsto [X,-] \in End\,\mathfrak{g}$.

<u>Proof</u> $ad_X(Y) = \frac{d}{dt}\Big|_{t=0} Ad_{\exp tX}(Y)$

$= \frac{d}{dt}\Big|_{t=0} d\,a_{\exp tX}(Y)$

$= \frac{d}{dt}\Big|_{t=0} d\,r_{\exp -tX}\, d\,l_{\exp tX} Y$

$= \frac{d}{dt}\Big|_{t=0} dr_{\exp -tX}\, Y_{\exp tX}$

$= [X,Y]$. $\blacksquare$

Corollary 1. The assignment

$$\mathrm{Lie} : \text{Lie groups} \longrightarrow \text{Lie algebras}$$
$$G \longmapsto \mathfrak{g} = T_1 G, [\,,\,]$$

is a functor: $G \xrightarrow{\varphi} H$ any homomorphism $\Rightarrow \mathfrak{g} \xrightarrow{\varphi'} \mathfrak{h}$ is a homomorphism of Lie algebras

2. $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ is a homomorphism of Lie algebras, i.e., a representation of $\mathfrak{g}$ on the vector space $\mathfrak{g}$.

Theorem G connected $\Rightarrow$ the center $Z(G)$ of G is the kernel of the adjoint representation

Proof $g \in Z(G) \Rightarrow \exp tX = g \exp tX g^{-1}$ $\forall x, t$

$\Rightarrow \mathrm{Ad}_g(x) = X, \ \forall x \in \mathfrak{g}$

$g \in \mathrm{Ker}(\mathrm{Ad}) \Rightarrow \exp tx = g \exp tx\, g^{-1}$

$\Rightarrow g$ commutes with a neighborhood $1 \in U \subset G$, but such V generate G. $\boxed{3}$

Theorem $\varphi : G \to H$ a homomorphism of Lie groups, $A = \ker \varphi$, $\mathfrak{a} = \ker \varphi'$. Then $A \subset G$ is a Lie subgroup with Lie algebra $\mathfrak{a}$.

Proof A is closed (defined by equations) $\Rightarrow$ Lie subgroup.

But $X \in \mathfrak{g}$ lies in Lie A $\iff$ $\exp tX \in A \ \forall t \in \mathbb{R}$

So check: $\exp tX \in A$ means $\varphi(\exp tX) = 1$ $\forall t \iff \varphi'(X) = 0$, ie $X \in \mathfrak{a}$. ∎

Corollary 1. $Z(G)$ is a Lie subgroup with Lie algebra $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, -] = 0\}$

2. G connected is abelian $\iff$ $\mathfrak{g}$ is.

Pf 1. clear

2. $Z(G)$ normal. If $G \neq Z(G)$

$G/Z(G)$ connected, with tangent space $\mathfrak{g}/Z(\mathfrak{g})$ $\Rightarrow$ so if $Z(\mathfrak{g}) = \mathfrak{g} \Rightarrow Z(G) = G$

Other direction is clear ∎

The Lie correspondence

Q: To what extent can we go backwards, from Lie algebras to Lie groups?

A: All the way.

First such result:

(*) <u>Theorem</u> $\mathfrak{h} < \mathfrak{g}$ Lie subalgebra of Lie G
$\Rightarrow \exists!$ connected Lie subgroup $H < G$ with Lie $H = \mathfrak{h}$.

When $\mathfrak{h} = \mathbb{R}\cdot v$ is one dimensional, this is the exponential map, ie solving ODE.

For general case we'll need a multivariable version of the $\exists\,\&\,!$ theorem for ODEs, the <u>Frobenius theorem</u>.

<u>Def</u> A rank k distribution on a manifold M is a rank k vector subbundle D of the tangent bundle TM: ie at each $x \in M$ we have a k-dim subspace $D_x \subset T_x M$ & locally on M

$$\begin{array}{ccc} TU & \cong & U \times \mathbb{R}^n \\ \cup & & \cup \\ D|_U & = & U \times \mathbb{R}^k \end{array}$$

Def A distribution is said to be <u>involutive</u> if for any two sections ξ, η of D (ie vector fields with $\xi_x, \eta_x \in D_x$ $\forall x$) we have $[\xi, \eta]_x \in D_x$ as well

[~ ie D is closed under bracket, its sections are a Lie subalgebra of Vect M]

Def A distribution is <u>integrable</u> if for all $x \in M$ there is a k-dimensional submanifold $x \in N \subset M$ with $TN_y = D_y$ $\forall y \in N$.

Theorem (Frobenius) A distribution is integrable iff it is involutive.

(rank 1 case $\Longleftrightarrow$ $\exists$ & ! Thm for ODE)

In fact one can find a coordinate system locally near each point for which the distribution $D = \text{Span}\left\langle \frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_k} \right\rangle$ is given by a coordinate system.

[Proof in Warner]

- easy to see <u>necessary</u>, since $\left\langle \frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_k} \right\rangle$ closed under bracket.

Proof that Frobenius $\Longrightarrow$ Thm $*$:

Given $\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra,
let $D \subset TG$ be the unique left invariant distribution with $D_1 = \mathfrak{h}$ —
ie $D_g = dl_g(\mathfrak{h})$

- D is spanned by left invariant vector fields $\xi \in \mathfrak{h} \subset \mathfrak{g}$.

Since $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ (ie $\mathfrak{h}$ Lie subalgebra)
$\Longrightarrow$ D is involutive $\Longrightarrow$ D is integrable.

Let H be the maximal connected integral submanifold of D through $1 \in G$.
For $h \in H$, $l_{h^{-1}} \cdot H$ is an integral submanifold for D through 1 (by left invariance) $\Longrightarrow$ (by maximality)
$l_{h^{-1}} H = H$ $\Longrightarrow$ H closed under $(\ )^{-1}$
& product $\Longrightarrow$ H is a Lie group
with $T_1 H = D_1 = \mathfrak{h}$ $\blacksquare$

Corollary $\exists$ 1:1 correspondence between connected Lie subgroups $H < G$ & Lie subalgebras $\mathfrak{h} < \mathfrak{g}$.

Proposition G, H connected Lie groups, $\varphi: G \to H$ homomorphism. Then φ is a covering map iff $d\varphi: \mathfrak{g} \to \mathfrak{h}$ is an isomorphism

Proof First suppose φ is a covering

$d\varphi$ injective - otherwise $\operatorname{Ker} d\varphi$ exponentiates to a connected subgroup in the kernel of φ, sets collapsed $\Rightarrow$ not covering!

$d\varphi$ surjective - otherwise image is contained in a proper submanifold...

Conversely suppose $d\varphi: \mathfrak{g} \xrightarrow{\sim} \mathfrak{h}$.
$\Rightarrow \varphi$ everywhere a local diffeomorphism.
- in particular surjective since maps onto neighborhood of identity.
Let $K = \operatorname{Ker} \varphi$, discrete subgroup
$H \simeq G/K$, locally looks like $U \times K$ for $0 \in U \subset \mathfrak{g}$ small so that $\exp U$ misses K.

$\blacksquare$

So we lose some information going from Lie groups to Lie algebras — namely, lose covering spaces. Correct by looking at simply connected groups:

Proposition Let $\mathfrak{g} = \mathrm{Lie}\, G$ with G simply connected, & $\psi : \mathfrak{g} \to \mathfrak{h} = \mathrm{Lie}\, H$ a Lie algebra hom.
$\Rightarrow \exists!\ \varphi : G \to H$ s.t. $d\varphi = \psi$

Proof Uniqueness was already established.
To prove existence we'll construct the graph $\Gamma(\varphi) \subset G \times H$ via the Frobenius theorem:
Let $D \subset T(G \times H)$ be the span for $\xi \in \mathfrak{g}$ of
$\langle \xi - \psi(\xi) \rangle \subset T_{(g,h)} G \times H \simeq \mathfrak{g} \oplus \mathfrak{h}$

Check D is involutive: the tangents to G & H directions commute, so

$$[\xi - \psi(\xi), \eta - \psi(\eta)] = [\xi, \eta] - \psi([\xi, \eta]). \checkmark$$

Let Γ be the maximal integral submanifold passing through $(1,1) \in G \times H$.

Then Γ is a connected Lie subgroup of $G \times H$, exponentiating the Lie subalgebra $\mathfrak{g} \xrightarrow[(id, -\psi)]{} \mathfrak{g} \oplus \mathfrak{h}$

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[rr, bend right] & G \times H \arrow[r, "\pi_2"'] & G \end{tikzcd} \quad \text{projection}$$

has differential which is an isomorphism

$\Longrightarrow$ π_2 is a covering homomorphism of Lie groups. G simply connected

$\Longrightarrow$ $\Gamma \simeq G$ & $\pi_1 : \Gamma \to H$ is the desired homomorphism! ∎

Corollary If G, H are simply connected Lie groups & $\mathrm{Lie}\, G \simeq \mathrm{Lie}\, H$ $\Longrightarrow$ $G \simeq H$.

In fact more is true: we'll quote the following:

Ado's Theorem Every [finite dimensional] Lie algebra has a faithful representation, ie. is isomorphic to a subalgebra of some $\mathfrak{gl}_n \mathbb{R}$. ····> has a Lie group ····> has a simply connected Lie group!

Corollary: The Lie Correspondence group!

The functor Lie : Lie Groups $\longrightarrow$ Lie Algebras gives rise to an equivalence

Connected, Simply connected Lie groups $\xrightarrow{\sim}$ Lie algebras ie up to π_0, π_1, the Lie algebra knows everything about a Lie group:

$$\begin{array}{ccc} \text{isom. classes of Lie algebras} & \overset{1:1}{\longleftrightarrow} & \text{isom classes of connected, simply conn Lie groups} \\ \updownarrow 1:1 & \overset{1:1}{\nearrow\!\!\!\swarrow} & \\ \text{isogeny classes of connected Lie groups} & & \end{array}$$

Some corollaries:

Exercise

- $H < G$ is normal iff $\mathfrak{h} < \mathfrak{g}$ is an ideal (ie $[x, h] \in \mathfrak{h}$ $\forall h \in \mathfrak{h}, x \in \mathfrak{g}$)

See book or think through

- $(G,G) = \langle aba^{-1}b^{-1}, a, b \in G \rangle$ is a connected Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}] < \mathfrak{g}$

- for any subsets $A \subset G$ or $\mathfrak{a} \subset \mathfrak{g}$

- $Z_G(A)$ is a Lie group, Lie $Z_G(A) = z_{\mathfrak{g}}(A)$
$= \{x \in \mathfrak{g} : \mathrm{Ad}\, a\,(x) = x \quad \forall a \in A\}$

- Lie $Z_G(\mathfrak{a}) = z_{\mathfrak{g}}(\mathfrak{a})$

- If Lie $A = \mathfrak{a}$ $\Rightarrow$ Lie $Z_G(A) = z_{\mathfrak{g}}(\mathfrak{a})$.

etc. . . .

Baker-Campbell-Hausdorff

One can in fact construct a small piece of a Lie group directly & algebraically from its Lie algebra! clearest argument uses algebraic geometry... we'll use Ado's theorem to reduce the problem to matrices, & quote Rossmann:

Recall $x, y \in \mathfrak{g} \rightsquigarrow x * y = \log(\exp x \exp y)$
$= x + y + \frac{1}{2}[x,y] + \cdots$

Explicitly for matrices can write, using series for log:

$$x * y = \sum_{1}^{\infty} \frac{(-1)^{k-1}}{k} \left\{ \left(\sum_{0}^{\infty} \frac{x^i}{i!} \right) \left(\sum \frac{y^j}{j!} \right) - 1 \right\}^k$$

$$= \sum \frac{(-1)^{k-1}}{k} \left\{ \sum_{\substack{i,j \geq 0 \\ i+j \geq 1}} \frac{x^i y^j}{i!\, j!} \right\}^k$$

$$= \sum \frac{(-1)^{k-1}}{k} \frac{x^{i_1} y^{j_1} x^{i_2} \cdots y^{j_k}}{i_1!\, j_1! \cdots j_k!}$$

Sum over all finite sequences of pos. integers with $i_r + j_r \geq 1$.

Not so helpful: uses matrix <u>products</u>, not <u>commutators</u>, so not Lie theoretic!

BCH Theorem (Dynkin)

$$X * Y = \sum \frac{(-1)^{k-1}}{k} \frac{1}{(i_1+j_1)+\dots+(i_k+j_k)} \frac{[X^{(i_1)} Y^{(j_1)} \dots Y^{(j_k)}]}{i_1!\, j_1! \dots i_k!\, j_k!}$$

where $[X^{(i_1)} Y^{(j_1)} \dots]$ means iterated bracket

$$[X,[X,[X, \dots [Y,[Y,[\dots \quad]]]]]]]$$

– formula not important, important it exists!

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]]$$
$$+ \dots$$

coefficients (organized into Taylor series in t) involve Bernoulli numbers ....